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S_n -equivariant sheaves and Koszul cohomology

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Abstract

Purpose: We give a new interpretation of Koszul cohomology, which is equivalent under the Bridgeland-King-Reid equivalence to Voisin's Hilbert scheme interpretation in dimensions 1 and 2 but is different in higher dimensions.

Methods: We show that an explicit resolution of a certain S_n -equivariant sheaf is equivalent to a resolution appearing in the theory of Koszul cohomology.

Results: Our methods easily show that the dimension $K_{p,q}(B, L)$ is a polynomial in d for $L = dA + P$ with A ample and d large enough.

Conclusions: This interpretation allows us to extract various pieces of information about asymptotic properties $K_{p,q}$ for fixed p, q .

Background

The Koszul cohomology of a line bundle L on an algebraic variety X was introduced by Green in [1]. Koszul cohomology is very closely related to the syzygies of the embedding defined by L (if L is very ample) and is thus related to a host of classical questions. We assume that the reader is already aware of these relations and the main theorems on Koszul cohomology; for a detailed discussion, the reader is referred to [1].

Recently, in [2], it was realized that a conjectural uniform asymptotic picture emerges as L becomes more positive. We give a new interpretation of Koszul cohomology which we believe will clarify this picture. As an application, we prove that $K_{p,q}(B, dA + P)$ grows polynomially in d . In particular, this gives a partial answer to Problem 7.2 from [2]. More precisely, we establish.

Theorem 1. *Let A and P be ample line bundles on a smooth projective variety X . Let B be a locally free sheaf on X . Then, there is a polynomial that equals $\dim K_{p,q}(B, dA + P)$ for d sufficiently large.*

The inspiration for our new interpretation came from Voisin's papers [3,4] on generic Green's conjecture. To prove generic Green's conjecture, Voisin writes Koszul cohomology for X a surface in terms of the sheaf cohomology of various sheaves on a Hilbert scheme of points of X . While this allows for the use of the geometry of Hilbert schemes in analyzing Koszul cohomology, this interpretation has multiple downsides.

The main downside is that it does not generalize well to higher dimensions, as the Hilbert scheme of points of X is not necessarily smooth (or even irreducible) unless X

has dimension at most 2. It is unclear to us if this difficulty can be circumvented through uniform use of the curvilinear Hilbert scheme, but even if it can, the loss of properness creates various technical difficulties.

Our construction can be thought of as replacing the Hilbert scheme with a noncommutative resolution of singularities (though the word noncommutative need not ever be mentioned). Specifically, we replace the cohomology of sheaves on Hilbert schemes with the S_n -invariants of the cohomology of S_n -equivariant sheaves on n th-fold products. This is motivated by the Bridgeland-King-Reid equivalence [5], which implies that for a curve or surface X , the derived category of coherent sheaves on the Hilbert scheme is equivalent to the derived category of S_n -equivariant coherent sheaves on X^n . We note that this had been previously used to compute the cohomology of sheaves on the Hilbert scheme, for instance in [6].

We would also like to note two other results related to ours. If one chooses to work with ordinary sheaves, as opposed to S_n -equivariant sheaves, one recovers in effect a theorem implicit in [7], which was first stated explicitly in [8]. We refer to the discussion after Theorem 3 for more detail. Finally, we note that Theorem C of [9] is essentially 4 for curves. The proof methods are very similar, and we are hopeful that Theorem 3 will help in extending their other results to higher dimensions.

'Methods' section of this paper is a short introduction to Koszul cohomology. The main goal is to describe the conjectural asymptotic story of [2]. We also describe (but do not prove) Voisin's interpretation of Koszul cohomology in terms of Hilbert schemes.

'Results and discussions' section contains our new interpretation. 'Conclusions' section then uses it to analyze the asymptotics of $K_{p,q}(dL + B)$. All the proofs in both sections are quite short.

Methods

Our discussion of Koszul cohomology will be quite terse. For details and motivation, see [1,2].

