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Some fixed point results for fuzzy homotopic mappings

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Abstract
The first purpose of this paper is to define a homotopy for fuzzy spaces. We continue our work by showing that the property of having a fixed point is invariant by this homotopy. These theorems generalize and improve well-known results.

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1 Introduction and preliminaries

The study of fuzzy metric spaces has been developing since 1971. The well-known fixed point theorem of Banach was extended by Grabiec [1]. On the other hand, a number of authors have studied the conditions under which the property of having a fixed point is invariant in metric spaces. For example, see [2, 3].

To seek completeness, we briefly recall some basic concepts used in the following.

Definition 1.1 ([4]) A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if $([0,1],*)$ is an abelian topological monoid with unit 1 such that $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for all $a,b,c,d \in [0,1]$.

Definition 1.2 ([5]) The 3-tuple $(X,M,*)$ is called a fuzzy metric space if X is an arbitrary non-empty set, $*$ is a continuous t-norm, M is a fuzzy set on $X^2 \times [0,\infty)$ satisfying the following conditions, for each $x,y,z \in X$ and $t,s > 0$,

(1) $M(x,y,t) > 0$,
(2) $M(x,y,t) = 1$ if and only if $x = y$,
(3) $M(x,y,t) = M(y,x,t)$,
(4) $M(x,y,t) * M(y,z,s) \le M(x,z,t+s)$,
(5) $M(x,y,t) : (0,\infty) \to [0,1]$ is continuous.

Example 1.3 ([6]) Let $X = \mathbb{N}$, define $a * b = ab$ for all $a,b \in [0,1]$, let M be a fuzzy set on $X^2 \times [0,\infty)$ as follows:

$$M(x,y,t) = \begin{cases} \frac{x+t}{y+t}, & x \le y, \\ \frac{y+t}{x+t}, & y > x. \end{cases}$$

Then $(X,M,*)$ is a fuzzy metric space.

Example 1.4 ([7]) Let (X,d) be a metric space. Define $a * b = ab$ for all $x, y \in X$ and $t > 0$,

$$M(x,y,t) = \frac{t}{t + d(x,y)}.$$

Then $(X,M,*)$ is a fuzzy metric space. We call this fuzzy metric M induced by the metric d the *standard fuzzy metric*. If (X,d) is a complete metric space, then also $(X,M,*)$ is complete.

Lemma 1.5 ([1]) *$M(x,y,\cdot)$ is non-decreasing for all $x,y \in X$.*

Remark 1.6 ([5])

(a) In a fuzzy metric space $(X,M,*)$, whenever $M(x,y,t) > 1 - r$ for x, y in X, $t > 0$, $0 < r < 1$, we can find $0 < t_0 < t$ such that $M(x,y,t_0) > 1 - r$.

(b) For any $r_1 > r_2$, we can find r_3 such that $r_1 * r_3 \geq r_2$, and for any r_4, we can find r_5 such that $r_5 * r_5 \geq r_4$ $(r_1, r_2, r_3, r_4, r_5 \in (0,1))$.

George and Veeramani introduced Hausdorff topology in fuzzy metric spaces. They showed that this topology is first countable.

Definition 1.7 ([5]) Let $(X,M,*)$ be a fuzzy metric space. For $t > 0$ and $0 < r < 1$, the open ball $B(x,r,t)$ with center $x \in X$ is defined by $B(x,r,t) = \{y \in X : M(x,y,t) > 1 - r\}$.

A subset $A \subseteq X$ is called open if for each $x \in A$ there exist $t > 0$ and $0 < r < 1$ such that $B(x,r,t) \subseteq A$. Let τ denote the family of all open subsets of X. Then τ is a topology on X induced by the fuzzy metric $(X,M,*)$. This topology also is metrizable (see [7]).

Definition 1.8 Let $(X,M,*)$ be a fuzzy metric space.

(1) A sequence $\{x_n\}$ is said to be convergent to a point $x \in X$ if $\lim_{n\to\infty} M(x_n,x,t) = 1$ for all $t > 0$.

(2) A sequence $\{x_n\}$ is said to be Cauchy sequence if

$$\lim_{n,m\to\infty} M(x_n,x_m,t) = 1$$

for all $t > 0$.

(3) A fuzzy metric space in which every Cauchy sequence is convergent to a point $x \in X$ is said to be complete.

