Jiang et al. *Advances in Difference Equations* (2016) 2016:171
DOI 10.1186/s13662-016-0902-7

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Oscillatory behavior of third-order nonlinear neutral delay differential equations

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Abstract

We provide an oscillation criterion for a class of third-order nonlinear neutral delay differential equations by using the double generalized Riccati substitutions. Our theorem complements and improves previous results. Two illustrative examples are included.

MSC: 34K11

Keywords: asymptotic behavior; oscillation; delay argument; third-order neutral differential equation

1 Introduction

This article is concerned with the oscillation and asymptotic behavior of a nonlinear third-order neutral delay differential equation

$$\bigl(r(t)\bigl(z''(t)\bigr)^{\alpha}\bigr)' + q(t)f\bigl(x\bigl(\sigma(t)\bigr)\bigr) = 0, \tag{1.1}$$

where $t \geq t_0 > 0$, $z(t) := x(t) + p(t)x(\tau(t))$, and $\alpha \geq 1$ is a ratio of odd positive integers. We also suppose that the following assumptions hold:

(A_1) $r \in \mathrm{C}^1([t_0,\infty),(0,\infty))$, $p, q \in \mathrm{C}([t_0,\infty),[0,\infty))$, $\tau \in \mathrm{C}^1([t_0,\infty),\mathbb{R})$, $\sigma \in \mathrm{C}([t_0,\infty),\mathbb{R})$, and q is not identically zero for large t;
(A_2) $\int_{t_0}^{\infty} r^{-1/\alpha}(t)\,\mathrm{d}t = \infty$ and $0 \leq p(t) \leq p_0 < \infty$;
(A_3) $\tau(t) \leq t$, $\sigma(t) \leq t$, and $\lim_{t\to\infty}\tau(t) = \lim_{t\to\infty}\sigma(t) = \infty$;
(A_4) $f \in \mathrm{C}(\mathbb{R},\mathbb{R})$ and there exists a positive constant k such that $f(u)/u^{\alpha} \geq k$ for all $u \neq 0$;
(A_5) $\tau'(t) \geq \tau_0 > 0$ and $\tau \circ \sigma = \sigma \circ \tau$.

By a solution to equation (1.1) we mean a function $x \in \mathrm{C}([T_x,\infty),\mathbb{R})$, $T_x \geq t_0$, which has the property $r(z'')^{\alpha} \in \mathrm{C}^1([T_x,\infty),\mathbb{R})$ and satisfies (1.1) on the interval $[T_x,\infty)$. We consider only those solutions to (1.1) which satisfy condition $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$ and assume that (1.1) possesses such solutions. A solution of (1.1) is called oscillatory if it has arbitrarily large zeros on $[T_x,\infty)$; otherwise, it is said to be nonoscillatory.

In recent years, the oscillation theory of functional differential equations has received much attention since it has a great number of applications in engineering and natural sciences. For some related contributions on the oscillatory behavior of various classes of functional differential equations, we refer the reader to [1–16] and the references cited therein. In the following, we provide some background details that motivated our study.

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Baculíková *et al.* [6] and Li and Rogovchenko [11] established several oscillation theorems for a second-order neutral differential equation

$$\left(r(t)\left|z'(t)\right|^{\alpha-1}z'(t)\right)' + q(t)f\left(x\left(\sigma(t)\right)\right) = 0, \quad z := x + p\cdot(x\circ\tau)$$

under the assumptions that (A_1)-(A_5) hold and $\alpha > 0$ is a constant. For oscillation of third-order neutral differential equations, Baculíková and Džurina [1, 2], Candan [7], and Džurina *et al.* [9] considered the couple of third-order neutral differential equations

$$\left(a(t)\left(\left(x(t) \pm p(t)x\left(\delta(t)\right)\right)''\right)^{\gamma}\right)' + q(t)x^{\gamma}\left(\tau(t)\right) = 0$$

