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A half-discrete Hilbert-type inequality with a homogeneous kernel and an extension

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Abstract
Using the way of weight functions and the technique of real analysis, a half-discrete Hilbert-type inequality with a general homogeneous kernel is obtained, and a best extension with two interval variables is given. The equivalent forms, the operator expressions, the reverses and some particular cases are considered.
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1 Introduction

Assuming that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f\ (\geq 0) \in L^p\ (R_+)$, $g(\geq 0) \in L^q\ (R_+)$, $\|f\|_p = \left\{\int_0^\infty f^p(x)\mathrm{d}x\right\}^{\frac{1}{p}} > 0$, $\|\ g\ \|_q > 0$, we have the following Hardy-Hilbert's integral inequality [1]:

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}\mathrm{d}x\mathrm{d}y < \frac{\pi}{sin(\pi/p)}||f||_p||g||_q, \tag{1}$$

where the constant factor $\frac{\pi}{sin(\pi/p)}$ is the best possible. If a_m, $b_n \geq 0$, $b = \{b_n\}_{n=1}^\infty \in l^q$, $b = \{b_n\}_{n=1}^\infty \in l^q$, $||a||_p = \left\{\sum_{m=1}^\infty a_m^p\right\}^{\frac{1}{p}} > 0$, $||\ b\ ||_q > 0$, then we still have the following discrete Hardy-Hilbert's inequality with the same best constant factor $\frac{\pi}{sin(\pi/p)}$:

$$\sum_{m=1}^\infty\sum_{n=1}^\infty \frac{a_m b_n}{m+n} < \frac{\pi}{sin(\pi/p)}||a||_p||b||_q. \tag{2}$$

For $p = q = 2$, the above two inequalities reduce to the famous Hilbert's inequalities. Inequalities (1) and (2) are important in analysis and its applications [2-4].

In 1998, by introducing an independent parameter $\lambda \in (0, 1]$, Yang [5] gave an extension of (1) for $p = q = 2$. Refinement and generalizing the results from [5], Yang [6] gave some best extensions of (1) and (2) as follows: If $\lambda_1, \lambda_2 \in \mathbf{R}$, $\lambda_1 + \lambda_2 = \lambda$, $k_\lambda(x,$

y) is a non-negative homogeneous function of degree - λ satisfying for any x, y, $t>0$, $k_\lambda(tx, ty) = t^{-\lambda} k_\lambda(x, y)$, $k(\lambda_1) = \int_0^\infty k_\lambda(t,1)t^{\lambda_1-1}\mathrm{d}t \in R_+$, $\phi(x) = x^{p(1-\lambda_1)-1}$,

$f(\geq 0) \in L_{p,\phi}(R_+) = \{f|||f||_{p,\phi} := \{\int_0^\infty \phi(x)|f(x)|^p\mathrm{d}x\}^{\frac{1}{p}} < \infty\}$,

$f(\geq 0) \in L_{p,\phi}(R_+) = \{f|||f||_{p,\phi} := \{\int_0^\infty \phi(x)|f(x)|^p\mathrm{d}x\}^{\frac{1}{p}} < \infty\}$, $g(\geq 0) \in L_{q,\psi}$, $||f||_{p,\varphi}$, $||g||_{q,\psi} > 0$, then we have

$$\int_0^\infty\int_0^\infty k_\lambda(x,y)f(x)g(y)\mathrm{d}x\mathrm{d}y < k(\lambda_1)||f||_{p,\phi}||g||_{q,\psi}, \tag{3}$$

where the constant factor $k(\lambda_1)$ is the best possible. Moreover, if $k_\lambda(x, y)$ is finite and $k_\lambda(x,y)x^{\lambda_1-1}(k_\lambda(x,y)y^{\lambda_2-1})$ is decreasing with respect to $x>0$($y>0$), then for $a_m,b_n \geq 0$, $a = \{a_m\}_{m=1}^\infty \in l_{p,\phi} = \{a|||a||_{p,\phi} := \{\sum_{n=1}^\infty \phi(n)|a_n|^p\}^{\frac{1}{p}} < \infty\}$, $b = \{b_n\}_{n=1}^\infty \in l_{q,\psi}$, $||a||_{p,\varphi}$, $||b||_{q,\psi} > 0$, we have

$$\sum_{m=1}^\infty\sum_{n=1}^\infty k_\lambda(m,n)a_mb_n < k(\lambda_1)||a||_{p,\phi}||b||_{q,\psi}, \tag{4}$$

with the best constant factor $k(\lambda_1)$. Clearly, for $\lambda = 1$, $k_1(x,y) = \frac{1}{x+y}$, $\lambda_1 = \frac{1}{q}$, $\lambda_2 = \frac{1}{p}$ (3) reduces to (1), and (4) reduces to (2). Some other results about Hilbert-type inequalities are provided by [7-15].

On half-discrete Hilbert-type inequalities with the non-homogeneous kernels, Hardy et al. provided a few results in Theorem 351 of [1]. But they did not prove that the constant factors are the best possible. And, Yang [16] gave a result by introducing an interval variable and proved that the constant factor is the best possible. Recently, Yang [17] gave the following half-discrete Hilbert's inequality with the best constant factor $B(\lambda_1, \lambda_2)$(λ, $\lambda_1 > 0$, $0 < \lambda_2 \leq 1$, $\lambda_1 + \lambda_2 = \lambda$):

$$\int_0^\infty f(x)\sum_{n=1}^\infty \frac{a_n}{(x+n)^\lambda}\mathrm{d}x < B(\lambda_1,\lambda_2)||f||_{p,\varphi}||a||_{q,\psi}. \tag{5}$$

In this article, using the way of weight functions and the technique of real analysis, a half-discrete Hilbert-type inequality with a general homogeneous kernel and a best constant factor is given as follows:

$$\int_0^\infty f(x)\sum_{n=1}^\infty k_\lambda(x,\ n)a_n\mathrm{d}x < k(\lambda_1)||f||_{p,\varphi}||a||_{q,\psi}, \tag{6}$$

which is a generalization of (5). A best extension of (6) with two interval variables, some equivalent forms, the operator expressions, the reverses and some particular cases are considered.

2 Some lemmas

We set the following conditions:

Condition (i) $v(y)(y \in [n_0 - 1, \infty))$ is strictly increasing with $v(n_0 - 1) \geq 0$ and for any fixed $x \in (b, c)$, $f(x, y)$ is decreasing for $y \in (n_0 - 1, \infty)$ and strictly decreasing in an interval of $(n_0 - 1, \infty)$.

Condition (ii) $v(y)(y \in [n_0 - \frac{1}{2}, \infty))$ is strictly increasing with $v\left(n_0 - \frac{1}{2}\right) \geq 0$ and for any fixed $x \in (b, c)$, $f(x, y)$ is decreasing and strictly convex for $y \in \left(n_0 - \frac{1}{2}, \infty\right)$.

Condition (iii) There exists a constant $\beta \geq 0$, such that $v(y)(y \in [n_0 - \beta, \infty))$ is strictly increasing with $v(n_0 - \beta) \geq 0$, and for any fixed $x \in (b, c)$, $f(x, y)$ is piecewise smooth satisfying

$$R(x) := \int_{n_0-\beta}^{n_0} f(x, y)\mathrm{d}y - \frac{1}{2}f(x, n_0) - \int_{n_0}^{\infty} \rho(y)f'_y(x, y)\mathrm{d}y > 0,$$

where $\rho(y)(= y - [y] - \frac{1}{2})$ is Bernoulli function of the first order.