Definition 1.9 Let $(X,M*)$ be a fuzzy metric space and $A \subseteq X$. Closure of the set A is the smallest closed set containing A, denoted by $\overline{A}$. Interior of the set A is the largest open set contained in A, denoted by A°. Obviously, having in mind the Hausdorff topology and the definition of converging sequences, we have that the next remark holds.

Remark 1.10 $x \in \overline{A}$ if and only if there exists a sequence $\{x_n\}$ in A such that $x_n \to x$.

We also need the following definitions.

Definition 1.11 Let $(X,M,*)$ be a fuzzy metric space, $A \subseteq X$ $\overline{A} \setminus A^\circ$ is called boundary of A and denoted by ∂A.

Definition 1.12 ([6]) Let A be a non-empty subset of fuzzy metric space $(X,M,*)$. For each $x \in X$ and $t > 0$, define

$$M(x,A,t) = \sup\bigl\{M(x,y,t) : y \in A\bigr\}.$$

The following lemma is essential in proving our result.

Lemma 1.13 ([8]) *Let $(X,M,*)$be a fuzzy metric space such that for every $x, y \in X$, $t > 0$ and $h > 1$,*

$$\lim_{n\to\infty} *_{i=n}^{\infty} M\bigl(x,y,th^i\bigr) = 1. \tag{1.13}$$

Suppose that $\{x_n\}$ is a sequence in X such that, for all $n \in \mathbb{N}$,

$$M(x_n, x_{n+1}, kt) \ge M(x_{n-1}, x_n, t),$$

where $0 < K < 1$, then $\{x_n\}$ is a Cauchy sequence.

Definition 1.14 Let $(X,M,*)$ be a fuzzy metric space. A map $F : X \to X$ is said to be fuzzy contraction if there exists a constant $0 < \alpha < 1$ with

$$M(Fx,Fy,\alpha t) \ge M(x,y,t).$$

Kiany and Amini proved the following improvement of Gregori and Sapena's fixed point theorem.

Theorem 1.15 ([8]) *Let $(X,M,*)$ be a complete fuzzy metric space. Suppose that $F : X \to X$ is a fuzzy contractive map. Furthermore, assume that $(X,M,*)$ satisfies* (1.13) *for some $x_0 \in X$, each $t > 0$ and $h > 1$. Then F has a fixed point.*

2 Main results

Let $(X,M,*)$ be a complete fuzzy metric space.

Lemma 2.1 *If $0 < a < 1$, $0 < p < 1$, $t, N > 0$ all are given, then there exists $\epsilon > 0$ such that, if we have $|\lambda - \lambda_0| \le \epsilon$, then*

$$\frac{at}{N|\lambda - \lambda_0| + at} \ge p.$$

Proof Put $0 < \epsilon \le \frac{at(1-p)}{pN}$. We have

$$\epsilon pN \le at(1-p),$$

$$\epsilon pN \le at - atp,$$

$$\epsilon pN + atp \le at,$$

$$p(\epsilon N + at) \le at,$$

$$p \le \frac{at}{\epsilon N + at}.$$

Since $|\lambda - \lambda_0| \leq \epsilon$, we get

$$p \leq \frac{at}{N|\lambda - \lambda_0| + at}$$

or

$$\frac{at}{N|\lambda - \lambda_0| + at} \geq p. \qquad \square$$

Definition 2.2 Let $(X, M, *)$ be a fuzzy metric space and A be a closed subset of X and $x_0 \notin A$, then we say X has a real distance if $\sup\{M(x, A, t) : \forall t > 0\} < 1$.

Example 2.3 Then $(X, M, *)$ is a complete *standard fuzzy metric*, then X has a real distance, because if A is a closed subset of X and $x_0 \notin A$, then $\inf\{d(x_0, A)\} > 0$.

Example 2.4 Suppose that $(X, M, *)$ is the same as in Example 1.3, $A \in X$ is an arbitrary subset of X and $x_0 \notin A$, then $M(x_0, A, t) \leq \frac{1}{2}$.