Let X be a smooth projective algebraic variety of dimension n over an algebraically closed field k of characteristic zero and let L be a line bundle on X . Form the graded ring $S = \text{Sym}^\bullet H^0(L)$. For any coherent sheaf B on X , we have a naturally graded S -module structure on $M = \bigoplus_{n \geq 0} H^0(B + nL)$. From this we can construct the bigraded vector space $\text{Tor}_S^{\bullet\bullet}(M, k)$. The (p, q) th Koszul cohomology of (B, L) is the $(p, p+q)$ -bigraded part of this vector space. We will call its dimension the (p, q) Betti number and the two-dimensional table of these the Betti table.

In this paper, we will always take B to be locally free.

The first theorem we describe is a duality theorem for Koszul cohomology, proven in Green's original paper [1]. (In fact, Green proves a slightly stronger result).

Theorem 2. *Assume that L is base-point-free and $H^i(B \otimes (q-i)L) = H^i(B \otimes (q-i-1)L) = 0$ for $i = 1, 2, \dots, n-1$. Then, we have a natural isomorphism between $K_{p,q}(B, L)^*$ and $K_{h^0(L)-n-p, n+1-q}(B^* \otimes K_X, L)$.*

To compute the Koszul cohomology, we can use either a free resolution of M or a free resolution of k . Taking the minimal free resolution of M relates the Koszul cohomology

to the degrees in which the syzygies of M lie. In particular, this description shows that $K_{p,q}$ is trivial for $q < 0$. Combining this with Theorem 2 and some elementary facts on Castelnuovo-Mumford regularity, it is possible to use this to show that $K_{p,q}$ is trivial for $q > n + 1$ and is nontrivial for $q = n + 1$ only when p is within a constant (depending only on B and X) of $h^0(L)$.

Extending this, in [2], Ein and Lazarsfeld study when the groups $K_{p,q}$ vanish for L very positive and $1 \leq q \leq n$. More precisely, let L be of the form $P + dA$, where P and A are fixed divisors with A ample. Then as d changes, Ein and Lazarsfeld prove that $K_{p,q}$ is nonzero if $p > O(d^{q-1})$ and $p < H^0(L) - O(d^{n-1})$. They conjecture that on the other hand, $K_{p,q}$ is trivial for $p < O(d^{q-1})$. It is known that if $q \geq 2$, then $K_{p,q}$ is trivial for $p < O(d)$. In particular, Theorem 4 only has content when q is 0 or 1.

For the purposes of computation, the syzygies of M are often hard to work with directly, so instead we will compute using a free resolution of k . The natural free resolution is the Koszul complex:

$$\cdots \wedge^2 H^0(L) \otimes S \rightarrow H^0(L) \otimes S \rightarrow S \rightarrow k \rightarrow 0.$$

Tensoring by M , we immediately arrive at the following lemma:

Lemma 1. *If $q \geq 1$, $K_{p,q}(B, L)$ is equal to the cohomology of the three-term complex*

$$H^0(B + (q-1)L) \otimes \wedge^{p+1} H^0(L) \rightarrow H^0(B + qL) \otimes \wedge^p H^0(L) \rightarrow H^0(B + (q+1)L) \otimes \wedge^{p-1} H^0(L).$$

On the other hand, for $q = 0$, $K_{p,q}(B, L)$ is equal to the kernel of the map

$$H^0(B) \otimes \wedge^p H^0(L) \rightarrow H^0(B + L) \otimes \wedge^{p-1} H^0(L).$$

Finally, we say a few words on Voisin's Hilbert scheme approach to Koszul cohomology (see [3,4]). Let X be a smooth curve or surface. Then, denote by $\text{Hilb}^n(X)$ the n th Hilbert scheme of points of X . We have a natural incidence subscheme I_n in $X \times \text{Hilb}^n(X)$. Let $p: I_n \rightarrow X$ and $q: I_n \rightarrow \text{Hilb}^n(X)$.