Lemma 1 *If* $\lambda_1, \lambda_2 \in \mathbf{R}$, $\lambda_1 + \lambda_2 = \lambda$, $k_\lambda(x, y)$ *is a non-negative finite homogeneous function of degree* - λ *in* R_+^2, $u(x)(x \in (b, c), -\infty \leq b < c \leq \infty)$ *and* $v(y)(y \in [n_0, \infty)$, $n_0 \in \mathbf{N})$ *are strictly increasing differential functions with* $u(b^+) = 0$, $v(n_0) > 0$, $u(c^-) = v(\infty) = \infty$, *setting* $K(x, y) = k_\lambda(u(x), v(y))$, *then we define weight functions* $\omega(n)$ *and* $\varpi(x)$ *as follows:*

$$\omega(n) := [v(n)]^{\lambda_2} \int_b^c K(x, n)[u(x)]^{\lambda_1 - 1}u'(x)\mathrm{d}x, n \geq n_0(n \in \mathbf{N}), \tag{7}$$

$$\varpi(x) := [u(x)]^{\lambda_1} \sum_{n=n_0}^{\infty} K(x, n)[v(n)]^{\lambda_2 - 1}v'(n), x \in (b, c). \tag{8}$$

It follows

$$\omega(n) = k(\lambda_1) := \int_0^{\infty} k_\lambda(t, 1)t^{\lambda_1 - 1}\mathrm{d}t. \tag{9}$$

Moreover, setting $f(x, y) := [u(x)]^{\lambda_1}K(x, y)[v(y)]^{\lambda_2 - 1}v'(y)$, *if* $k(\lambda_1) \in R_+$ *and one of the above three conditions is fulfilled, then we still have*

$$\varpi(x) < k(\lambda_1)(x \in (b, c)). \tag{10}$$

Proof. Setting $t = \frac{u(x)}{v(n)}$ in (7), by calculation, we have (9).

(i) If Condition (i) is fulfilled, then we have

$$\varpi(x) = \sum_{n=n_0}^{\infty} f(x,n) < [u(x)]^{\lambda_1} \int_{n_0-1}^{\infty} K(x,y)[v(y)]^{\lambda_2-1} v'(y)\mathrm{d}y$$

$$\overset{t=u(x)/v(y)}{=} \int_0^{\frac{u(x)}{v(n_0-1)}} k_\lambda(t,1)t^{\lambda_1-1}\mathrm{d}t \le k(\lambda_1).$$

(ii) If Condition (ii) is fulfilled, then by Hadamard's inequality [18], we have

$$\varpi(x) = \sum_{n=n_0}^{\infty} f(x,n) < \int_{n_0-\frac{1}{2}}^{\infty} f(x,y)\mathrm{d}y$$

$$\overset{t=u(x)/v(y)}{=} \int_0^{\frac{u(x)}{v(n_0-\frac{1}{2})}} k_\lambda(t,1)t^{\lambda_1-1}\mathrm{d}t \le k(\lambda_1).$$

(iii) If Condition (iii) is fulfilled, then by Euler-Maclaurin summation formula [6], we have

$$\begin{aligned}
\varpi(x) &= \sum_{n=n_0}^{\infty} f(x,n) \\
&= \int_{n_0}^{\infty} f(x,y)\mathrm{d}y + \frac{1}{2} f(x,n_0) + \int_{n_0}^{\infty} \rho(y) f'_y(x,y)\mathrm{d}y \\
&= \int_{n_0-\beta}^{\infty} f(x,y)\mathrm{d}y - R(x) \\
&= \int_0^{\frac{u(x)}{v(n_0-\beta)}} k_\lambda(t,1)t^{\lambda_1-1}\mathrm{d}t - R(x) \\
&\le k(\lambda_1) - R(x) < k(\lambda_1).
\end{aligned}$$

The lemma is proved. ■

Lemma 2 *Let the assumptions of Lemma 1 be fulfilled and additionally,* $p > 0 (p \ne 1)$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_n \ge 0$, $n \ge n_0 (n \in \mathbf{N})$, $f(x)$ *is a non-negative measurable function in* (b, c). *Then, (i) for* $p > 1$, *we have the following inequalities:*

$$\begin{aligned}
J_1 :&= \left\{ \sum_{n=n_0}^{\infty} \frac{v'(n)}{[v(n)]^{1-p\lambda_2}} \left[\int_d^c K(x,n) f(x)\mathrm{d}x \right]^p \right\}^{\frac{1}{p}} \\
&\le [k(\lambda_1)]^{\frac{1}{q}} \left\{ \int_d^c \varpi(x) \frac{[u(x)]^{p(1-\lambda_1)-1}}{[u'(x)]^{p-1}} f^p(x)\mathrm{d}x \right\}^{\frac{1}{p}},
\end{aligned} \tag{11}$$

$$L_1 := \left\{ \int_b^c \frac{[\varpi(x)]^{1-q} u'(x)}{[u(x)]^{1-q\lambda_1}} \left[\sum_{n=n_0}^{\infty} K(x,n) a_n \right]^q \mathrm{d}x \right\}^{\frac{1}{q}} \leq \left\{ k(\lambda_1) \sum_{n=n_0}^{\infty} \frac{[v(n)]^{q(1-\lambda_2)-1}}{[v'(n)]^{q-1}} a_n^q \right\}^{\frac{1}{q}}; \tag{12}$$

(ii) for $0 < p < 1$, *we have the reverses of (11) and (12).*

Proof (i) By Hölder's inequality with weight [18] and (9), it follows

$$\begin{aligned}
\left[\int_b^c K(x,n) f(x) \mathrm{d}x \right]^p &= \left\{ \int_b^c K(x,n) \left[\frac{[u(x)]^{(1-\lambda_1)/q}}{[v(n)]^{(1-\lambda_2)/p}} \frac{[v'(n)]^{1/p}}{[u'(x)]^{1/q}} f(x) \right] \right. \\
&\quad \times \left. \left[\frac{[v(n)]^{(1-\lambda_2)/p}}{[u(x)]^{(1-\lambda_1)/q}} \frac{[u'(x)]^{1/q}}{[v'(n)]^{1/p}} \right] \mathrm{d}x \right\}^p \\
&\leq \int_b^c K(x,n) \frac{[u(x)]^{(1-\lambda_1)(p-1)} v'(n)}{[v(n)]^{1-\lambda_2} [u'(x)]^{p-1}} f^p(x) \mathrm{d}x \\
&\quad \times \left\{ \int_b^c K(x,n) \frac{[v(n)]^{(1-\lambda_2)(q-1)} u'(x)}{[u(x)]^{1-\lambda_1} [v'(n)]^{q-1}} \mathrm{d}x \right\}^{p-1} \\
&= \left\{ \frac{\omega(n)[v(n)]^{q(1-\lambda_2)-1}}{[v'(n)]^{q-1}} \right\}^{p-1} \int_b^c K(x,n) \frac{[u(x)]^{(1-\lambda_1)(p-1)} v'(n) f^p(x)}{[v(n)]^{1-\lambda_2} [u'(x)]^{p-1}} \mathrm{d}x \\
&= \frac{[k(\lambda_1)]^{p-1}}{[v(n)]^{p\lambda_2 - 1} v'(n)} \int_b^c K(x,n) \frac{[u(x)]^{(1-\lambda_1)(p-1)} v'(n)}{[v(n)]^{1-\lambda_2} [u'(x)]^{p-1}} f^p(x) \mathrm{d}x.
\end{aligned}$$

Then, by Lebesgue term-by-term integration theorem [19], we have

$$\begin{aligned}
J_1 &\leq [k(\lambda_1)]^{\frac{1}{q}} \left\{ \sum_{n=n_0}^{\infty} \int_b^c K(x,n) \frac{[u(x)]^{(1-\lambda_1)(p-1)} v'(n) f^p(x)}{[v(n)]^{1-\lambda_2} [u'(x)]^{p-1}} \mathrm{d}x \right\}^{\frac{1}{p}} \\
&= [k(\lambda_1)]^{\frac{1}{q}} \left\{ \int_b^c \sum_{n=n_0}^{\infty} K(x,n) \frac{[u(x)]^{(1-\lambda_1)(p-1)} v'(n) f^p(x)}{[v(n)]^{1-\lambda_2} [u'(x)]^{p-1}} \mathrm{d}x \right\}^{\frac{1}{p}} \\
&= [k(\lambda_1)]^{\frac{1}{q}} \left\{ \int_b^c \varpi(x) \frac{[u(x)]^{p(1-\lambda_1)-1}}{[u'(x)]^{p-1}} f^p(x) \mathrm{d}x \right\}^{\frac{1}{p}},
\end{aligned}$$

and (11) follows.