Definition 2.5 Let $F : \overline{U} \to X$ and $G : \overline{U} \to X$ be two fuzzy contractions. We say that F and G are fuzzy *homotopic* • maps if there exists $H : \overline{U} \times [0, 1] \to X$ with the following properties:

(a) $H(\cdot, 0) = G$ and $H(\cdot, 1) = F$;
(b) $x \neq H(x, s)$ for $x \in \partial U$ and $s \in [0, 1]$;
(c) there exists K, $0 < K < 1$, such that $M(H(x, s), H(y, s), Kt) \geq M(x, y, t)$ for every $x, y \in \overline{U}$, $s \in [0, 1]$ and $t > 0$;
(d) there exists N, $N \geq 0$, such that $M(H(x, s_0), H(y, s_1), t) \geq \frac{t}{t+N|s_0-s_1|}$ for every $x \in \overline{U}$, $t > 0$ and $s_0, s_1 \in [0, 1]$.

Theorem 2.6 *Let* $(X, M, *)$ *be a fuzzy complete metric space and* U *be an open subset of* X. *Suppose that* $F : \overline{U} \to X$ *and* $G : \overline{U} \to X$ *are two homotopic fuzzy maps and G has a fixed point in* U. *Assume that* $(X, M, *)$ *satisfies* (1.13) *for some* $x_0 \in X$ *and also* X *has a real distance, then F has a fixed point in* U.

Proof Consider the set

$$A = \{\lambda \in [0, 1] : x = H(x, \lambda) \text{ for some } x \in U\},$$

where H is a homotopy between F and G as described in Definition 2.5. Notice that A is non-empty since G has a fixed point, that is, $0 \in A$. We will show that A is both open and closed in $[0, 1]$ and, by connectedness, we have that $A = [0, 1]$. As a result, F has a fixed point in U. We break the argument into two steps.

Step one. A is open in $[0, 1]$.

Since A is non-empty, there exists $x_0 \in U$ with $x_0 = H(x_0, \lambda_0)$. Since X has a real distance, there exists $0 < r^* < 1$ such that

$$M(x_0, \partial U, t) > 1 - r^*.$$

So we can choose r, $0 < r < 1$, such that $1 - r^* > 1 - r$.

Now if

$$x \in \overline{B(x_0, r, t)} \quad \Longrightarrow \quad M(x_0, x, t) > 1 - r.$$

From Remark 1.6(a) we can find t_0, $0 < t_0 < t$, such that $M(x_0, x, t_0) > 1 - r$. Let

$$r_0 = M(x_0, x, t) > 1 - r. \tag{1}$$

Since $r_0 > 1 - r$, we can find s, $0 < s < 1$, such that

$$r_0 > 1 - s > 1 - r. \tag{2}$$

Now, for given r_0 and s, from Remark 1.6(b) we can find p, $0 < p < 1$, such that

$$r_0 * p \geq 1 - s. \tag{3}$$

Now consider Lemma 2.1 with $a = (1 - K)$, p, N, t_0, we can find ϵ such that if $|\lambda - \lambda_0| \leq \epsilon$, then we have

$$\frac{(1-K)t_0}{(1-K)t_0 + N|\lambda - \lambda_0|} \geq p. \tag{4}$$

Thus, for each fixed $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$ and $x \in \overline{\bullet B(x_0, r, t)}$, we have

$$\begin{aligned} M\bigl(x_0, H(x, \lambda), t\bigr) &\geq M\bigl(x_0, H(x, \lambda), t_0\bigr) \\ &\geq M\bigl(H(x_0, \lambda_0), H(x_0, \lambda), (1-K)t_0\bigr) * M\bigl(H(x_0, \lambda), H(x, \lambda), Kt_0\bigr). \end{aligned} \tag{5}$$

By Definition 2.5(d) we know that

$$M\bigl(H(x_0, \lambda_0), H(x_0, \lambda), (1-K)t_0\bigr) \geq \frac{(1-K)t_0}{(1-K)t_0 + N|\lambda - \lambda_0|}.$$

Also by Definition 2.5(c) we know that

$$M\bigl(H(x_0, \lambda), H(x, \lambda), Kt_0\bigr) \geq M(x_0, x, t_0).$$

Substitution of these expressions into (5) reveals

$$M\bigl(x_0, H(x, \lambda), t\bigr) \geq \frac{(1-K)t_0}{(1-K)t_0 + N|\lambda - \lambda_0|} * M(x_0, x, t_0). \tag{6}$$