Theorem 2. *Let $L^{[n]}$ be $\wedge^n \text{pr}_{2*} p^* L$. Then for $q \geq 1$, we have*

$$K_{p,q}(B, L) \cong \text{coker} \left(H^0(B + (q-1)L) \otimes H^0(L^{[p+1]}) \rightarrow H^0((B + (q-1)L) \boxtimes L^{[p+1]}|_{I_{p+1}}) \right).$$

Furthermore,

$$K_{p,0} \cong \ker \left(H^0(B) \otimes H^0(L^{[p+1]}) \rightarrow H^0(B \boxtimes L^{[p+1]}|_{I_{p+1}}) \right).$$

In fact, we have $H^0(L^{[n]}) = \wedge^n H^0(L)$ and $H^0((B + (q-1)L) \boxtimes L^{[n]}|_{I_n}) = \ker(H^0(B + qL) \otimes H^0(L^{[n-1]}) \rightarrow H^0(B + (q+1)L) \otimes H^0(L^{[n-2]}))$. We thus see that Theorem 2 is a 'geometrized' version of Theorem 1.

We would like to comment that one of our original motivations was to rewrite this in terms of the q th or $(q-1)$ th sheaf cohomology of some sheaf. This is accomplished by the formalism of the next section combined with the results of [6].

Results and discussions

In this section, we prove an analogue of Theorem 2, but with the category of coherent sheaves on the Hilbert scheme replaced by the category of S_n -equivariant sheaves on X^n . Here we take X to be smooth projective, X^n to be its n th Cartesian power, and B to be a locally free sheaf on X .

An S_n -equivariant sheaf on X^n is a sheaf on X^n together with S_n -linearization in the sense of [10], Chapter 5. Note that for any map $X^n \rightarrow Y$ that is fixed under the S_n action on X^n , we have an ' S_n -invariant pushforward' functor from the category of S_n -equivariant quasicoherent sheaves on X^n to the category of quasicoherent sheaves on Y . The usual pushforward is equipped with a natural S_n action, and the S_n -invariant pushforward is defined just to be the invariants under this action. The S_n -invariant cohomology $H_{S_n}^i$ is defined to be the derived functor of S_n -invariant pushforward to $\text{Spec } k$.

We start by defining $L^{[n]}$ to be the sheaf $L \boxtimes \cdots \boxtimes L$ on X^n with S_n acting via the alternating action. By definition, we have $H_{S_n}^\bullet(L^{[n]}) = \wedge^n H^\bullet(L)$. In the next section, we will also use the sheaf D_L , defined to be the sheaf $L \boxtimes \cdots \boxtimes L$ with S_n acting via the trivial action. From the definitions, it is clear that $L^{[n]} = \mathcal{O}^{[n]} \otimes D_L$.

Let Δ_i be the subvariety of $X \times X^n$ defined as the set of points $(x, x_1, \dots, x_n)$ with $x = x_i$. Let Z be the scheme theoretic union of the Δ_i . Then Z is equipped with two projections $p: Z \rightarrow X$ and $q: Z \rightarrow X^n$. We will make crucial use of a long exact sequence

$$0 \rightarrow \mathcal{O}_Z \rightarrow \oplus_{i_1} \mathcal{O}_{\Delta_{i_1}} \rightarrow \oplus_{i_1 < i_2} \mathcal{O}_{\Delta_{i_1, i_2}} \rightarrow \cdots \rightarrow \mathcal{O}_{\Delta_{1, 2, \dots, n}}.$$

We will prove the following theorem.

Theorem 3. Assume that $H^i(L) = H^i(B + mL) = 0$ for all $i, m > 0$. Then for $q > 1$, $K_{p,q}(B, L) \cong H_{S_{p+q}}^{q-1}(p^*B \otimes q^*L^{[p+q]})$. We also have an exact sequence

$$0 \rightarrow K_{p+1,0}(B, L) \rightarrow H^0(B) \otimes H_{S_{p+1}}^0(L^{[p+1]}) \rightarrow H_{S_{p+1}}^0(p^*B \otimes q^*L^{[p+1]}) \rightarrow K_{p,1}(B, L) \rightarrow 0.$$

Note that when B has no higher cohomology, we can write this more aesthetically as $K_{p,q}(B, L) = H_{S_{p+q}}^q((B \boxtimes L^{[p+q]}) \otimes \mathcal{I}_Z)$, where $\mathcal{I}_Z$ is the ideal sheaf of Z .