Still by Hölder's inequality, we have

$$\begin{aligned}
\left[\sum_{n=n_0}^{\infty} K(x,n) a_n \right]^q &= \left\{ \sum_{n=n_0}^{\infty} K(x,n) \left[\frac{[u(x)]^{(1-\lambda_1)/q}}{[v(n)]^{(1-\lambda_2)/p}} \frac{[v'(n)]^{1/p}}{[u'(x)]^{1/q}} \right] \right. \\
&\quad \times \left. \left[\frac{[v(n)]^{(1-\lambda_2)/p}}{[u(x)]^{(1-\lambda_1)/q}} \frac{[u'(x)]^{1/q}}{[v'(n)]^{1/p}} a_n \right] \right\}^q \\
&\leq \left\{ \sum_{n=n_0}^{\infty} K(x,n) \frac{[u(x)]^{(1-\lambda_1)(p-1)}}{[v(n)]^{1-\lambda_2}} \frac{v'(n)}{[u'(x)]^{p-1}} \right\}^{q-1} \\
&\quad \times \sum_{n=n_0}^{\infty} K(x,n) \frac{[v(n)]^{(1-\lambda_2)(q-1)}}{[u(x)]^{1-\lambda_1}} \frac{u'(x)}{[v'(n)]^{q-1}} a_n^q \\
&= \frac{[u(x)]^{1-q\lambda_1}}{[\varpi(x)]^{1-q} u'(x)} \sum_{n=n_0}^{\infty} K(x,n) \frac{u'(x)}{[u(x)]^{1-\lambda_1}} \frac{[v(n)]^{(q-1)(1-\lambda_2)}}{[v'(n)]^{q-1}} a_n^q.
\end{aligned}$$

Then, by Lebesgue term-by-term integration theorem, we have

$$L_1 \leq \left\{ \int_b^c \sum_{n=n_0}^{\infty} K(x,n) \frac{u'(x)}{[u(x)]^{1-\lambda_1}} \frac{[v(n)]^{(q-1)(1-\lambda_2)}}{[v'(n)]^{q-1}} a_n^q \mathrm{d}x \right\}^{\frac{1}{q}}$$

$$= \left\{ \sum_{n=n_0}^{\infty} \left[[v(n)]^{\lambda_2} \int_b^c K(x,n) \frac{u'(x)\mathrm{d}x}{[u(x)]^{1-\lambda_1}} \right] \frac{[v(n)]^{q(1-\lambda_2)-1}}{[v'(n)]^{q-1}} a_n^q \right\}^{\frac{1}{q}}$$

$$= \left\{ \sum_{n=n_0}^{\infty} \omega(n) \frac{[v(n)]^{q(1-\lambda_2)-1}}{[v'(n)]^{q-1}} a_n^q \right\}^{\frac{1}{q}},$$

and then in view of (9), inequality (12) follows.

(ii) By the reverse Hölder's inequality [18] and in the same way, for $q < 0$, we have the reverses of (11) and (12). ■

3 Main results

We set $\Phi(x) := \frac{[u(x)]^{p(1-\lambda_1)-1}}{[u'(x)]^{p-1}} (x \in (b,c))$, $\Psi(n) := \frac{[v(n)]^{q(1-\lambda_2)-1}}{[v'(n)]^{q-1}} (n \geq n_0, n \in \mathbf{N})$, wherefrom $[\Phi(x)]^{1-q} = \frac{u'(x)}{[u(x)]^{1-q\lambda_1}}$, $[\Psi(n)]^{1-p} = \frac{v'(n)}{[v(n)]^{1-p\lambda_2}}$.

Theorem 1 *Suppose that $\lambda_1, \lambda_2 \in \mathbf{R}$, $\lambda_1 + \lambda_2 = \lambda$, $k_\lambda(x, y)$ is a non-negative finite homogeneous function of degree - λ, in R_+^2, $u(x) (x \in (b,c), -\infty \leq b < c \leq \infty)$ and $v(y)$ ($y \in [n_0, \infty)$, $n_0 \in \mathbf{N}$ are strictly increasing differential functions with $u(b^+) = 0$, $v(n_0) > 0$, $u(c^-) = v(\infty) = \infty$, $\varpi(x) < k(\lambda_1) \in R_+ (x \in (b, c))$. If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(x)$, $a_n \geq 0$, f ⎣ $L_{p\Phi}(b, c)$, $a = \{a_n\}_{n=n_0}^{\infty} \in l_{q,\Psi}$, $|| f ||_{p,\Phi} > 0$ and $|| a ||_{q,\Psi} > 0$, then we have the following equivalent inequalities:*

$$I := \sum_{n=n_0}^{\infty} a_n \int_b^c K(x,n) f(x) \mathrm{d}x = \int_b^c f(x) \sum_{n=n_0}^{\infty} K(x,n) a_n \mathrm{d}x < k(\lambda_1) ||f||_{p,\Phi} ||a||_{q,\Psi}, \tag{13}$$

$$J := \left\{ \sum_{n=n_0}^{\infty} [\Psi(n)]^{1-p} \left[\int_b^c K(x,n) f(x) \mathrm{d}x \right]^p \right\}^{\frac{1}{p}} < k(\lambda_1) ||f||_{p,\Phi}, \tag{14}$$

$$L := \left\{ \int_b^c [\Phi(x)]^{1-q} \left[\sum_{n=n_0}^{\infty} K(x,n) a_n \right]^q \mathrm{d}x \right\}^{\frac{1}{q}} < k(\lambda_1) ||a||_{q,\Psi}. \tag{15}$$

Moreover, if $\frac{v'(y)}{v(y)} (y \geq n_0)$ is decreasing and there exist constants $\delta < \lambda_1$ and $M > 0$, such that $k_\lambda(t, 1) \leq \frac{M}{t^\delta} \left(t \in \left(0, \frac{1}{v(n_0)} \right] \right)$, then the constant factor $k(\lambda_1)$ in the above inequalities is the best possible.

Proof By Lebesgue term-by-term integration theorem, there are two expressions for I in (13). In view of (11), for $\varpi(x) < k(\lambda_1) \in R_+$, we have (14). By Hölder's inequality, we have

$$I = \sum_{n=n_0}^{\infty} \left[\Psi^{\frac{-1}{q}}(n) \int_b^c K(x,\ n) f(x) \mathrm{d}x \right] [\Psi^{\frac{1}{q}}(n) a_n] \leq J||a||_{q,\Psi}. \tag{16}$$

Then, by (14), we have (13). On the other hand, assuming that (13) is valid, setting

$$a_n := [\Psi(n)]^{1-p} \left[\int_b^c K(x,\ n) f(x) \mathrm{d}x \right]^{p-1}, n \geq n_0,$$

then $J^{p-1} = || a ||_{q,\Psi}$. By (11), we find $J <\infty$. If $J = 0$, then (14) is naturally valid; if J >0, then by (13), we have

$$||a||_{q,\Psi}^q = J^p = I < k(\lambda_1)||f||_{p,\Phi}||a||_{q,\Psi}, ||a||_{q,\Psi}^{q-1} = J < k(\lambda_1)||f||_{p,\Phi},$$

and we have (14), which is equivalent to (13).

