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A reliable method for the space-time fractional Burgers and time-fractional Cahn-Allen equations via the FRDTM

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Abstract

We propose a new method called the fractional reduced differential transform method (FRDTM) to solve nonlinear fractional partial differential equations such as the space-time fractional Burgers equations and the time-fractional Cahn-Allen equation. The solutions are given in the form of series with easily computable terms. Numerical solutions are calculated for the fractional Burgers and Cahn-Allen equations to show the nature of solutions as the fractional derivative parameter is changed. The results prove that the proposed method is very effective and simple for obtaining approximate solutions of nonlinear fractional partial differential equations.

MSC: 35J05; 35J10; 35A22; 65M12; 65M15; 65M20

Keywords: fractional reduced differential transform method; Caputo fractional derivative; Burgers equations; Cahn-Allen equation

1 Introduction

The space-fractional Burgers equation describes the physical processes of unidirectional propagation of weakly nonlinear acoustic waves through a gas-filled pipe. They are also connected with applications in acoustic phenomena and have been used to model turbulence and certain steady-state viscous flows. Moreover, Burgers equations are used to model the formation and decay of nonplanar shock waves, where the variable x is a coordinate moving with the wave at the speed of sound and the dependent variable u represents the velocity fluctuations. The Burgers equations occur in various areas of applied sciences and physical applications, such as modeling of fluid mechanics and financial mathematics, and the equation has still interesting applications in physics and astrophysics.

The fractional differential equations (FDE) appear more and more frequently in different research areas and engineering applications. There are many physical applications in science and engineering that can be represented by models using fractional differential equations [1–10], which are quite useful for many physical problems. These equations are represented by fractional linear and nonlinear PDEs, and solving such fractional differential equations is very important [11–21].

Many approximation and numerical techniques have been used to solve fractional differential equations [12, 16, 22–25]. Lately, many new approaches to fractional differential equations have been proposed, a few of these methods are as follows: the fractional

differential transform method (FDTM) [25–28], the fractional Adomian decomposition method (FADM) [2], the fractional variational iteration method (FVIM) [4], the fractional sub-equation method [23], the fractional natural decomposition method [17, 29] and the fractional homotopy perturbation method (FHPM) [22, 30]. Kurulay [26] found approximate and exact solutions of the space- and time-fractional Burgers equations. Bekir et al. [23] found exact solutions of the time-fractional Cahn-Allen equation. Khan et al. [22] used the generalized differential transform method (GDTM) and the homotopy perturbation method (HPM) to solve the time-fractional Burgers and coupled Burgers equations. Recently, Rawashdeh [16, 31] used the FRDTM to solve nonlinear fractional partial differential equations.

The general response expression contains a parameter describing the order of the fractional derivative that can be varied to obtain various responses. Note that we call Eq. (1.1) the time-fractional Burgers and the space-fractional Burgers equation in the case $0 < \alpha \leq 1, \eta = 0$ and $0 < \beta \leq 1, \alpha = 1$, respectively.

In this work, we first consider the nonlinear fractional generalized Burgers equation with time- and space-fractional derivatives of the form [30]

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \eta \frac{\partial^\beta u}{\partial x^\beta} + v \frac{\partial^2 u}{\partial x^2} - \varepsilon u \frac{\partial u}{\partial x} = 0, \quad x, t > 0, 0 < \alpha, \beta \leq 1, \tag{1.1}$$

where ε, v and η are parameters and α and β are parameters describing the order of the fractional time- and space-derivatives, respectively. The function $u(x, t)$ is a function of x and t and $u(x, t)$ will vanish when $t < 0$ and $x < 0$.

Second, we consider the time-fractional Cahn-Allen equation [23, 32]

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^2 u}{\partial x^2} + u^3 - u = 0, \quad t > 0, 0 < \alpha \leq 1. \tag{1.2}$$

The rest of this paper is divided into six sections. In Section 2, we give a background of fractional calculus. In Section 3, the RDTM is introduced. Section 4 is devoted to application of the FRDTM to three test problems and presentation of graphs to show the effectiveness of the FRDTM for some values of x and t . In Section 5, we present tables for Examples 4.1, 4.2 and 4.3. Section 6 is for discussion and conclusion of this paper.

2 Background of fractional calculus

Here are some definitions and facts that we shall use in our work. Some of these basic definitions are due to Liouville [3, 4, 33, 34].

Definition 2.1 A real function $f(x), x > 0$, is said to be in the space $C_\mu, \mu \in \mathbb{R}$, if there exists a real number $q(> \mu)$ such that $f(x) = x^q g(x)$, where $g(x) \in C[0, \infty)$, and it is said to be in the space C_μ^m if $f^{(m)} \in C_\mu, m \in \mathbb{N}$.

Definition 2.2 For an integrable function $f \in C_\mu$, the Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ is defined as

$$\begin{cases} J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, & \text{when } \alpha > 0, x > 0, \\ J^0 f(x) = f(x). \end{cases}$$

Definition 2.3 The fractional derivative of $f \in C_{-1}^m$ in the Caputo sense can be defined as

$$D^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t) dt, \quad \text{where } m-1 < \alpha \leq m, m \in \mathbb{N}, x > 0.$$

Lemma 2.4 ([6]) *If $m-1 < \alpha \leq m, m \in \mathbb{N}$ and $f \in C_\mu^m, \mu \geq -1$, then*

$$\begin{cases} D^\alpha J^\alpha f(x) = f(x), & \text{if } x > 0, \\ J^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, & \text{if } m-1 < \alpha < m. \end{cases}$$

We use the Caputo fractional derivative because it allows traditional initial and boundary conditions to be included in the formulation of our work.