Now from substitution of (1) and (4) into the (6) we have

$$M\bigl(x_0, H(x, \lambda), t\bigr) \geq p * r_0.$$

From (3) we get

$$M\bigl(x_0, H(x, \lambda), t\bigr) \geq 1 - s.$$

Then from (2) we get

$$M\bigl(x_0, H(x,\lambda), t\bigr) \geq 1 - r.$$

Thus, for each fixed $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$,

$$H(\cdot,\lambda) : \overline{B(x_0,r,t)} \to \overline{B(x_0,r,t)}.$$

We can apply Theorem 1.15 to deduce that $H(\cdot,\lambda)$ has a fixed point in U. Thus $\lambda \in A$ for any $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$ and therefore A is open in $[0,1]$.

Step two. A is closed in $[0,1]$.

To see this, let

$$\{\lambda_n\}_{n=1}^{\infty} \subseteq A \quad \text{with } \lambda_n \to \lambda \in [0,1] \text{ as } n \to \infty.$$

We must show that $\lambda \in A$. Since $\lambda_n \in A$ for $n = 1, 2, \ldots$, there exists $x_n \in U$ with $x_n = H(x_n, \lambda_n)$. Hence, by Lemma 1.13, we know $\{x_n\}$ is a Cauchy sequence. Since $(X, M, *)$ is a fuzzy complete metric space, then there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$, that means

$$\lim_{n\to\infty} M(x_n, x, t) = 1 \quad \text{for each } t > 0. \tag{7}$$

On the other hand, from $\lambda_n \to \lambda$, we have

$$\frac{(1-K)t}{N|\lambda_n - \lambda| + (1-K)t} \to 1.$$

In addition, $x = H(x,\lambda)$ since

$$\begin{aligned} M\bigl(x_n, H(x,\lambda), t\bigr) &= M\bigl(H(x_n,\lambda_n), H(x,\lambda), t\bigr) \\ &\geq M\bigl(H(x_n,\lambda_n), H(x_n,\lambda), (1-K)t\bigr) * M\bigl(H(x_n,\lambda), H(x,\lambda), Kt\bigr). \end{aligned} \tag{8}$$

By Definition 2.5(d) we have

$$M\bigl(H(x_n,\lambda_n), H(x_n,\lambda), (1-K)t\bigr) \geq \frac{(1-K)t}{(1-K)t + N|\lambda_n - \lambda|}. \tag{9}$$

Also, by Definition 2.5(c), we have

$$M\bigl(H(x_n,\lambda), H(x,\lambda), Kt\bigr) \geq M(x_n, x, t). \tag{10}$$

Now from substitution of (9) and (10) in (8) we get

$$M\bigl(x_n, H(x,\lambda), t\bigr) \geq \frac{(1-K)t}{(1-K)t + N|\lambda_n - \lambda|} * M(x_n, x, t).$$

As seen above, on the left-hand side of this inequality, both limits exist and are equal to one. So, for each $t > 0$, we must have

$$\lim_{n\to\infty} M\bigl(x_n, H(x,\lambda), t\bigr) = 1.$$

From (7) we get $H(x,\lambda) = x$. Thus $\lambda \in A$, and A is closed in $[0,1]$. □

Example 2.7 Let $X = \mathbb{R}$, $M(x,y,t) = \frac{t}{t+|x-y|}$, $a * b = ab$, then $(X, M, *)$ is a complete fuzzy metric space. Also $(X, M, *)$ satisfies (1.13). Let $N > 0$ be a fixed real number and $f(x) : X \to X$ be given by

$$f(x) = \begin{cases} \frac{x}{2}, & 0 \leq x \leq 2N; \\ N, & \text{else.} \end{cases}$$

Also define

$$g(x) = (1-\beta)f(x) \quad \text{for } 0 < \beta < 1.$$

It is easy to show $|f(x) - f(y)| \leq \frac{1}{2}|x-y|$ for all $x, y \in X$. Now we have $M(f(x), f(y), \frac{t}{2}) \geq M(x,y,t)$. Because f is a fuzzy contraction ($\alpha = \frac{1}{2}$), g is a fuzzy contraction, too. Let $s, s_0, s_1 \in [0,1]$, $t > 0$. We define $H : X \times [0,1] \to X$