In view of (12), for $[\varpi(x)]^{1-q} > [k(\lambda_1)]^{1-q}$, we have (15). By Hölder's in equality, we find

$$I = \int_b^c [\Phi^{\frac{1}{p}}(x) f(x)] \left[\Phi^{\frac{-1}{p}}(x) \sum_{n=n_0}^{\infty} K(x,n) a_n \right] \mathrm{d}x \leq \ ||f||_{p,\Phi} L. \tag{17}$$

Then, by (15), we have (13). On the other hand, assuming that (13) is valid, setting

$$f(x) := [\Phi(x)]^{1-q} \left[\sum_{n=n_0}^{\infty} K(x,n) a_n \right]^{q-1}, x \in (b,c),$$

then $L^q{}_{-1} = || f ||_{p\Phi}$. By (12), we find $L <\infty$. If $L = 0$, then (15) is naturally valid; if L >0, then by (13), we have

$$||f||_{p,\Phi}^p = L^q = I < k(\lambda_1)||f||_{p,\Phi}||a||_{q,\Psi}, ||f||_{p,\Phi}^{p-1} = L < k(\lambda_1)||a||_{q,\Psi},$$

and we have (15) which is equivalent to (13).

Hence, inequalities (13), (14) and (15) are equivalent.

There exists an unified constant $d \in (b, c)$, satisfying $u(d) = 1$. For $0 < \varepsilon < p(\lambda_1 - \delta)$, setting $\tilde{f}(x) = 0$, $x \in (b,\ d)$; $\tilde{f}(x) = [u(x)]^{\lambda_1 - \frac{\varepsilon}{p} - 1} u'(x)$, $x \in (d,\ c)$, and $\tilde{a}_n = [v(n)]^{\lambda_2 - \frac{\varepsilon}{q} - 1} v'(n)$, $n \geq n_0$, if there exists a positive number $k(\leq k(\lambda_1))$, such that (13) is valid as we replace $k(\lambda_1)$ by k, then in particular, we find

$$\begin{aligned} \tilde{I} &:= \sum_{n=n_0}^{\infty} \int_b^c K(x,n) \tilde{a}_n \tilde{f}(x) \mathrm{d}x < k||\tilde{f}||_{p,\Phi}||\tilde{a}||_{q,\Psi} \\ &= k \left\{ \int_d^c \frac{u'(x)}{[u(x)]^{\varepsilon+1}} \mathrm{d}x \right\}^{\frac{1}{p}} \left\{ \frac{v'(n_0)}{[v(n_0)]^{\varepsilon+1}} + \sum_{n=n_0+1}^{\infty} \frac{v'(n)}{[v(n)]^{\varepsilon+1}} \right\}^{\frac{1}{q}} \\ &< k \left(\frac{1}{\varepsilon} \right)^{\frac{1}{p}} \left\{ \frac{v'(n_0)}{[v(n_0)]^{\varepsilon+1}} + \int_{n_0}^{\infty} [v(y)]^{-\varepsilon-1} v'(y) \mathrm{d}y \right\}^{\frac{1}{q}} \\ &= \frac{k}{\varepsilon} \left\{ \varepsilon \frac{v'(n_0)}{[v(n_0)]^{\varepsilon+1}} + [v(n_0)]^{-\varepsilon} \right\}^{\frac{1}{q}} \end{aligned} \tag{18}$$

$$\begin{aligned}
\tilde{I} &= \sum_{n=n_0}^{\infty} [v(n)]^{\lambda_2-\frac{\varepsilon}{q}-1} v'(n) \int_d^c K(x,n)[u(x)]^{\lambda_1-\frac{\varepsilon}{p}-1} u'(x)\mathrm{d}x \\
&\overset{t=u(x)/v(n)}{=} \sum_{n=n_0}^{\infty} [v(n)]^{-\varepsilon-1} v'(n) \int_{1/v(n)}^{\infty} k_\lambda(t,\ 1) t^{\lambda_1-\frac{\varepsilon}{p}-1}\mathrm{d}t \\
&= k\left(\lambda_1 - \frac{\varepsilon}{p}\right) \sum_{n=n_0}^{\infty} [v(n)]^{-\varepsilon-1} v'(n) - A(\varepsilon) \\
&> k\left(\lambda_1 - \frac{\varepsilon}{p}\right) \int_{n_0}^{\infty} [v(y)]^{-\varepsilon-1} v'(y)\mathrm{d}y - A(\varepsilon) \\
&= \frac{1}{\varepsilon} k\left(\lambda_1 - \frac{\varepsilon}{p}\right) [v(n_0)]^{-\varepsilon} - A(\varepsilon), \\
A(\varepsilon) &:= \sum_{n=n_0}^{\infty} [v(n)]^{-\varepsilon-1} v'(n) \int_0^{1/v(n)} k_\lambda(t,1) t^{\lambda_1-\frac{\varepsilon}{p}-1}\mathrm{d}t.
\end{aligned} \tag{19}$$

For $k_\lambda(t,1) \le M\left(\frac{1}{t^\delta}\right)$ $(\delta < \lambda_1; t \in (0, 1/v(n_0)])$, we find

$$\begin{aligned}
0 < A(\varepsilon) &\le M \sum_{n=n_0}^{\infty} [v(n)]^{-\varepsilon-1} v'(n) \int_0^{1/v(n)} t^{\lambda_1-\delta-\frac{\varepsilon}{p}-1}\mathrm{d}t \\
&= \frac{M}{\lambda_1-\delta-\frac{\varepsilon}{p}} \sum_{n=n_0}^{\infty} [v(n)]^{-\lambda_1+\delta-\frac{\varepsilon}{q}-1} v'(n) \\
&= \frac{M}{\lambda_1-\delta-\frac{\varepsilon}{p}} \left[\frac{v'(n_0)}{[v(n_0)]^{\lambda_1-\delta+\frac{\varepsilon}{q}+1}} + \sum_{n=n_0+1}^{\infty} \frac{v'(n)}{[v(n)]^{\lambda_1-\delta+\frac{\varepsilon}{q}+1}} \right] \\
&\le \frac{M}{\lambda_1-\delta-\frac{\varepsilon}{p}} \left[\frac{v'(n_0)}{[v(n_0)]^{\lambda_1-\delta+\frac{\varepsilon}{q}+1}} + \int_{n_0}^{\infty} \frac{v'(y)}{[v(y)]^{\lambda_1-\delta+\frac{\varepsilon}{q}+1}}\mathrm{d}y \right] \\
&= \frac{M}{\lambda_1-\delta-\frac{\varepsilon}{p}} \left[\frac{v'(n_0)}{[v(n_0)]^{\lambda_1-\delta+\frac{\varepsilon}{q}+1}} + \frac{[v(n_0)]^{-\lambda_1+\delta-\frac{\varepsilon}{q}}}{\lambda_1-\delta+\frac{\varepsilon}{q}} \right] < \infty,
\end{aligned}$$

namely $A(\varepsilon) = O(1)(\varepsilon \to 0^+)$. Hence, by (18) and (19), it follows

$$k\left(\lambda_1 - \frac{\varepsilon}{p}\right)[v(n_0)]^{-\varepsilon} - \varepsilon O(1) < k\left\{\varepsilon \frac{v'(n_0)}{[v(n_0)]^{\varepsilon+1}} + [v(n_0)]^{-\varepsilon}\right\}^{\frac{1}{q}}. \tag{20}$$

By Fatou Lemma [19], we have $k(\lambda_1) \le \underline{lim}_{\varepsilon\to 0^+} k\left(\lambda_1 - \frac{\varepsilon}{p}\right)$, then by (20), it follows $k(\lambda_1) \le k(\varepsilon \to 0^+)$. Hence, $k = k(\lambda_1)$ is the best value of (12).