3 Analysis of the FRDTM

We present the methodology of the FRDTM as follows. Consider a function $u(x, t)$ which is analytic and k -times continuously differentiable with respect to time t and space x in the domain of the interest. Now one can represent $u(x, t)$ as a product of two single-variable functions such as $u(x, t) = f(x)g(t)$. Thus the function can be represented as

$$u(x, t) = \left(\sum_{i=0}^{\infty} F(i)x^i \right) \left(\sum_{j=0}^{\infty} G(j)t^j \right) = \sum_{k=0}^{\infty} U_k(x)t^k. \tag{3.1}$$

Definition 3.1 If $u(x, t)$ is analytic and continuously differentiable with respect to space variable x and time t in the domain of interest, then the t -dimensional spectrum function

$$U_k(x) = \frac{1}{\Gamma(\alpha k + 1)} \left[\frac{\partial^{\alpha k}}{\partial t^{\alpha k}} u(x, t) \right]_{t=t_0} \tag{3.2}$$

is the reduced transformed function of $u(x, t)$, where α is a parameter which describes the order of time-fractional derivative.

Throughout this paper, $u(x, t)$ represents the original function and $U_k(x)$ represents the reduced transformed function. The differential inverse transform of $U_k(x)$ is given by

$$u(x, t) = \sum_{k=0}^{\infty} U_k(x)(t - t_0)^{\alpha k}. \tag{3.3}$$

From Eqs. (3.3) and (3.2) one can deduce

$$u(x, t) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + 1)} \left[\frac{\partial^{\alpha k}}{\partial t^{\alpha k}} u(x, t) \right]_{t=t_0} (t - t_0)^{\alpha k}. \tag{3.4}$$

Note that when $t_0 = 0$, Eq. (3.4) becomes

$$u(x, t) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + 1)} \left[\frac{\partial^{\alpha k}}{\partial x^{\alpha k}} u(x, t) \right]_{t=0} t^{\alpha k}. \tag{3.5}$$

Table 1 Basic operations of the FRDTM [39]

Functional form	Transformed form
$u(x, t)$	$\frac{1}{\Gamma(k\alpha+1)} [\frac{\partial^{\alpha k}}{\partial t^{\alpha k}} u(x, t)]_{t=0}$
$\gamma u(x, t) \pm \beta v(x, t)$	$\gamma U_k(x) \pm \beta V_k(x)$, where γ and β are constants
$u(x, t).v(x, t)$	$\sum_{i=0}^k U_i(x)V_{k-i}(x)$
$u(x, t).v(x, t).w(x, t)$	$\sum_{i=0}^k \sum_{j=0}^i U_j(x)V_{i-j}(x)W_{k-i}(x)$
$\frac{\partial^{n\alpha}}{\partial t^{n\alpha}} u(x, t)$	$\frac{\Gamma(k\alpha+n\alpha+1)}{\Gamma(k\alpha+1)} U_{k+n}(x)$
$\frac{\partial^n}{\partial x^n} u(x, t)$	$\frac{\partial^n}{\partial x^n} U_k(x)$
$x^m t^n u(x, t)$	$x^m U_{k-n}(x)$
$x^m t^n$	$x^m \delta(k\alpha - n)$, where $\delta(k\alpha - n) = \begin{cases} 1, & \alpha k = n \\ 0, & \alpha k \neq n \end{cases}$

Some basic operations of the reduced differential transformation [35–38] obtained from Eqs. (3.2) and (3.4) are given in Table 1, and the proofs of some of the properties can be found in [39].

Remark 3.2 In Table 1, Γ represents the gamma function, where $\Gamma(z + 1) = z\Gamma(z)$, $z > 0$.

3.1 Methodology

To explain how the FRDTM works, we consider the general fractional nonlinear nonhomogeneous partial differential equation

$$L(u(x, t)) + R(u(x, t)) + N(u(x, t)) = L(h(x, t)) \tag{3.6}$$

subject to the initial conditions

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x). \tag{3.7}$$

Here the $L = D_t^\alpha$, R is the linear differential operator, N represents the general nonlinear operator and $h(x, t)$ is the nonhomogeneous source term.

From Table 1 and Eq. (3.6), we can get the following:

$$\frac{\Gamma(\alpha k + \alpha + 1)}{\Gamma(\alpha k + 1)} U_{k+1}(x) = H_k(x) - R(U_k(x)) - N(U_k(x)), \tag{3.8}$$

where $U_k(x)$, $R(U_k(x))$, $N(U_k(x))$ and $H_k(x)$ are the transformations of the functions $L(u(x, t))$, $R(u(x, t))$, $N(u(x, t))$ and $L(h(x, t))$, respectively.