in the case where

$$0 \le p(t) \le p_0 < 1. \tag{1.2}$$

Baculíková and Džurina [5] studied a class of third-order neutral differential equations

$$\left(a(t)\left(\left(x(t) + p(t)x\left(\delta(t)\right)\right)'\right)^{\gamma}\right)'' + q(t)x^{\gamma}\left(\tau(t)\right) = 0,$$

whereas Baculíková and Džurina [4], Jiang and Li [10], Li and Rogovchenko [12], Li *et al.* [14], and Xing *et al.* [16] considered a third-order neutral differential equation

$$\left(r(t)\left(x(t) + p(t)x\left(\tau(t)\right)\right)''\right)' + q(t)x\left(\sigma(t)\right) = 0. \tag{1.3}$$

In particular, using the comparison method, Xing *et al.* [16] obtained the following result for equation (1.3); see ([16], Corollary 2.8).

Theorem 1.1 *Assume that conditions* (A_1)-(A_3) *and* (A_5) *are satisfied, and let* $\sigma^{-1} \in \mathrm{C}^1([t_0,\infty),\mathbb{R})$, $(\sigma^{-1}(t))' \ge \sigma_0 > 0$, *and* $\sigma(t) < \tau(t) \le t$. *If*

$$r' \ge 0 \tag{1.4}$$

and

$$\liminf_{t\to\infty}\int_{\tau^{-1}(\sigma(t))}^{t}\frac{s^2\bar{Q}(s)}{r(s)}\,\mathrm{d}s > \frac{2(\tau_0+p_0)}{\sigma_0\tau_0\mathrm{e}},$$

where $\bar{Q}(t) := \min\{q(\sigma^{-1}(t)), q(\sigma^{-1}(\tau(t)))\}$, τ^{-1} *and* σ^{-1} *denote the inverse functions of* τ *and* σ, *respectively, then every solution* x *of* (1.3) *is either oscillatory or satisfies* $\lim_{t\to\infty}x(t) = 0$.

It should be noted that assumptions (1.2) and (1.4) are restrictive conditions in the study of oscillation of (1.1) and research in this paper was strongly motivated by the recent contributions of Li and Rogovchenko [11, 12], Li *et al.* [14], and Xing *et al.* [16]. Our principal goal is to establish an oscillation criterion for a nonlinear third-order neutral delay differential equation (1.1) which can be applied in the case when $p_0 > 1$ as well and without requiring condition (1.4). In the sequel, we use the following notation:

$$f_+(t) := \max\left\{0, f(t)\right\}, \qquad Q(t) := \min\left\{q(t), q\left(\tau(t)\right)\right\}, \qquad R(t) := \max\left\{r(t), r\left(\tau(t)\right)\right\},$$

and all functional inequalities are tacitly assumed to hold for all t large enough, unless mentioned otherwise.

2 Lemmas

Lemma 2.1 *Assume that conditions* (A_1)-(A_4) *hold and* x *is a positive solution of* (1.1). *Then there are only the following two possible cases for* z:

(I) $z(t) > 0$, $z'(t) > 0$, $z''(t) > 0$, *and* $(r(z'')^\alpha)'(t) \le 0$;

(II) $z(t) > 0$, $z'(t) < 0$, $z''(t) > 0$, *and* $(r(z'')^\alpha)'(t) \le 0$,

where $t \ge T$, $T \ge t_0$ *is sufficiently large.*

Proof The proof is similar to that of Baculíková and Džurina ([1], Lemma 1), and thus is omitted. □

Lemma 2.2 *Assume that conditions* (A_1)-(A_5) *are satisfied. Let* x *be a positive solution of* (1.1) *and the corresponding* z *satisfy case* (II) *in Lemma* 2.1. *If*

$$\int_{t_0}^{\infty} \xi \left(\frac{1}{R(\xi)} \int_{\xi}^{\infty} Q(s)\,\mathrm{d}s \right)^{1/\alpha} \mathrm{d}\xi = \infty, \tag{2.1}$$

then $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} z(t) = 0$.