By the equivalence, the constant factor $k(\lambda_1)$ in (14) and (15) is the best possible, otherwise we can imply a contradiction by (16) and (17) that the constant factor in (13) is not the best possible. ■

Remark 1 (i) Define a half-discrete Hilbert's operator $T: L_{p,\Phi}(b,c) \to l_{p,\Psi^{1-p}}$ as: for $f \in L_{p\Phi}(b, c)$, we define $Tf \in l_{p,\Psi^{1-p}}$, satisfying

$$Tf(n) = \int_b^c K(x,n)f(x)dx, \quad n \geq n_0.$$

Then, by (14), it follows $||Tf||_{p,\Psi^{1-p}} \leq k(\lambda_1)||f||_{p,\Phi}$ and then T is a bounded operator with $||\ T\ || \leq k(\lambda_1)$. Since, by Theorem 1, the constant factor in (14) is the best possible, we have $||\ T\ || = k(\lambda_1)$.

(ii) Define a half-discrete Hilbert's operator $\tilde{T}: l_{q,\Psi} \to L_{q,\Phi^{1-q}}(b,c)$ as: for $a \in l_{q\Psi}$, we define $\tilde{T}a \in L_{q,\Phi^{1-q}}(b,c)$, satisfying

$$\tilde{T}a(x) = \sum_{n=n_0}^{\infty} K(x,n)a_n, x \in (b,c).$$

Then, by (15), it follows $||\tilde{T}a||_{q,\Phi^{1-q}} \leq k(\lambda_1)||a||_{q,\Psi}$ and then $\tilde{T}$ is a bounded operator with $||\tilde{T}|| \leq k(\lambda_1)$. Since, by Theorem 1, the constant factor in (15) is the best possible, we have $||\tilde{T}|| = k(\lambda_1)$.

In the following theorem, for $0 < p < 1$, or $p < 0$, we still use the formal symbols of $||f||_{p,\tilde{\Phi}}$ and $||a||_{q,\Psi}$ *and so on.* ■

Theorem 2 *Suppose that* $\lambda_1, \lambda_2 \in \mathbf{R}$, $\lambda_1 + \lambda_2 = \lambda$, $k_\lambda(x, y)$ *is a non-negative finite homogeneous function of degree* $-\lambda$ *in* $\mathbf{R}_+^2$, $u(x)(x \in (b, c), -\infty \leq b < c \leq \infty)$ *and* $v(y)(y \in [n_0, \infty)$, $n_0 \in \mathbf{N})$ *are strictly increasing differential functions with* $u(b^+) = 0$, $v(n_0) > 0$, $u(c^-) = v(\infty) = \infty$, $k(\lambda_1) \in R_+$, $\theta_\lambda(x) \in (0, 1)$, $k(\lambda_1)(1 - \theta_\lambda(x)) < \varpi(x) < k(\lambda_1)(x \in (b, c))$. *If* $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(x)$, $a_n \geq 0$, $\tilde{\Phi}(x) := (1 - \theta_\lambda(x))\Phi(x)(x \in (b,c))$, $0 < ||f||_{q,\tilde{\Phi}} < \infty$ *and* $0 < ||a||_{q,\Psi} < \infty$. *Then, we have the following equivalent inequalities:*

$$\begin{aligned} I \quad &:= \quad \sum_{n=n_0}^{\infty} \int_b^c K(x,n)a_n f(x)\mathrm{d}x = \int_b^c \sum_{n=n_0}^{\infty} K(x,n)a_n f(x)\mathrm{d}x \\ &> \quad k(\lambda_1)||f||_{p,\tilde{\Phi}}||a||_{q,\Psi}, \end{aligned} \tag{21}$$

$$J \quad := \quad \left\{ \sum_{n=n_0}^{\infty} [\Psi(n)]^{1-p} \left[\int_b^c K(x,n)f(x)\mathrm{d}x \right]^p \right\}^{\frac{1}{p}} > k(\lambda_1)||f||_{p,\tilde{\Phi}}, \tag{22}$$

$$\tilde{L} \quad := \quad \left\{ \int_b^c [\tilde{\Phi}(x)]^{1-q} \left[\sum_{n=n_0}^{\infty} K(x,n)a_n \right]^q \mathrm{d}x \right\}^{\frac{1}{q}} > k(\lambda_1)||a||_{q,\Psi}. \tag{23}$$

Moreover, if $\frac{v'(y)}{v(y)}(y \geq n_0)$ *is decreasing and there exist constants* δ, $\delta_0 > 0$, *such that* $\theta_\lambda(x) = O(\frac{1}{[u(x)]^\delta})(x \in (d,c))$ *and* $k(\lambda_1 - \delta_0) \in \mathbf{R}_+$, *then the constant factor* $k(\lambda_1)$ *in the above inequalities is the best possible.*

Proof. In view of (9) and the reverse of (11), for $\varpi(x) > k(\lambda_1)(1 - \theta_\lambda(x))$, we have (22). By the reverse Hölder's inequality, we have

$$I = \sum_{n=n_0}^{\infty}\left[\Psi^{\frac{-1}{q}}(n)\int_b^c K(x,n)f(x)\mathrm{d}x\right][\Psi^{\frac{1}{q}}(n)a_n] \geq J||a||_{q,\Psi}. \tag{24}$$

Then, by (22), we have (21). On the other hand, assuming that (21) is valid, setting a_n as Theorem 1, then $J^{p-1} = ||a||_{q\Psi}$. By the reverse of (11), we find $J > 0$. If $J = \infty$, then (24) is naturally valid; if $J < \infty$, then by (21), we have

$$||a||_{q,\Psi}^q = J^p = I > k(\lambda_1)||f||_{p,\tilde{\Phi}}||a||_{q,\Psi},\ ||a||_{q,\Psi}^{q-1} = J > k(\lambda_1)||f||_{p,\tilde{\Phi}},$$

and we have (22) which is equivalent to (21).

In view of (9) and the reverse of (12), for $[\varpi(x)]^{1-q} > [k(\lambda_1)(1 - \theta_\lambda(x))]^{1-q}$ ($q < 0$), we have (23). By the reverse Hölder's inequality, we have

$$I = \int_b^c [\tilde{\Phi}^{\frac{1}{p}}(x)f(x)]\left[\tilde{\Phi}^{\frac{-1}{p}}(x)\sum_{n=n_0}^{\infty}K(x,n)a_n\right]\mathrm{d}x \geq ||f||_{p,\tilde{\Phi}}\tilde{L}. \tag{25}$$

Then, by (23), we have (21). On the other hand, assuming that (21) is valid, setting

$$f(x) := [\tilde{\Phi}(x)]^{1-q}\left[\sum_{n=n_0}^{\infty}K(x,n)a_n\right]^{q-1}, \quad x \in (b,c),$$

then $\tilde{L}^{q-1} = ||f||_{p,\tilde{\Phi}}$. By the reverse of (12), we find $\tilde{L} > 0$. If $\tilde{L} = \infty$, then (23) is naturally valid; if $\tilde{L} < \infty$, then by (21), we have

$$||f||_{p,\tilde{\Phi}}^p = \tilde{L}^q = I > k(\lambda_1)||f||_{p,\tilde{\Phi}}||a||_{q,\Psi},\ ||f||_{p,\tilde{\Phi}}^{p-1} = \tilde{L} > k(\lambda_1)||a||_{q,\Psi},$$

and we have (23) which is equivalent to (21). ■

Hence, inequalities (21), (22) and (23) are equivalent.