Now from Eq. (3.7) we can get

$$U_0(x) = f(x); \quad U_1(x) = g(x). \tag{3.9}$$

To find all other iterations, we first substitute Eq. (3.10) into Eq. (3.9) to find the remaining values of $U_k(x)$. Finally, we apply the inverse transformation to all the values of $\{U_k(x)\}_{k=0}^n$ to obtain

$$\widehat{u}(x, t) = \sum_{k=0}^n U_k(x)t^{\alpha k}. \tag{3.10}$$

Finally, the exact solution of the problem is given by $u(x, t) = \lim_{n \rightarrow \infty} \widehat{u}^n(x, t)$.

4 Worked examples

We shall employ the FRDTM to three different applications to illustrate the accuracy and efficiency of the method.

4.1 The time-fractional Burgers equation

We consider the following time-fractional Burgers equation [30]:

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \nu \frac{\partial^2 u}{\partial x^2} + \varepsilon u \frac{\partial u}{\partial x} = 0, \quad x > 0, t > 0, 0 < \alpha \leq 1, \tag{4.1}$$

subject to the initial condition

$$u(x, 0) = \frac{\mu + \sigma + (\sigma - \mu)e^\gamma}{1 + e^\gamma}, \tag{4.2}$$

where $\gamma = \frac{\mu}{\nu}(x - \lambda)$, and the parameters μ, σ, λ and ν are arbitrary constants.

Using Table 1, Eq. (4.1) and Eq. (4.2) become

$$U_{k+1}(x) = \frac{\Gamma(k\alpha + 1)}{\Gamma(\alpha k + \alpha + 1)} \left(\nu \frac{\partial^2}{\partial x^2} U_k(x) - \varepsilon \left(\sum_{i=0}^k U_i(x) \frac{\partial}{\partial x} U_{k-i}(x) \right) \right), \tag{4.3}$$

where

$$U_0(x) = \frac{\mu + \sigma + (\sigma - \mu)e^\gamma}{1 + e^\gamma}. \tag{4.4}$$

Substitute Eq. (4.4) into Eq. (4.3) to get

$$\begin{aligned} U_1(x) &= \frac{2e^\gamma \mu^2 (\sigma(1 + e^\gamma))}{\nu(1 + e^\gamma)^3 (\alpha + 1)}, \\ U_2(x) &= \frac{2e^\gamma \mu^3 (\sigma^2(e^\gamma - 1)(1 + e^\gamma)^2)}{\nu^2(\alpha + 1)(2\alpha + 1)(1 + e^\gamma)^5}. \end{aligned} \tag{4.5}$$

We proceed in this way to get

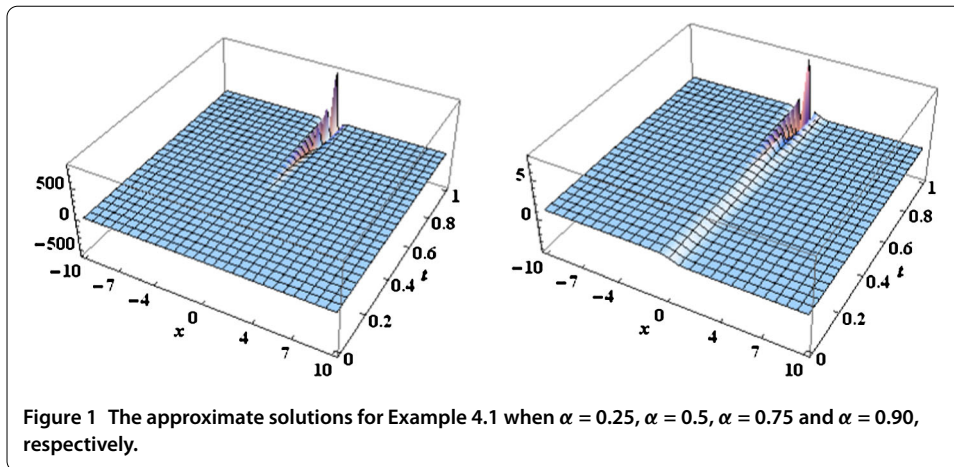
$$\sum_{k=0}^{\infty} U_k(x) t^{\alpha k} = U_0(x) + U_1(x) t^\alpha + U_2(x) t^{2\alpha} + \dots \tag{4.6}$$

Thus, we have the solution of Eq. (4.1) in a series form for $\alpha = 1$ and $\varepsilon = 1$

$$\begin{aligned} u(x, t) &= \frac{\mu + \sigma + (\sigma - \mu)e^\gamma}{1 + e^\gamma} + \frac{2e^\gamma \mu^2 (\sigma(1 + e^\gamma))}{\nu(1 + e^\gamma)^3 (\alpha + 1)} t \\ &\quad + \frac{2e^\gamma \mu^3 (\sigma^2(e^\gamma - 1)(1 + e^\gamma)^2)}{\nu^2(\alpha + 1)(2\alpha + 1)(1 + e^\gamma)^5} t^2 + \dots \end{aligned} \tag{4.7}$$

Hence

$$u(x, t) = \frac{\mu + \sigma + (\sigma - \mu) \exp[\frac{\mu}{\nu}(x - \sigma t - \lambda)]}{1 + \exp[\frac{\mu}{\nu}(x - \sigma t - \lambda)]}. \tag{4.8}$$



This is the exact solution given by the ADM in [2] and the VIM in [4], where $\gamma = \frac{\mu}{\nu}(x - \lambda)$ and the parameters μ, σ, λ and ν are arbitrary constants.