Proof Note that there exist three constants c_1, c_2, and c_3 such that $\lim_{t\to\infty} r(t)(z''(t))^\alpha = c_1 \ge 0$, $\lim_{t\to\infty} z'(t) = c_2 \le 0$, and $\lim_{t\to\infty} z(t) = c_3 \ge 0$. A similar analysis to that in ([12], Theorem 15) leads to the conclusion that $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} z(t) = 0$. □

3 Main results

Theorem 3.1 *Assume that conditions* (A_1)-(A_5) *and* (2.1) *are satisfied. If there exist two functions* $\rho \in C^1([t_0,\infty),(0,\infty))$ *and* $\delta \in C^1([t_0,\infty),[0,\infty))$ *such that*

$$\int_{t_*}^{\infty} \left[2^{1-\alpha} k \rho(t) Q(t) \left(\frac{\int_{t_2}^{\sigma(t)} \int_{t_1}^{s} r^{-1/\alpha}(u)\,\mathrm{d}u\,\mathrm{d}s}{\int_{t_1}^{t} r^{-1/\alpha}(u)\,\mathrm{d}u} \right)^{\alpha} - G(t) \right] \mathrm{d}t = \infty \tag{3.1}$$

for a sufficiently large $t_1 \ge t_0$ *and for some* $t_* > t_2 > t_1$, *where*

$$\begin{aligned} G(t) := {} & \rho(t) \Big[\big(\big(r(t)\delta(t)\big)' - r(t)\delta^{1+1/\alpha}(t) \big) \\ & + \frac{p_0^\alpha}{\tau_0} \big(\big(r\big(\tau(t)\big)\delta\big(\tau(t)\big)\big)' - r\big(\tau(t)\big)\tau'(t)\delta^{1+1/\alpha}\big(\tau(t)\big) \big) \Big] \\ & + \frac{\rho(t) r(t)}{(\alpha+1)^{\alpha+1}} \left(\frac{\rho'_+(t)}{\rho(t)} + (\alpha+1)\delta^{1/\alpha}(t) \right)^{\alpha+1} \\ & + \frac{p_0^\alpha}{\tau_0^{\alpha+1}} \frac{\rho(t) r(\tau(t))}{(\alpha+1)^{\alpha+1}} \left(\frac{\rho'_+(t)}{\rho(t)} + (\alpha+1)\delta^{1/\alpha}\big(\tau(t)\big)\tau'(t) \right)^{\alpha+1}, \end{aligned} \tag{3.2}$$

then every solution x *of* (1.1) *is either oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof Suppose that x is a nonoscillatory solution of (1.1) which, without loss of generality, is eventually positive. Then there exists a $t_1 \ge t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and

$x(\sigma(t)) > 0$ for $t \geq t_1$. By Lemma 2.1, we observe that z satisfies either (I) or (II) for $t \geq T$, where $T \geq t_1$ is large enough. We consider each of the two cases separately.

Assume first that case (I) holds. By virtue of (1.1) and (A_4),

$$\big(r(t)\big(z''(t)\big)^{\alpha}\big)' = -q(t)f\big(x\big(\sigma(t)\big)\big) \leq -kq(t)x^{\alpha}\big(\sigma(t)\big) \leq 0. \tag{3.3}$$

It follows from $(r(\tau(t))(z''(\tau(t)))^{\alpha})' = (r(z'')^{\alpha})'(\tau(t))\tau'(t)$ that there exists a $t_2 \geq T$ such that, for $t \geq t_2$,

$$p_0^{\alpha}\frac{(r(\tau(t))(z''(\tau(t)))^{\alpha})'}{\tau'(t)} \leq -p_0^{\alpha}kq\big(\tau(t)\big)x^{\alpha}\big(\sigma\big(\tau(t)\big)\big).$$