For $0 < \varepsilon < p\delta_0$, setting $\tilde{f}(x)$ and $\tilde{a}_n$ as Theorem 1, if there exists a positive number $k(\geq k(\lambda_1))$, such that (21) is still valid as we replace $k(\lambda_1)$ by k, then in particular, for $q < 0$, in view of (9) and the conditions, we have

$$\begin{aligned}
\tilde{I} :&= \int_b^c \sum_{n=n_0}^{\infty} K(x,n)\tilde{a}_n\tilde{f}(x)\mathrm{d}x > k||\tilde{f}||_{p,\tilde{\Phi}}||\tilde{a}||_{q,\Psi} \\
&= k\left\{\int_d^c \left(1 - O\left(\frac{1}{[u(x)]^{\delta}}\right)\right)\frac{u'(x)\mathrm{d}x}{[u(x)]^{\varepsilon+1}}\right\}^{\frac{1}{p}}\left\{\sum_{n=n_0}^{\infty}\frac{v'(n)}{[v(n)]^{\varepsilon+1}}\right\}^{\frac{1}{q}} \\
&= k\left\{\frac{1}{\varepsilon} - O(1)\right\}^{\frac{1}{p}}\left\{\frac{v'(n_0)}{[v(n_0)]^{\varepsilon+1}} + \sum_{n=n_0+1}^{\infty}\frac{v'(n)}{[v(n)]^{\varepsilon+1}}\right\}^{\frac{1}{q}} \\
&> k\left\{\frac{1}{\varepsilon} - O(1)\right\}^{\frac{1}{p}}\left\{\frac{v'(n_0)}{[v(n_0)]^{\varepsilon+1}} + \int_{n_0}^{\infty}\frac{v'(y)}{[v(y)]^{+1}}\mathrm{d}y\right\}^{\frac{1}{q}} \\
&= \frac{k}{\varepsilon}\{1 - \varepsilon O(1)\}^{\frac{1}{p}}\left\{\varepsilon\frac{v'(n_0)}{[v(n_0)]^{\varepsilon+1}} + [v(n_0)]^{-\varepsilon}\right\}^{\frac{1}{q}},
\end{aligned} \tag{26}$$

$$\begin{aligned}
\tilde{I} &= \sum_{n=n_0}^{\infty}[v(n)]^{\lambda_2-\frac{\varepsilon}{q}-1}v'(n)\int_d^c K(x,\ n)[u(x)]^{\lambda_1-\frac{\varepsilon}{p}-1}u'(x)\mathrm{d}x \\
&\le \sum_{n=n_0}^{\infty}[v(n)]^{\lambda_2-\frac{\varepsilon}{q}-1}v'(n)\int_b^c K(x,\ n)\ [u(x)]^{\lambda_1-\frac{\varepsilon}{p}-1}u'(x)\mathrm{d}x \\
&\overset{t=u(x)/v(n)}{=} \sum_{n=n_0}^{\infty}[v(n)]^{-\varepsilon-1}v'(n)\int_0^{\infty} k_\lambda(t,\ 1)t^{(\lambda_1-\frac{\varepsilon}{p})-1}\mathrm{d}t \\
&\le k\left(\lambda_1-\frac{\varepsilon}{p}\right)\left[\frac{v'(n_0)}{[v(n_0)]^{\varepsilon+1}}+\int_{n_0}^{\infty}[v(y)]^{-\varepsilon-1}v'(y)\mathrm{d}y\right] \\
&= \frac{1}{\varepsilon}k\left(\lambda_1-\frac{\varepsilon}{p}\right)\left[\varepsilon\frac{v'(n_0)}{[v(n_0)]^{\varepsilon+1}}+[v(n_0)]^{-\varepsilon}\right].
\end{aligned} \tag{27}$$

Since we have $k_\lambda(t,\ 1)t^{\lambda_1-\frac{\varepsilon}{p}-1} \le k_\lambda(t,\ 1)t^{\lambda_1-\delta_0-1}$, $t \in (0, 1]$ and

$$\int_0^1 k_\lambda(t,\ 1)t^{\lambda_1-\delta_0-1}\mathrm{d}t \le k(\lambda_1-\delta_0) < \infty,$$

then by Lebesgue control convergence theorem [19], it follows

$$\begin{aligned}
k\left(\lambda_1-\frac{\varepsilon}{p}\right) &\le \int_1^{\infty} k_\lambda(t,1)t^{\lambda_1-1}\mathrm{d}t+\int_0^1 k_\lambda(t,\ 1)t^{\lambda_1-\frac{\varepsilon}{p}-1}\mathrm{d}t \\
&= k(\lambda_1)+o(1)(\varepsilon \to 0^+).
\end{aligned}$$

By (26) and (27), we have

$$(k(\lambda_1)+o(1))\left[\varepsilon\frac{v'(n_0)}{[v(n_0)]^{\varepsilon+1}}+[v(n_0)]^{-\varepsilon}\right] > k\{1-\varepsilon O(1)\}^{\frac{1}{p}}\left[\varepsilon\frac{v'(n_0)}{[v(n_0)]^{\varepsilon+1}}+[v(n_0)]^{-\varepsilon}\right]^{\frac{1}{q}},$$

and then $k(\lambda_1) \ge k(\varepsilon \to 0^+)$. Hence, $k = k(\lambda_1)$ is the best value of (21).

By the equivalence, the constant factor $k(\lambda_1)$ in (22) and (23) is the best possible, otherwise we can imply a contradiction by (24) and (25) that the constant factor in (21) is not the best possible. ■

In the same way, for $p < 0$, we also have the following theorem.

Theorem 3 *Suppose that* $\lambda_1, \lambda_2 \in \mathbf{R}$, $\lambda_1 + \lambda_2 = \lambda$, $k_\lambda(x, y)$ *is a non-negative finite homogeneous function of degree* $-\lambda$, *in* $\mathbf{R}_+^2$, $u(x)(x \in (b, c), -\infty \le b < c \le \infty)$ *and* $v(y)(y \in [n_0, \infty)$, $n_0 \in \mathbf{N})$ *are strictly increasing differential functions with* $u(b^+) = 0$, $v(n_0) > 0$, $u(c^-) = v(\infty) = \infty$, $\varpi(x) < k(\lambda_1) \in R_+$ $(x \in (b, c))$. *If* $p < 0$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(x)$, $a_n \ge 0$, $0 < ||f||_{p\Phi} < \infty$ *and* $0 < ||a||_{q,\Psi} < \infty$. *Then, we have the following equivalent inequalities:*

$$\begin{aligned}
I := \sum_{n=n_0}^{\infty}\int_b^c K(x,n)a_n f(x)\mathrm{d}x &= \int_b^c\sum_{n=n_0}^{\infty}K(x,n)a_n f(x)\mathrm{d}x \\
&> k(\lambda_1)||f||_{p,\Phi}||a||_{q,\Psi},
\end{aligned} \tag{28}$$

$$J := \left\{ \sum_{n=n_0}^{\infty} [\Psi(n)]^{1-p} \left[\int_b^c K(x,n) f(x) \mathrm{d}x \right]^p \right\}^{\frac{1}{p}} > k(\lambda_1) ||f||_{p,\Phi}, \tag{29}$$

$$\tilde{L} := \left\{ \int_b^c [\Phi(x)]^{1-q} \left[\sum_{n=n_0}^{\infty} K(x,n) a_n \right]^q \mathrm{d}x \right\}^{\frac{1}{q}} > k(\lambda_1) ||a||_{q,\Psi}. \tag{30}$$