Before proving the theorem, we need an exact sequence. Define $\Delta_{i_1, i_2, \dots, i_m}$ to be the subvariety of $X \times X^n$ such that $(x, x_1, \dots, x_n)$ is a point of $\Delta_{i_1, i_2, \dots, i_m}$ if and only if $x = x_{i_1} = \cdots = x_{i_m}$. Now note that we have a long exact sequence of S_n -equivariant sheaves on $X \times X^n$

$$0 \rightarrow \mathcal{O}_Z \rightarrow \oplus_{i_1} \mathcal{O}_{\Delta_{i_1}} \rightarrow \oplus_{i_1 < i_2} \mathcal{O}_{\Delta_{i_1, i_2}} \rightarrow \cdots \rightarrow \mathcal{O}_{\Delta_{1, 2, \dots, n}}.$$

We define the map from $\mathcal{O}_{\Delta_{i_1, i_2, \dots, i_m}}$ to $\mathcal{O}_{\Delta_{j_1, j_2, \dots, j_{m+1}}}$ to be nonzero if and only if $\Delta_{j_1, \dots, j_{m+1}}$ is a subvariety of $\Delta_{i_1, i_2, \dots, i_m}$, in which case we define it to be the natural map induced by the inclusion, up to sign. If the difference between $\{i_1, \dots, i_m\}$ and $\{j_1, \dots, j_{m+1}\}$ is j_k , then we modify this map by a factor of $(-1)^{k-1}$.

We need to be careful to define our S_n -equivariant structure on our complex in a way compatible with this modification. For any element $\sigma \in S_n$, we have $\sigma_* \mathcal{O}_{\Delta_{i_1, i_2, \dots, i_m}} \cong \mathcal{O}_{\Delta_{h_1, h_2, \dots, h_m}}$, where $h_1 < h_2 < \cdots < h_m$ and $\{h_1, \dots, h_m\} = \{\sigma i_1, \dots, \sigma i_m\}$. This gives each term of our complex a natural S_n -action. To make our maps S_n -equivariant, we modify the action by the sign of the permutation sending $h_1, h_2, \dots, h_m$ to $\sigma i_1, \dots, \sigma i_m$.

After these definitions, it is quite easy to check that we have indeed defined an S_n -equivariant complex. Exactness is harder; a proof can be found in Appendix A of [6].

We now prove Theorem 3.

Proof. Tensor our exact sequence by $B \boxtimes L^{[n]}$. Each term of the complex is of the form $\oplus_{i_1 < i_2 < \cdots < i_m} B \boxtimes L^{[n]}|_{\Delta_{i_1, i_2, \dots, i_m}}$. We compute the S_n -invariant cohomology of this sheaf. This is the same as the $S_m \times S_{n-m}$ invariant cohomology of $B \boxtimes L^{[n]}|_{\Delta_{1, 2, \dots, m}}$. Now note that

there is a natural S_{n-m} -equivariant isomorphism between $\Delta_{1,2,\dots,m}$ and $X \times X^{n-m}$. This sends $B \boxtimes L^{[n]}|_{\Delta_{1,2,\dots,m}}$ to $(B + mL) \boxtimes L^{[n-m]}$. The action of S_m on $H^0(B + mL)$ is trivial, as the alternating actions coming from our modified S_n -equivariance and $L^{[n]}$ cancel, and thus we can simply consider the S_{n-m} -invariant cohomology. By our assumptions and the fact that $H_{S_n}^\bullet(L^{[n]}) = {}^\wedge^n H^\bullet(L)$, we see that the higher S_{n-m} -invariant cohomology of $B \boxtimes L^{[n]}|_{\Delta_{1,2,\dots,m}}$ vanishes and that $H_{S_{n-m}}^0(B \boxtimes L^{[n]}|_{\Delta_{1,2,\dots,m}}) \cong H^0(B + mL) \otimes {}^\wedge^{n-m} H^0(L)$.