Using the latter inequality and condition $\tau'(t) \geq \tau_0 > 0$, we get, for $t \geq t_2$,

$$\frac{p_0^{\alpha}}{\tau_0}\big(r\big(\tau(t)\big)\big(z''\big(\tau(t)\big)\big)^{\alpha}\big)' \leq -p_0^{\alpha}kq\big(\tau(t)\big)x^{\alpha}\big(\sigma\big(\tau(t)\big)\big). \tag{3.4}$$

Combining (3.3) and (3.4) and using the assumption $\sigma \circ \tau = \tau \circ \sigma$, we conclude that

$$\begin{aligned}
&\big(r(t)\big(z''(t)\big)^{\alpha}\big)' + \frac{p_0^{\alpha}}{\tau_0}\big(r\big(\tau(t)\big)\big(z''\big(\tau(t)\big)\big)^{\alpha}\big)' \\
&\quad \leq -k\big(q(t)x^{\alpha}\big(\sigma(t)\big) + p_0^{\alpha}q\big(\tau(t)\big)x^{\alpha}\big(\sigma\big(\tau(t)\big)\big)\big) \\
&\quad \leq -k\min\big\{q(t), q\big(\tau(t)\big)\big\}\big(x^{\alpha}\big(\sigma(t)\big) + p_0^{\alpha}x^{\alpha}\big(\tau\big(\sigma(t)\big)\big)\big) \\
&\quad = -kQ(t)\big(x^{\alpha}\big(\sigma(t)\big) + p_0^{\alpha}x^{\alpha}\big(\tau\big(\sigma(t)\big)\big)\big).
\end{aligned} \tag{3.5}$$

Using condition $0 \leq p(t) \leq p_0 < \infty$ and the inequality (see ([3], Lemma 1))

$$x_1^{\alpha} + x_2^{\alpha} \geq \frac{1}{2^{\alpha-1}}(x_1 + x_2)^{\alpha},$$

where $\alpha \geq 1$, $x_1 \geq 0$, and $x_2 \geq 0$, we have

$$x^{\alpha}\big(\sigma(t)\big) + p_0^{\alpha}x^{\alpha}\big(\tau\big(\sigma(t)\big)\big) \geq \frac{(x(\sigma(t)) + p_0x(\tau(\sigma(t))))^{\alpha}}{2^{\alpha-1}} \geq \frac{z^{\alpha}(\sigma(t))}{2^{\alpha-1}}. \tag{3.6}$$

Substitution of (3.6) into (3.5) implies that, for $t \geq t_2$,

$$\big(r(t)\big(z''(t)\big)^{\alpha}\big)' + \frac{p_0^{\alpha}}{\tau_0}\big(r\big(\tau(t)\big)\big(z''\big(\tau(t)\big)\big)^{\alpha}\big)' \leq -\frac{k}{2^{\alpha-1}}Q(t)\big(z\big(\sigma(t)\big)\big)^{\alpha}. \tag{3.7}$$

For $t \geq t_2$, define a function ω by

$$\omega(t) := \rho(t)\left[\frac{r(t)(z''(t))^{\alpha}}{(z'(t))^{\alpha}} + r(t)\delta(t)\right]. \tag{3.8}$$

Then $\omega(t) > 0$ for $t \geq t_2$. Differentiation of (3.8) yields

$$\begin{aligned}
\omega'(t) &= \rho'(t)\left[\frac{r(t)(z''(t))^{\alpha}}{(z'(t))^{\alpha}} + r(t)\delta(t)\right] + \rho(t)\left[\frac{r(t)(z''(t))^{\alpha}}{(z'(t))^{\alpha}} + r(t)\delta(t)\right]' \\
&= \frac{\rho'(t)}{\rho(t)}\omega(t) + \rho(t)\big(r(t)\delta(t)\big)' + \rho(t)\frac{(r(t)(z''(t))^{\alpha})'}{(z'(t))^{\alpha}} - \alpha\rho(t)\frac{r(t)(z''(t))^{\alpha+1}}{(z'(t))^{\alpha+1}}.
\end{aligned} \tag{3.9}$$