Moreover, if $\frac{v'(y)}{v(y)} (y \geq n_0)$ *is decreasing and there exists constant* $\delta_0 > 0$, *such that* $k(\lambda_1 + \delta_0) \in \mathbf{R}_+$, *then the constant factor* $k(\lambda_1)$ *in the above inequalities is the best possible.*

Remark 2 (i) For $n_0 = 1$, $b = 0$, $c = \infty$, $u(x) = v(x) = x$, if

$$\varpi(x) = x^{\lambda_1} \sum_{n=1}^{\infty} k_\lambda(x,n) n^{\lambda_2 - 1} < k(\lambda_1) \in \mathbf{R}_+ (x \in (0,\infty)),$$

then (13) reduces to (6). In particular, for $k_\lambda(x,n) = \frac{1}{(x+n)^\lambda} (\lambda = \lambda_1 + \lambda_2, \lambda_1 > 0, 0 < \lambda_2 \leq 1)$,(6) reduces to (5).

(ii) For $n_0 = 1$, $b = 0$, $c = \infty$, $u(x) = v(x) = x^\alpha (\alpha > 0)$, $k_\lambda(x,y) = \frac{1}{(max\{x,y\})^\lambda} (\lambda, \lambda_1 > 0, 0 < \alpha\lambda_2 \leq 1)$, since

$$f(x,y) = \frac{\alpha x^{\alpha\lambda_1} y^{\alpha\lambda_2 - 1}}{(max\{x^\alpha, y^\alpha\})^\lambda} = \begin{cases} \alpha x^{-\alpha\lambda_2} y^{\alpha\lambda_2 - 1}, y \leq x \\ \alpha x^{\alpha\lambda_1} y^{-\alpha\lambda_1 - 1}, y > x \end{cases}$$

is decreasing for $y \in (0, \infty)$ and strictly decreasing in an interval of $(0, \infty)$, then by Condition (i), it follows

$$\varpi(x) < \alpha x^{\alpha\lambda_1} \int_0^\infty \frac{1}{(max\{x^\alpha, y^\alpha\})^\lambda} y^{\alpha(\lambda_2 - 1)} y^{\alpha - 1} \mathrm{d}y$$
$$\overset{t=(y/x)^\alpha}{=} \int_0^\infty \frac{t^{\lambda_2 - 1}}{(max\{1,t\})^\lambda} \mathrm{d}t = \frac{\lambda}{\lambda_1 \lambda_2} = k(\lambda_1).$$

Since for $\delta = \frac{\lambda_1}{2} < \lambda_1$, $k_\lambda(t,1) = 1 \leq \frac{1}{t^\delta} (t \in (0,1])$, then by (13), we have the following inequality with the best constant factor $\frac{\lambda}{\alpha\lambda_1\lambda_2}$:

$$\sum_{n=1}^{\infty} a_n \int_0^\infty \frac{1}{(max\{x^\alpha, n^\alpha\})^\lambda} f(x) \mathrm{d}x$$
$$< \frac{\lambda}{\alpha\lambda_1\lambda_2} \left\{ \int_0^\infty x^{p(1-\alpha\lambda_1)-1} f^p(x) \mathrm{d}x \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} n^{q(1-\alpha\lambda_2)-1} a_n^q \right\}^{\frac{1}{q}}. \tag{31}$$

(iii) For $n_0 = 1$, $b = \beta$, $c = \infty$, $u(x) = v(x) = x - \beta (0 \leq \beta \leq \frac{1}{2})$, $k_\lambda(x,y) = \frac{ln(x/y)}{x^\lambda - y^\lambda} (\lambda, \lambda_1 > 0, 0 < \lambda_2 \leq 1)$, since for any fixed $x \in (\beta, \infty)$,

$$f(x,y) = (x-\beta)^{\lambda_1} \frac{ln[(x-\beta)/(y-\beta)]}{(x-\beta)^{\lambda}-(y-\beta)^{\lambda}}(y-\beta)^{\lambda_2-1}$$

is decreasing and strictly convex for $y \in (\frac{1}{2}, \infty)$, then by Condition (ii), it follows

$$\begin{aligned} \varpi(x) &< (x-\beta)^{\lambda_1} \int_{\frac{1}{2}}^{\infty} \frac{ln[(x-\beta)/(y-\beta)]}{(x-\beta)^{\lambda}-(y-\beta)^{\lambda}}(y-\beta)^{\lambda_2-1}\mathrm{d}y \\ &\overset{t=[(y-\beta)/(x-\beta)]^{\lambda}}{=} \frac{1}{\lambda^2} \int_{[\frac{\frac{1}{2}-\beta}{x-\beta}]^{\lambda}}^{\infty} \frac{lnt}{t-1} t^{(\lambda_2/\lambda)-1}\mathrm{d}t \\ &\leq \frac{1}{\lambda^2} \int_0^{\infty} \frac{lnt}{t-1} t^{(\lambda_2/\lambda)-1}\mathrm{d}t = \left[\frac{\pi}{\lambda sin(\frac{\pi\lambda_1}{\lambda})}\right]^2. \end{aligned}$$

Since for $\delta = \frac{\lambda_1}{2} < \lambda_1$, $k_{\lambda}(t,1) = \frac{lnt}{t^{\lambda}-1} \leq \frac{M}{t^{\delta}} (t \in (0,1])$, then by (13), we have the following inequality with the best constant factor $[\frac{\pi}{\lambda sin(\frac{\pi\lambda_1}{\lambda})}]^2$:

$$\begin{aligned} &\sum_{n=1}^{\infty} a_n \int_{\beta}^{\infty} \frac{ln[(x-\beta)/(n-\beta)]}{(x-\beta)^{\lambda}-(n-\beta)^{\lambda}} f(x)\mathrm{d}x < \left[\frac{\pi}{\lambda sin(\frac{\pi\lambda_1}{\lambda})}\right]^2 \\ &\times \left\{\int_{\beta}^{\infty} (x-\beta)^{p(1-\lambda_1)-1} f^p(x)\mathrm{d}x\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} (n-\beta)^{q(1-\lambda_2)-1} a_n^q\right\}^{\frac{1}{q}}. \end{aligned} \tag{32}$$

(iv) For $n_0 = 1$, $b = 1 - \beta = \gamma$, $c = \infty$, $u(x) = v(x) = (x - \gamma)$, $k_{\lambda}(x,y) = \frac{1}{x^{\lambda}+y^{\lambda}} (0 < \lambda \leq 4)$, $k_{\lambda}(x,y) = \frac{1}{x^{\lambda}+y^{\lambda}} (0 < \lambda \leq 4)$, $\lambda_1 = \lambda_2 = \frac{\lambda}{2}$, we have

$$\begin{aligned} k\left(\frac{\lambda}{2}\right) &= \int_0^{\infty} k_{\lambda}(t,1) t^{\frac{\lambda}{2}-1}\mathrm{d}t = \frac{2}{\lambda}\int_0^{\infty} \frac{1}{t^{\lambda}+1}\mathrm{d}t^{\frac{\lambda}{2}} \\ &= \frac{2}{\lambda} \arctan t^{\frac{\lambda}{2}} |_0^{\infty} = \frac{\pi}{\lambda} \in \mathbf{R}_+ \end{aligned}$$

and

$$f(x,y) = \frac{(x-\gamma)^{\frac{\lambda}{2}}(y-\gamma)^{\frac{\lambda}{2}-1}}{(x-\gamma)^{\lambda}+(y-\gamma)^{\lambda}} (x, y \in (\gamma, \infty)).$$