For a special case, we consider the case when $\nu = 0.1, \mu = 0.4, \sigma = 0.6, \lambda = 0.125, \epsilon = 1$ to obtain

$$\begin{aligned}
 U_0(x) &= \frac{1 + 0.2e^{4(x-0.125)}}{1 + e^{4(x-0.125)}}, \\
 U_1(x) &= \frac{(5.219101e^{4x} + 3.165545e^{8x})\Gamma(\alpha)}{(1.648721 + e^{4x})\Gamma(1 + \alpha)}, \\
 U_2(x) &= \frac{(-92.554154e^{4x} - 168.410797e^{8x} - 68.097541e^{12x})\Gamma(\alpha)\Gamma(2\alpha)}{(1.64872127 + e^{4x})^7\Gamma(\alpha + 1)\Gamma(2\alpha + 1)} \\
 &\quad + \frac{(41.303246e^{16x} + 37.577528e^{20x} + 7.597308e^{24x})\Gamma(\alpha)\Gamma(2\alpha)}{(1.64872127 + e^{4x})^7\Gamma(\alpha + 1)\Gamma(2\alpha + 1)}.
 \end{aligned}$$

If we proceed in this direction, the differential inverse transform of $\{U_k(x)\}_{k=0}^\infty$ is given by

$$\widehat{u}(x, t) = \sum_{k=0}^n U_k(x)t^{\alpha k} = U_0(x) + U_1(x)t^\alpha + U_2(x)t^{2\alpha} + \dots \tag{4.9}$$

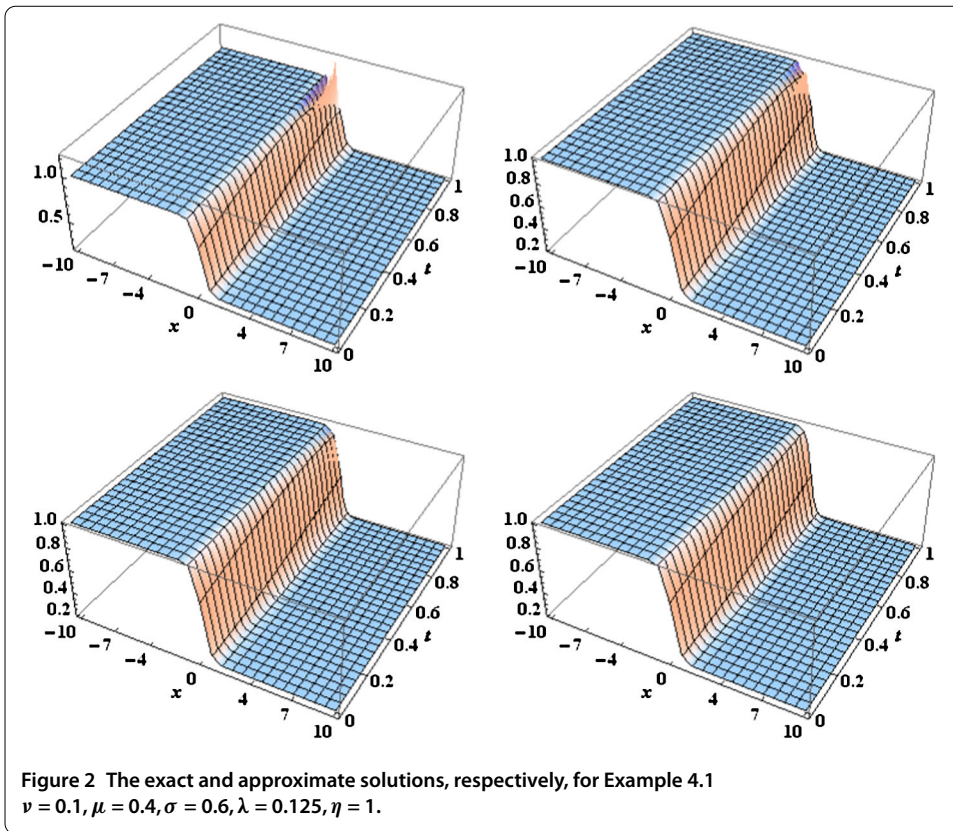
Thus, the exact solution of the problem is given by $u(x, t) = \lim_{n \rightarrow \infty} \widehat{u}_n(x, t)$.

Remark 4.1 The sketches of the approximate solutions and the exact solution $u(x, t)$ of Eq. (4.1) given by Momani [40] for the values of $\alpha = 0.25, \alpha = 0.5, \alpha = 0.75, \alpha = 1$ and different values of x and t are shown in Figure 1. Also from Figure 2 one can observe that the exact and approximate solutions for Example 4.1 are very close when $\nu = 0.1, \mu = 0.4, \sigma = 0.6, \lambda = 0.125, \eta = 1$.