By virtue of (3.8),

$$\left(\frac{z''(t)}{z'(t)}\right)^{\alpha+1} = \left(\frac{\omega(t)}{\rho(t)r(t)} - \delta(t)\right)^{(\alpha+1)/\alpha}. \tag{3.10}$$

Substituting (3.10) into (3.9), we conclude that

$$\begin{aligned}\omega'(t) &= \frac{\rho'(t)}{\rho(t)}\omega(t) + \rho(t)\bigl(r(t)\delta(t)\bigr)' + \rho(t)\frac{(r(t)(z''(t))^\alpha)'}{(z'(t))^\alpha} \\ &\quad - \alpha\rho(t)r(t)\left(\frac{\omega(t)}{\rho(t)r(t)} - \delta(t)\right)^{(\alpha+1)/\alpha} \\ &\le \frac{\rho'_+(t)}{\rho(t)}\omega(t) + \rho(t)\bigl(r(t)\delta(t)\bigr)' + \rho(t)\frac{(r(t)(z''(t))^\alpha)'}{(z'(t))^\alpha} \\ &\quad - \frac{\alpha}{(\rho(t)r(t))^{1/\alpha}}\bigl(\omega(t) - \rho(t)r(t)\delta(t)\bigr)^{1+1/\alpha}. \end{aligned} \tag{3.11}$$

Let

$$A := \omega(t) \quad \text{and} \quad B := \rho(t)r(t)\delta(t).$$

Using the inequality (see ([13], Lemma 1) and note that $\alpha \ge 1$ is a ratio of odd integers)

$$A^{1+1/\alpha} - (A-B)^{1+1/\alpha} \le B^{1/\alpha}\left[\left(1+\frac{1}{\alpha}\right)A - \frac{1}{\alpha}B\right], \quad AB \ge 0, \tag{3.12}$$

we have

$$\begin{aligned}\bigl(\omega(t) - \rho(t)r(t)\delta(t)\bigr)^{1+1/\alpha} &\ge \omega^{1+1/\alpha}(t) + \frac{1}{\alpha}\bigl(\rho(t)r(t)\delta(t)\bigr)^{1+1/\alpha} \\ &\quad - \left(1+\frac{1}{\alpha}\right)\omega(t)\bigl(\rho(t)r(t)\delta(t)\bigr)^{1/\alpha}. \end{aligned} \tag{3.13}$$

Combining (3.11) and (3.13), we get

$$\begin{aligned}\omega'(t) &\le \frac{\rho'_+(t)}{\rho(t)}\omega(t) + \rho(t)\bigl(r(t)\delta(t)\bigr)' + \rho(t)\frac{(r(t)(z''(t))^\alpha)'}{(z'(t))^\alpha} - \frac{\alpha}{(\rho(t)r(t))^{1/\alpha}} \\ &\quad \times \left[\omega^{(\alpha+1)/\alpha}(t) + \frac{1}{\alpha}\bigl(\rho(t)r(t)\delta(t)\bigr)^{(\alpha+1)/\alpha} - \left(1+\frac{1}{\alpha}\right)\omega(t)\bigl(\rho(t)r(t)\delta(t)\bigr)^{1/\alpha}\right] \\ &= \rho(t)\frac{(r(t)(z''(t))^\alpha)'}{(z'(t))^\alpha} + \rho(t)\bigl(r(t)\delta(t)\bigr)' - \rho(t)r(t)\delta^{(\alpha+1)/\alpha}(t) \\ &\quad + \left(\frac{\rho'_+(t)}{\rho(t)} + (\alpha+1)\delta^{1/\alpha}(t)\right)\omega(t) - \frac{\alpha}{(\rho(t)r(t))^{1/\alpha}}\omega^{(\alpha+1)/\alpha}(t). \end{aligned} \tag{3.14}$$