$$H(x,s) = sf(x) + (1-s)g(x).$$

It is obvious that H satisfies Definition 2.5(a) and Definition 2.5(b). We need only check that Definition 2.5(c) and Definition 2.5(d) are true. For Definition 2.5(c), we have

$$\begin{aligned} \bigl|H(x,s) - H(y,s)\bigr| &= \bigl|sf(x) + (1-s)g(x) - sf(y) - (1-s)g(y)\bigr| \\ &= \bigl|s\bigl(f(x) - f(y)\bigr) + (1-s)(1-\beta)\bigl(f(x) - f(y)\bigr)\bigr| \\ &= \bigl|f(x) - f(y)\bigl(s + (1-s)(1-\beta)\bigr)\bigr| \\ &\leq \bigl|f(x) - f(y)\bigl(s + (1-s)\bigr)\bigr| \\ &= \bigl|f(x) - f(y)\bigr| \leq \left|\frac{1}{2}(x-y)\right|. \end{aligned}$$

For $K = \frac{1}{2}$, we have

$$M\bigl(H(x,s), H(y,s), Kt\bigr) \geq M(x,y,t).$$

For Definition 2.5(d), we have

$$\begin{aligned} \bigl|H(x,s_0) - H(x,s_1)\bigr| &= \bigl|s_0 f(x) + (1-s_0)g(x) - s_1 f(x) - (1-s_1)g(x)\bigr| \\ &= \bigl|(s_0 - s_1)\bigl(f(x) - g(x)\bigr)\bigr| \\ &= \bigl|(s_0 - s_1)\bigl(f(x) - f(x) + \beta f(x)\bigr)\bigr| \\ &= \bigl|(s_0 - s_1)\beta f(x)\bigr| \\ &\leq \bigl|(s_0 - s_1)\beta N\bigr| \\ &\leq \bigl|(s_0 - s_1)N\bigr|. \end{aligned}$$

So

$$M\bigl(H(x,s_0), H(x,s_1), t\bigr) = \frac{t}{t + |H(x,s_0) - H(x,s_1)|} \geq \frac{t}{t + N|s_0 - s_1|}.$$

Now f and g are two fuzzy homotopic contractive maps. Notice that f has a fixed point in zero. We can now apply Theorem 2.6 to deduce that there exists x with $x = g(x)$.

Now, as a result of Theorem 2.6, we can prove the following theorem due to Fournier [3].

Theorem 2.8 *Let* (X, d) *be a complete metric space and* U *be an open subset of* X. *Suppose that* $F : U \to X$ *and* $G : U \to X$ *if there exists* $H : \overline{U} \times [0,1] \to X$ *with the following properties*:

(a) $H(\cdot, 0) = G$ *and* $H(\cdot, 1) = F$;
(b) $x \neq H(x, s)$ *for* $x \in \partial U$ *and* $s \in [0,1]$;
(c) *there exists* K, $0 \leq K < 1$, *such that* $d(H(x,s), H(y,s)) \leq Kd(x,y)$ *for every* $x, y \in \overline{U}$, $s \in [0,1]$;
(d) *there exists* N, $N \geq 0$, *such that* $d(H(x,s), H(y,p)) \leq N|s - p|$ *for every* $x, y \in \overline{U}$ *and* $s, p \in [0,1]$. *Suppose that* F *and* G *are two contractive maps and* G *has a fixed point in* U, *then* F *has a fixed point in* U.

Proof Let $(X, M, *)$ be a standard fuzzy metric space induced by the metric d with $a * b = \min\{a, b\}$. Notice that F and G are two contractive maps, so they are fuzzy contractive maps in the induced fuzzy metric space. Now we can see that condition (1.13) is satisfied. Also X has a real distance. Since (X, d) is a complete metric space, $(X, M, *)$ is a complete fuzzy metric space. It is easy to see that $(X, M, *)$ satisfies all the conditions Definition 2.5(a), Definition 2.5(b), Definition 2.5(c) and Definition 2.5(d). We can apply Theorem 2.6 to deduce that F has a fixed point. □

3 Conclusions

Motivated by the results of Frigon, I slightly modified the definition of homotopic contractive maps. I proved that the property of having a fixed point is invariant by homotopy for fuzzy contractive maps. This investigation could be extended to a fuzzy quasi-metric space with possible application to the study of analysis of probabilistic metric spaces.

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Competing interests
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Authors' contributions
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