4.2 The space-fractional Burgers equation

Consider the space-fractional Burgers equation [30]

$$\frac{\partial u}{\partial t} + \eta \frac{\partial^\beta u}{\partial x^\beta} + \epsilon u \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} = 0, \quad x > 0, t > 0, 0 < \beta \leq 1, \tag{4.10}$$



where ϵ, ν, η are parameters, and the boundary conditions are as follows:

$$u(0, t) = 0, \quad u_x(0, t) = \frac{1}{t} - \frac{\pi^2}{2\nu t^2}. \tag{4.11}$$

Apply the FRDTM operator to Eq. (4.10) and Eq. (4.11) to get

$$U_{k+2}(t) = \frac{1}{\nu(k+2)(k+1)} \left(\eta \frac{\Gamma(k\beta + \beta + 1)}{\Gamma(k\beta + 1)} U_{k+1}(t) + \frac{\partial}{\partial t} U_k(t) \right) + \frac{1}{\nu(k+2)(k+1)} \left(\epsilon \sum_{i=0}^k (i+1) U_{i+1}(t) U_{k-i}(t) \right) \tag{4.12}$$

and

$$U_0(t) = 0, \quad U_1(t) = \frac{1}{t} - \frac{\pi^2}{2t^2}. \tag{4.13}$$

Substitute Eq. (4.13) into Eq. (4.12) to obtain

$$U_2(t) = \frac{(2t - \pi^2)\eta\Gamma(\beta + 1)}{4t^2\nu},$$

$$U_3(t) = \frac{(4\pi^2t(1 - \epsilon) + 4t^2(\epsilon - 1) + \epsilon\pi^4)\nu + t^2(2t - \pi^2)\eta^2\Gamma(2\beta + 1)}{24t^4\nu^2}.$$

We proceed, and after the 7th iteration we get the following approximate solution:

$$\widehat{u}(x, t) = \sum_{k=0}^n U_k(t)x^{\beta k} = U_0(t) + U_1(t)x^\beta + U_2(t)x^{2\beta} + U_3(t)x^{3\beta} + \dots$$

Thus, when $\beta = 1$ and $\varepsilon = 1$, the approximate solution becomes

$$\begin{aligned} \widehat{u}(x, t) = & \left(\frac{1}{t} - \frac{\pi^2}{2t^2}\right)x + \left(\frac{(2t - \pi^2)\eta}{4t^2\nu}\right)x^2 + \left(\frac{\pi^4\nu + t^2(2t - \pi^2)\eta^2}{12t^4\nu^2}\right)x^3 \\ & + \left(\frac{\eta((9\pi^4 - 24\pi^2t + 24t^2)\nu + 3(\pi^4\nu + 2(2t^3 - \pi^2t^2)\eta^2))}{288t^4\nu^3}\right)x^4 \\ & + \dots \end{aligned}$$

This is the approximate solution of Eq. (4.10) in a series form. Thus the exact solution in the special case when $\eta = 0$ is given by

$$\begin{aligned} u(x, t) &= \sum_{k=0}^{\infty} U_k(t)x^k \\ &= U_0(t) + U_1(t)x + U_2(t)x^2 + \dots \\ &= \left(\frac{1}{t} - \frac{\pi^2}{2t^2}\right)x + \left(\frac{\pi^4}{12t^4\nu}\right)x^3 + \dots \\ &= \frac{x}{t} - \frac{\pi}{t} \tanh\left(\frac{\pi x}{2\nu t}\right). \end{aligned}$$

This is the exact solution of Eq. (4.10) which was given in [30].

Remark 4.2 The sketches of the approximate solutions and the exact solution $u(x, t)$ of Eq. (4.10) given by Momani [12] for the values of $\beta = 0.25, \beta = 0.5, \beta = 0.75, \beta = 0.90$ and different values of x and t when $\eta = 1, \nu = 1, \varepsilon = 1$ are shown in Figure 3 and 4.

4.3 The time-fractional Cahn-Allen equation

Consider the following nonlinear time-fractional Cahn-Allen equation [23, 32]:

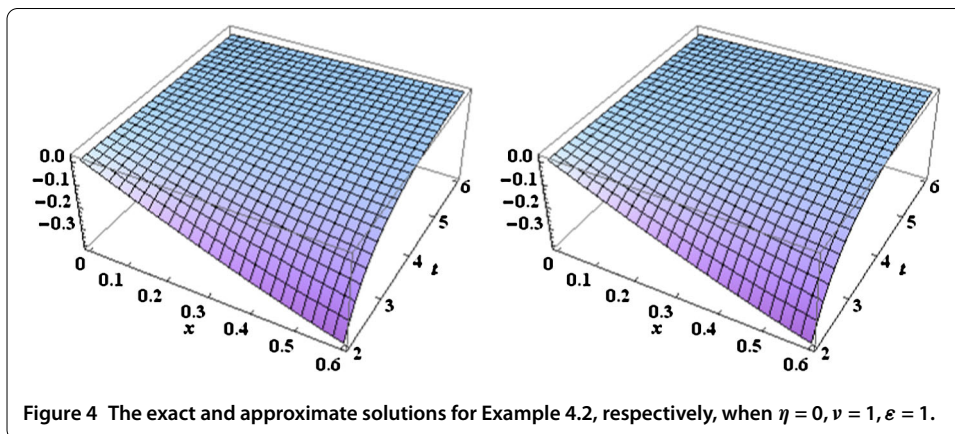
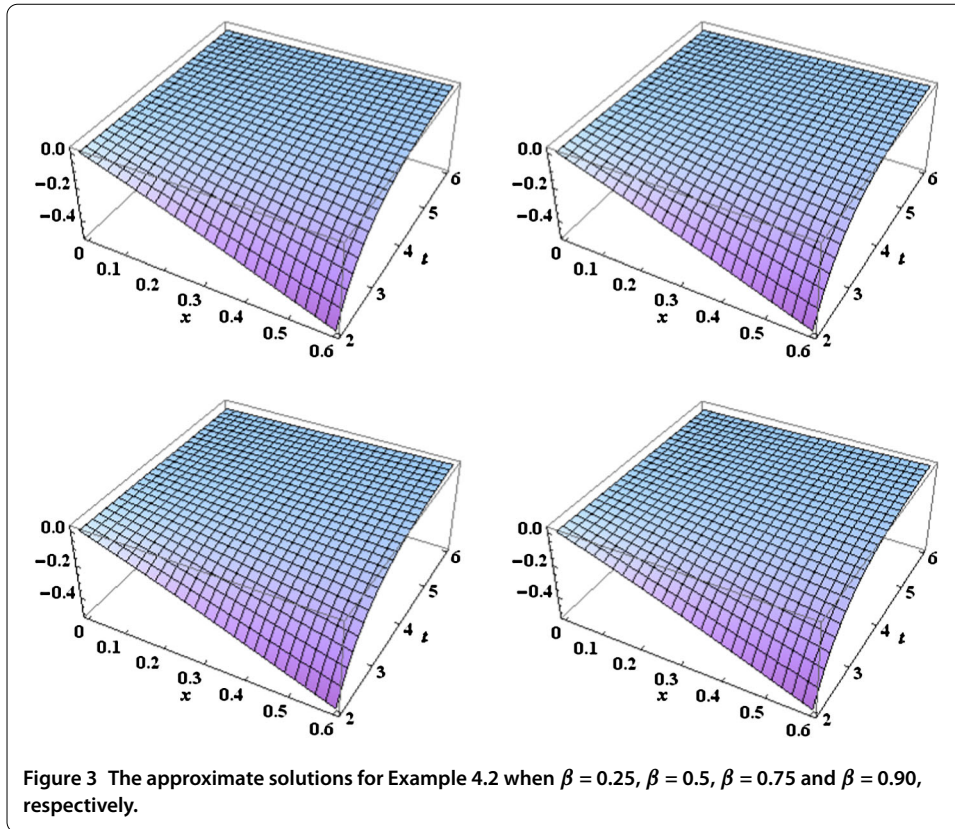
$$\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^2 u}{\partial x^2} + u^3 - u = 0, \quad t > 0, 0 < \alpha \leq 1, \tag{4.14}$$

subject to

$$u(x, 0) = \frac{1}{1 + e^{\left(\frac{x}{\sqrt{2}}\right)}}. \tag{4.15}$$

Apply the FRDTM to Eq. (4.14) to get

$$\begin{aligned} U_{k+1}(x) &= \frac{\Gamma(k\alpha + \alpha + 1)}{\Gamma(k\alpha + 1)} \left(U_k(x) + \frac{\partial^2 U_k(x)}{\partial x^2} \right) \\ &\quad - \frac{\Gamma(k\alpha + \alpha + 1)}{\Gamma(k\alpha + 1)} \left(\sum_{i=0}^k \sum_{j=0}^i U_{i-j}(x)U_j(x)U_{k-i}(x) \right). \end{aligned} \tag{4.16}$$



Using Table 1 and Eq. (4.15), we conclude

$$U_0(x) = \frac{1}{1 + e^{\left(\frac{x}{\sqrt{2}}\right)}}. \tag{4.17}$$

Substitute Eq. (4.17) into Eq. (4.16) to obtain

$$U_1(x) = \frac{3}{4\alpha(1 + \cosh(\frac{x}{\sqrt{2}}))}, \quad U_2(x) = \frac{9 \operatorname{csch}^3(\frac{x}{\sqrt{2}}) \sinh^4(\frac{x}{2\sqrt{2}})}{4\alpha^2},$$

$$U_3(x) = \frac{9(\cosh(\frac{x}{\sqrt{2}}) - 2) \operatorname{sech}^4(\frac{x}{2\sqrt{2}})}{128\alpha^3},$$

$$U_4(x) = \frac{27(\cosh(\frac{x}{\sqrt{2}}) - 5) \operatorname{sech}^4(\frac{x}{2\sqrt{2}}) \tanh(\frac{x}{2\sqrt{2}})}{1024\alpha^4}.$$

We proceed in this way, and after the 4th iteration the approximate solution is given by

$$\widehat{u}(x, t) = \sum_{k=0}^{\infty} U_k(x)t^{\alpha k}$$

$$= U_0(x) + U_1(x)t^{\alpha} + U_2(x)t^{2\alpha} + U_3(x)t^{3\alpha} + \dots.$$

Thus

$$u(x, t) = \frac{1}{1 + e^{\frac{x}{\sqrt{2}}}} + \frac{3t^{\alpha}}{4\alpha(1 + \cosh(\frac{x}{\sqrt{2}}))}$$

$$+ \frac{9 \operatorname{csch}^3(\frac{x}{\sqrt{2}}) \sinh^4(\frac{x}{2\sqrt{2}})}{4\alpha^2} t^{2\alpha}$$

$$+ \frac{9(\cosh(\frac{x}{\sqrt{2}}) - 2) \operatorname{sech}^4(\frac{x}{2\sqrt{2}})}{128\alpha^3} t^{3\alpha} + \dots. \tag{4.18}$$