Let

$$C := \frac{\rho'_+(t)}{\rho(t)} + (\alpha+1)\delta^{1/\alpha}(t), \qquad D := \frac{\alpha}{(\rho(t)r(t))^{1/\alpha}}, \quad \text{and} \quad u := \omega(t).$$

Using the inequality (see [11])

$$Cu - Du^{(\alpha+1)/\alpha} \le \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{C^{\alpha+1}}{D^\alpha}, \quad D > 0, \tag{3.15}$$

we deduce from (3.14) that

$$\begin{aligned} \omega'(t) \le{}& \rho(t)\frac{(r(t)(z''(t))^\alpha)'}{(z'(t))^\alpha} + \rho(t)\bigl(r(t)\delta(t)\bigr)' - \rho(t)r(t)\delta^{1+1/\alpha}(t) \\ &+ \frac{\rho(t)r(t)}{(\alpha+1)^{\alpha+1}}\left(\frac{\rho'_+(t)}{\rho(t)} + (\alpha+1)\delta^{1/\alpha}(t)\right)^{\alpha+1}. \end{aligned} \tag{3.16}$$

Define another function ν by

$$\nu(t) := \rho(t)\left[\frac{r(\tau(t))(z''(\tau(t)))^\alpha}{(z'(\tau(t)))^\alpha} + r\bigl(\tau(t)\bigr)\delta\bigl(\tau(t)\bigr)\right]. \tag{3.17}$$

Then $\nu(t) > 0$ for $t \ge t_2$. Differentiation of (3.17) implies that

$$\begin{aligned} \nu'(t) ={}& \rho'(t)\left[\frac{r(\tau(t))(z''(\tau(t)))^\alpha}{(z'(\tau(t)))^\alpha} + r\bigl(\tau(t)\bigr)\delta\bigl(\tau(t)\bigr)\right] \\ &+ \rho(t)\left[\frac{r(\tau(t))(z''(\tau(t)))^\alpha}{(z'(\tau(t)))^\alpha} + r\bigl(\tau(t)\bigr)\delta\bigl(\tau(t)\bigr)\right]' \\ ={}& \frac{\rho'(t)}{\rho(t)}\nu(t) + \rho(t)\bigl(r\bigl(\tau(t)\bigr)\delta\bigl(\tau(t)\bigr)\bigr)' + \rho(t)\frac{(r(\tau(t))(z''(\tau(t)))^\alpha)'}{(z'(\tau(t)))^\alpha} \\ &- \alpha\rho(t)\frac{r(\tau(t))(z''(\tau(t)))^{\alpha+1}\tau'(t)}{(z'(\tau(t)))^{\alpha+1}} \\ ={}& \frac{\rho'(t)}{\rho(t)}\nu(t) + \rho(t)\bigl(r\bigl(\tau(t)\bigr)\delta\bigl(\tau(t)\bigr)\bigr)' + \rho(t)\frac{(r(\tau(t))(z''(\tau(t)))^\alpha)'}{(z'(\tau(t)))^\alpha} \\ &- \alpha\rho(t)r\bigl(\tau(t)\bigr)\tau'(t)\left(\frac{\nu(t)}{\rho(t)r(\tau(t))} - \delta\bigl(\tau(t)\bigr)\right)^{(\alpha+1)/\alpha} \\ \le{}& \frac{\rho'_+(t)}{\rho(t)}\nu(t) + \rho(t)\bigl(r\bigl(\tau(t)\bigr)\delta\bigl(\tau(t)\bigr)\bigr)' + \rho(t)\frac{(r(\tau(t))(z''(\tau(t)))^\alpha)'}{(z'(\tau(t)))^\alpha} \\ &- \frac{\alpha\tau'(t)}{(\rho(t)r(\tau(t)))^{1/\alpha}}\bigl(\nu(t) - \rho(t)r\bigl(\tau(t)\bigr)\delta\bigl(\tau(t)\bigr)\bigr)^{1+1/\alpha}. \end{aligned} \tag{3.18}$$