Hence, $v(y) (y \in [\gamma, \infty))$ is strictly increasing with $v(1 - \beta) = v(\gamma) = 0$, and for any fixed $x \in (\gamma, \infty)$, $f(x, y)$ is smooth with

$$f'_y(x,y) = -\left(1+\frac{\lambda}{2}\right)\frac{(x-\gamma)^{\frac{\lambda}{2}}(y-\gamma)^{\frac{\lambda}{2}-2}}{(x-\gamma)^{\lambda}+(y-\gamma)^{\lambda}} + \frac{\lambda(x-\gamma)^{\frac{3\lambda}{2}}(y-\gamma)^{\frac{\lambda}{2}-2}}{[(x-\gamma)^{\lambda}+(y-\gamma^{\lambda}]^2}.$$

We set

$$R(x) := \int_{\gamma}^{1} f(x,y)\mathrm{d}y - \frac{1}{2}f(x,1) - \int_{1}^{\infty} \rho(y)f'_{y}(x,y)\mathrm{d}y. \tag{33}$$

For $x \in (\gamma, \infty)$, $0 < \lambda \le 4$, by (33) and the following improved Euler-Maclaurin summation formula [6]:

$$\frac{-1}{8}g(1) < \int_{1}^{\infty} \rho(y)g(y)\mathrm{d}y < 0((-1)^{i}g^{(i)}(y) > 0, g^{(i)}(\infty) = 0,\ i = 0, 1),$$

we have

$$\begin{aligned}
R(x) &= \int_{\gamma}^{1} \frac{(x-\gamma)^{\frac{\lambda}{2}}(y-\gamma)^{\frac{\lambda}{2}-1}}{(x-\gamma)^{\lambda}+(y-\gamma)^{\lambda}}\mathrm{d}y - \frac{1}{2}\frac{(x-\gamma)^{\frac{\lambda}{2}}(1-\gamma)^{\frac{\lambda}{2}-1}}{(x-\gamma)^{\lambda}+(1-\gamma)^{\lambda}} \\
&\quad + \left(1+\frac{\lambda}{2}\right)\int_{1}^{\infty} \rho(y)\frac{(x-\gamma)^{\frac{\lambda}{2}}(y-\gamma)^{\frac{\lambda}{2}-2}}{(x-\gamma)^{\lambda}+(y-\gamma)^{\lambda}}\mathrm{d}y \\
&\quad - \int_{1}^{\infty} \rho(y)\frac{\lambda(x-\gamma)^{\frac{3\lambda}{2}}(y-\gamma)^{\frac{\lambda}{2}-2}}{[(x-\gamma)^{\lambda}+(y-\gamma^{\lambda}]^{2}}\mathrm{d}y \\
&> \frac{2}{\lambda}\arctan\left(\frac{1-\gamma}{x-\gamma}\right)^{\frac{\lambda}{2}} - \frac{(1-\gamma)^{\frac{\lambda}{2}-1}(x-\gamma)^{\frac{\lambda}{2}}}{2[(1-\gamma)^{\lambda}+(x-\gamma^{\lambda}]} \\
&\quad - \frac{1}{8}\left(1+\frac{\lambda}{2}\right)\frac{(1-\gamma)^{\frac{\lambda}{2}-2}(x-\gamma)^{\frac{\lambda}{2}}}{(1-\gamma)^{\lambda}+(x-\gamma)^{\lambda}} + 0 \\
&= h(x) := \frac{2}{\lambda}\arctan\left(\frac{1-\gamma}{x-\gamma}\right)^{\frac{\lambda}{2}} \\
&\quad - \left[\frac{1-\gamma}{2} + \frac{1}{8}(1+\frac{\lambda}{2})\right]\frac{(1-\gamma)^{\frac{\lambda}{2}-2}(x-\gamma)^{\frac{\lambda}{2}}}{(1-\gamma)^{\lambda}+(x-\gamma)^{\lambda}}.
\end{aligned}$$

Since for $\gamma \le 1 - \frac{1}{8}[\lambda + \sqrt{\lambda(3\lambda+4)}]$, i.e. $1-\gamma \ge \frac{1}{8}[\lambda + \sqrt{\lambda(3\lambda+4)}](0 < \lambda \le 4)$,

$$\begin{aligned}
h'(x) &= \frac{(1-\gamma)^{\frac{\lambda}{2}}(x-\gamma)^{-\frac{\lambda}{2}-1}}{(1-\gamma)^{\lambda}+(x-\gamma)^{\lambda}} - \left[\frac{1-\gamma}{2}+\frac{1}{8}(1+\frac{\lambda}{2})\right](1-\gamma)^{\frac{\lambda}{2}-2} \\
&\quad \times \left\{\frac{\lambda(x-\gamma)^{\frac{\lambda}{2}-1}}{2[(1-\gamma)^{\lambda}+(x-\gamma)^{\lambda}]} - \frac{\lambda(x-\gamma)^{\frac{3\lambda}{2}-1}}{[(1-\gamma)^{\lambda}+(x-\gamma)^{\lambda}]^{2}}\right\} \\
&= -\left[(1-\gamma)^{2} - \frac{\lambda}{4}(1-\gamma) - \frac{\lambda}{16}(1+\frac{\lambda}{2})\right]\frac{(1-\gamma)^{\frac{\lambda}{2}-2}(x-\gamma)^{\frac{\lambda}{2}-1}}{(1-\gamma)^{\lambda}+(x-\gamma)^{\lambda}} \\
&\quad - \left[\frac{1-\gamma}{2}+\frac{1}{8}(1+\frac{\lambda}{2})\right]\frac{\lambda(1-\gamma)^{\frac{3\lambda}{2}-2}(x-\gamma)^{\frac{\lambda}{2}-1}}{[(1-\gamma)^{\lambda}+(x-\gamma)^{\lambda}]^{2}} < 0,
\end{aligned}$$

then $h(x)$ is strictly decreasing and $R(x) > h(x) > h(\infty) = 0$.

Then, by Condition (iii), it follows $\varpi(x) < k(\frac{\lambda}{2}) = \frac{\pi}{\lambda}(x \in (\gamma, \infty))$. For $\delta = 0 < \frac{\lambda}{2}$, it follows

$$k_\lambda(t, 1) = \frac{1}{t^\lambda + 1} \le 1 = \frac{1}{t^\delta}, \quad t \in \left(0, \frac{1}{1-\gamma}\right),$$

and by (13), we have the following inequality with the best constant factor $\frac{\pi}{\lambda}$:

$$\sum_{n=1}^{\infty} a_n \int_{\gamma}^{\infty} \frac{1}{(x-\gamma)^\lambda + (n-\gamma)^\lambda} f(x)\mathrm{d}x < \frac{\pi}{\lambda}\left\{\int_{\gamma}^{\infty} (x-\gamma)^{p(1-\frac{\lambda}{2})-1} f^p(x)\mathrm{d}x\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} (n-\gamma)^{q(1-\frac{\lambda}{2})-1} a_n^q\right\}^{\frac{1}{q}}, \tag{34}$$

where $\gamma \le 1 - \frac{1}{8}[\lambda + \sqrt{\lambda(3\lambda + 4)}](0 < \lambda \le 4)$.

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Authors' contributions
BY wrote and reformed the article. QC conceived of the study, and participated in its design and coordination. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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