Hence the exact solution when $\alpha = 1$ is given by

$$\widehat{u}(x, t) = \sum_{k=0}^{\infty} U_k(x)x^k$$

$$= \frac{1}{1 + e^{\frac{x}{\sqrt{2}}}} + \frac{3t}{4(1 + \cosh(\frac{x}{\sqrt{2}}))}$$

$$+ \frac{9 \operatorname{csch}^3(\frac{x}{\sqrt{2}}) \sinh^4(\frac{x}{2\sqrt{2}})}{4} t^2 + \dots$$

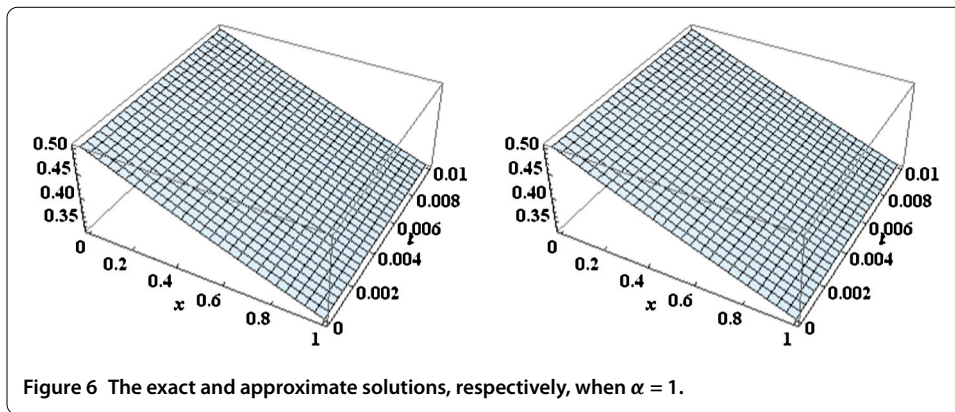
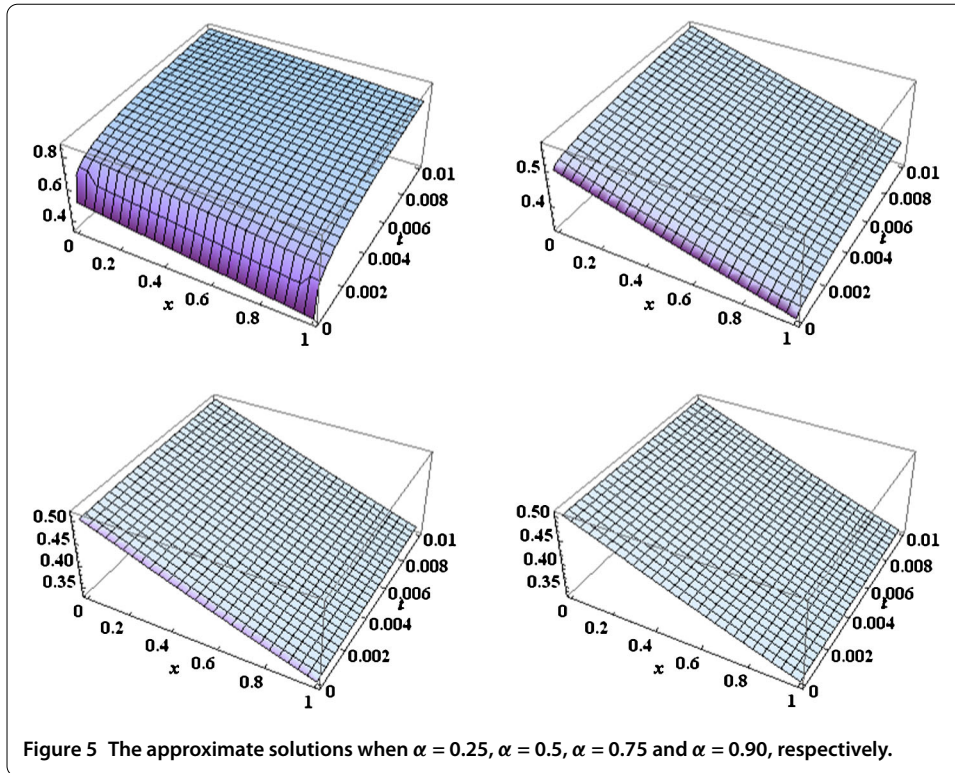
$$= \frac{1}{1 + e^{\frac{x}{\sqrt{2}} - \frac{3t}{2}}}. \tag{4.19}$$

This is the exact solution of Eq. (4.14) which was given in [23, 32].

Remark 4.3 The sketches of the approximate solutions and the exact solution $u(x, t)$ of Eq. (4.19) are given in Figures 5 and 6 for a constant value of $a = 4$ and for the values of $\alpha = 0.25, \alpha = 0.5, \alpha = 0.75, \alpha = 0.90$ and for different values of x and t . Note that the accuracy increases as the order of approximation increases.