Let

$$A := \nu(t) \quad \text{and} \quad B := \rho(t)r\bigl(\tau(t)\bigr)\delta\bigl(\tau(t)\bigr).$$

Using inequality (3.12), we obtain

$$\begin{aligned} \bigl(\nu(t) - \rho(t)r\bigl(\tau(t)\bigr)\delta\bigl(\tau(t)\bigr)\bigr)^{1+1/\alpha} \ge{}& \nu^{1+1/\alpha}(t) + \frac{1}{\alpha}\bigl(\rho(t)r\bigl(\tau(t)\bigr)\delta\bigl(\tau(t)\bigr)\bigr)^{1+1/\alpha} \\ &- \left(1+\frac{1}{\alpha}\right)\nu(t)\bigl(\rho(t)r\bigl(\tau(t)\bigr)\delta\bigl(\tau(t)\bigr)\bigr)^{1/\alpha}. \end{aligned} \tag{3.19}$$

Substituting (3.19) into (3.18), we get

$$\begin{aligned}
\nu'(t) \le{}& \frac{\rho'_+(t)}{\rho(t)}\nu(t) + \rho(t)\big(r\big(\tau(t)\big)\delta\big(\tau(t)\big)\big)' + \rho(t)\frac{(r(\tau(t))(z''(\tau(t)))^\alpha)'}{(z'(\tau(t)))^\alpha} - \frac{\alpha\tau'(t)}{(\rho(t)r(\tau(t)))^{1/\alpha}} \\
&\times\bigg[\nu^{1+1/\alpha}(t) + \frac{1}{\alpha}\big(\rho(t)r\big(\tau(t)\big)\delta\big(\tau(t)\big)\big)^{1+1/\alpha} \\
&- \bigg(1+\frac{1}{\alpha}\bigg)\nu(t)\big(\rho(t)r\big(\tau(t)\big)\delta\big(\tau(t)\big)\big)^{1/\alpha}\bigg] \\
\le{}& \rho(t)\frac{(r(\tau(t))(z''(\tau(t)))^\alpha)'}{(z'(\tau(t)))^\alpha} + \rho(t)\big(r\big(\tau(t)\big)\delta\big(\tau(t)\big)\big)' - \rho(t)r\big(\tau(t)\big)\delta^{1+1/\alpha}\big(\tau(t)\big)\tau'(t) \\
&+ \bigg(\frac{\rho'_+(t)}{\rho(t)} + (\alpha+1)\delta^{1/\alpha}\big(\tau(t)\big)\tau'(t)\bigg)\nu(t) - \frac{\alpha\tau'(t)}{(\rho(t)r(\tau(t)))^{1/\alpha}}\nu^{(\alpha+1)/\alpha}(t).
\end{aligned}\tag{3.20}$$

Let

$$C := \frac{\rho'_+(t)}{\rho(t)} + (\alpha+1)\delta^{1/\alpha}\big(\tau(t)\big)\tau'(t), \qquad D := \frac{\alpha\tau'(t)}{(\rho(t)r(\tau(t)))^{1/\alpha}}, \quad \text{and} \quad u := \nu(t).$$

Using inequality (3.15), we deduce from (3.20) that

$$\begin{aligned}
\nu'(t) \le{}& \rho(t)\frac{(r(\tau(t))(z''(\tau(t)))^\alpha)'}{(z'(\tau(t)))^\alpha} + \rho(t)\big(r\big(\tau(t)\big)\delta\big(\tau(t)\big)\big)' - \rho(t)r\big(\tau(t)\big)\tau'(t)\delta^{1+1/\alpha}\big(\tau(t)\big) \\
&+ \frac{\rho(t)r(\tau(t))}{(\alpha+1)^{\alpha+1}(\tau'(t))^\alpha}\bigg(\frac{\rho'_+(t)}{\rho(t)} + (\alpha+1)\delta^{1/\alpha}\big(\tau(t)\big)\tau'(t)\bigg)^{\alpha+1}.
\end{aligned}\tag{3.21}$$