Our long exact sequence now immediately shows that the S_n -invariant cohomology of $B \boxtimes L^{[n]}|_Z$ is given by the cohomology of the complex

$$0 \rightarrow H^0(B + L) \otimes {}^\wedge^{n-1} H^0(L) \rightarrow H^0(B + 2L) \otimes {}^\wedge^{n-2} H^0(L) \rightarrow \dots$$

and, setting $n = p + q$, our result immediately follows from Lemma 1. $\square$

We note that in particular, if $H^{q-1}(p^*B \otimes q^*L^{[p+q]}) = 0$ (note that this is noninvariant cohomology!), then $K_{p,q}(B, L)$ vanishes. A very similar statement was first claimed in the proof of Theorem 3.2 of [7], with a minor mistake corrected by Inamdar in [8]. Since then, this statement has been used in various places (e.g., [11,12]).

Note that for any map $X^n \rightarrow Y$ that is fixed under the S_n action on X^n , we have a S_n -invariant pushforward functor from the category of S_n -equivariant quasicoherent sheaves on X^n to the category of quasicoherent sheaves on Y . The usual pushforward is equipped with a natural S_n action, and the S_n -invariant pushforward is defined just to be the invariants under this action. The S_n -invariant cohomology is defined to be the S_n -invariant pushforward to $\text{Spec } k$.

Conclusions

In this section, we will let $L_d = dA + P$, where A and P are line bundles with A ample. We allow B to be any locally free sheaf on X . Our main theorem is the following.

Theorem 4. *For d sufficiently large, $\dim K_{p,q}(B, L_d)$ is a polynomial in d .*

Proof. We start by showing that $K_{p,q}(B, L_d)$ is trivial for $q \geq 2$ and d large enough. This has been known at least since [13], but we will reprove it here for the sake of self-containedness. Let $\mathcal{F}$ be the sheaf $\text{pr}_{2*} \mathcal{O}_Z \otimes p^{[p+q]}$. As q is finite, pushforward along it is exact. Therefore, by Theorem 3, we know that

$$K_{p,q}(B, L_d) \cong H_{S_{p+q}}^{q-1}(\mathcal{F} \otimes dD_A)$$

(see the beginning of the previous section for the definition of D_A). But D_A is ample, so this is zero for large enough d .

By Theorem 3, it suffices to show that the dimensions of $K_{p+1,0}(B, L_d)$, $H^0(B) \otimes H_{S_{p+1}}^0(L_d^{[p+1]})$, and $H_{S_{p+1}}^0(p^*B \otimes q^*L_d^{[p+1]})$ all grow polynomially in d . Using $H_{S_n}^\bullet(L_d^{[n]}) \cong {}^\wedge^n H^\bullet(L_d)$, we immediately see that $\dim H^0(B) \otimes H_{S_{p+1}}^0(L_d^{[p+1]})$ grows polynomially in d .

It follows immediately from the second part of Theorem 3 that $K_{p+1,0}(B, L_d) \cong H_{S_{p+1}}^0(B \boxtimes L_d^{[p+1]} \otimes \mathcal{I}_Z)$. Letting $\mathcal{G}$ denote the sheaf $\text{pr}_{2*}(B \boxtimes L_d^{[p+1]} \otimes \mathcal{I}_Z)$, we see that

$$K_{p+1,0} \cong H_{S_{p+1}}^0(\mathcal{G} \otimes dD_A).$$

To proceed, we take the S_{p+1} -invariant pushforward of this to $\mathrm{Sym}^{p+1}(X)$. Let the S_{p+1} -invariant pushforward of $\mathcal{G}$ be $\mathcal{H}$. Note that D_A is the pullback (with the natural S_{p+1} -action) of an ample line bundle A' on $\mathrm{Sym}^{p+1}(X)$. We thus see that $K_{p+1,0}(B, L_d) = H^0(\mathcal{H} \otimes dA')$, and by the existence of the Hilbert polynomial, its dimension is a polynomial for large enough d . An identical argument shows that $\dim H_{S_{p+1}}^0(p^*B \otimes q^*L_d^{[p+1]})$ is a polynomial for large enough d . $\square$

Note that this theorem answers the first part of Problem 7.2 in [2] in the affirmative. Their specific question asks if the triviality of a specific Koszul cohomology group is independent of d if d is large. As any nonzero polynomial has only finitely many roots, Theorem 4 implies that the answer is yes.

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