5 Tables of numerical calculations

In this section, we present tables to show the comparison of results of the FRDTM approximate solutions and the exact solution for different values of α and β . In Table 2 we use different values of x and t and $\nu = 0.1, \mu = 0.4, \sigma = 0.6, \lambda = 0.125, \eta = 1$, and in Table 3 we use different values of x and t and $\eta = 1, \nu = 1, \varepsilon = 1$. Finally, we present Table 4 for different values of x and t and different values for α with only four iterations. It is important to mention that for Example 4.3 we only used $n = 4$, i.e., four iterations, and we obtained



better approximate values, while in [32] the authors used five iterations. This shows that the FRDTM converges faster than the existing methods in the literature.

6 Conclusion

In this paper, we successfully implemented the FRDTM to find approximate solutions of the space-time fractional Burgers equations and the time-fractional Cahn-Allen equation for different values of α and β and the results we obtained in Examples 4.1, 4.2 and 4.3 were in excellent agreement with the exact solutions. The FRDTM introduces a significant improvement in the field over the existing methods.

Competing interests

The author declares that he has no competing interests.

Table 2 The results obtained by the FRDTM for different values of α for Example 4.1

x	t	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$	Exact
		Numerical	Numerical	Numerical	Numerical	
-10	2	1	1	1	1	1
	4	1	1	1	1	1
	6	1	1	1	1	1
	8	1	1	1	1	1
-5	2	1.00014	1.00014	1	1	1
	4	1.02059	1.00261	1.00001	1	1
	6	1.37155	1.02059	1.00014	1.00002	1
	8	3.86303	1.02059	1.00112	1.00014	1
5	2	0.200786	0.20001	0.200001	0.2	0.2
	4	0.280682	0.200786	0.200059	0.20001	0.20004
	6	1.48828	0.211558	0.200786	0.200123	0.204848
	8	9.53999	0.280682	0.205256	0.200786	0.540446
10	2	0.2	0.2	0.2	0.2	0.2
	4	0.2	0.2	0.2	0.2	0.2
	6	0.2	0.2	0.2	0.2	0.2
	8	0.2	0.2	0.2	0.2	0.2

Table 3 The results obtained by the FRDTM for different values of β for Example 4.2

x	t	$\beta = 0.25$	$\beta = 0.5$	$\beta = 0.75$	$\beta = 1$	Exact
		Numerical	Numerical	Numerical	Numerical	
0.1	2	-0.0764845	-0.0764173	-0.0765568	-0.076893	-0.073117
	3	-0.0224361	-0.0224164	-0.0224575	-0.0225564	-0.0214477
	4	-0.00609528	-0.00608992	-0.00610107	-0.00612793	-0.00582667
	6	0.00309842	0.00309573	0.00310145	0.00311518	0.00296201
0.3	2	-0.243822	-0.24332	-0.244843	-0.248451	-0.21341
	3	-0.0721999	-0.0720561	-0.0725197	-0.0736135	-0.0631537
	4	-0.0195606	-0.019521	-0.0196462	-0.019942	-0.0171018
	6	0.0102373	0.0102193	0.0102888	0.0104509	0.008961
0.5	2	-0.419375	-0.418177	-0.4228	-0.433774	-0.336983
	3	-0.126549	-0.126227	-0.127726	-0.131255	-0.101393
	4	-0.034099	-0.0340076	-0.0344084	-0.0353558	-0.0272609
	6	0.0188912	0.0188622	0.0191169	0.0197044	0.0151832
0.6	2	-0.505904	-0.504266	-0.511001	-0.527077	-0.389893
	3	-0.154611	-0.154193	-0.156467	-0.161845	-0.118574
	4	-0.0414992	-0.041377	-0.0419815	-0.0434176	-0.0317049
	6	0.0238959	0.0238714	0.0242898	0.0252544	0.0184231

Table 4 The results obtained by the FRDTM for different values of α for Example 4.3

x	t	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$	Exact
		Numerical	Numerical	Numerical	Numerical	
0.1	0.002	0.768025	0.515858	0.487054	0.483079	0.483079
	0.003	0.79101	0.523384	0.488734	0.483453	0.483453
	0.004	0.807412	0.529721	0.490276	0.483828	0.483828
	0.006	0.830055	0.540329	0.493102	0.484577	0.484577
0.3	0.002	0.7419	0.480518	0.451845	0.447907	0.447907
	0.003	0.766709	0.488048	0.453511	0.448278	0.448278
	0.004	0.784528	0.494401	0.455041	0.448649	0.448649
	0.006	0.809302	0.505062	0.457846	0.449391	0.449391
0.5	0.002	0.713958	0.445372	0.417112	0.413248	0.413248
	0.003	0.740586	0.452832	0.418748	0.413612	0.413612
	0.004	0.759881	0.459137	0.420251	0.413976	0.413976
	0.006	0.787034	0.469743	0.423008	0.414704	0.414704
0.6	0.002	0.699334	0.427979	0.400028	0.396214	0.396214
	0.003	0.726851	0.435377	0.401642	0.396573	0.396573
	0.004	0.746891	0.441635	0.403127	0.396932	0.396932
	0.006	0.775296	0.452175	0.40585	0.397651	0.397651

Acknowledgements

The author would like to thank the editor and the anonymous referees for their comments and suggestions on this paper.

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Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 9 February 2017 Accepted: 21 March 2017 Published online: 31 March 2017

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