Since $z''(t) > 0$ and $\tau(t) \le t$, we have $z'(\tau(t)) \le z'(t)$. Inequality (3.21) yields

$$\begin{aligned}
\nu'(t) \le{}& \rho(t)\frac{(r(\tau(t))(z''(\tau(t)))^\alpha)'}{(z'(t))^\alpha} + \rho(t)\big(r\big(\tau(t)\big)\delta\big(\tau(t)\big)\big)' - \rho(t)r\big(\tau(t)\big)\tau'(t)\delta^{1+1/\alpha}\big(\tau(t)\big) \\
&+ \frac{\rho(t)r(\tau(t))}{(\alpha+1)^{\alpha+1}(\tau'(t))^\alpha}\bigg(\frac{\rho'_+(t)}{\rho(t)} + (\alpha+1)\delta^{1/\alpha}\big(\tau(t)\big)\tau'(t)\bigg)^{\alpha+1}.
\end{aligned}\tag{3.22}$$

Combining (3.16) and (3.22) and utilizing (3.7), we obtain

$$\begin{aligned}
\omega'(t) + \frac{p_0^\alpha}{\tau_0}\nu'(t) \le{}& \rho(t)\frac{(r(t)(z''(t))^\alpha)' + p_0^\alpha/\tau_0(r(\tau(t))(z''(\tau(t)))^\alpha)'}{(z'(t))^\alpha} \\
&+ \rho(t)\big(\big(r(t)\delta(t)\big)' - r(t)\delta^{1+1/\alpha}(t)\big) \\
&+ \frac{\rho(t)r(t)}{(\alpha+1)^{\alpha+1}}\bigg(\frac{\rho'_+(t)}{\rho(t)} + (\alpha+1)\delta^{1/\alpha}(t)\bigg)^{\alpha+1} \\
&+ \frac{p_0^\alpha}{\tau_0}\rho(t)\big(\big(r\big(\tau(t)\big)\delta\big(\tau(t)\big)\big)' - r\big(\tau(t)\big)\tau'(t)\delta^{1+1/\alpha}\big(\tau(t)\big)\big) \\
&+ \frac{p_0^\alpha}{\tau_0}\frac{\rho(t)r(\tau(t))}{(\alpha+1)^{\alpha+1}(\tau'(t))^\alpha}\bigg(\frac{\rho'_+(t)}{\rho(t)} + (\alpha+1)\delta^{1/\alpha}\big(\tau(t)\big)\tau'(t)\bigg)^{\alpha+1} \\
\le{}& -2^{1-\alpha}k\rho(t)Q(t)\frac{(z(\sigma(t)))^\alpha}{(z'(t))^\alpha} + \rho(t)\big(\big(r(t)\delta(t)\big)' - r(t)\delta^{1+1/\alpha}(t)\big) \\
&+ \frac{p_0^\alpha}{\tau_0}\rho(t)\big(\big(r\big(\tau(t)\big)\delta\big(\tau(t)\big)\big)' - r\big(\tau(t)\big)\tau'(t)\delta^{1+1/\alpha}\big(\tau(t)\big)\big)
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\rho(t)r(t)}{(\alpha+1)^{\alpha+1}}\left(\frac{\rho'_+(t)}{\rho(t)} + (\alpha+1)\delta^{1/\alpha}(t)\right)^{\alpha+1} \\
&+ \frac{p_0^\alpha}{\tau_0^{\alpha+1}}\frac{\rho(t)r(\tau(t))}{(\alpha+1)^{\alpha+1}}\left(\frac{\rho'_+(t)}{\rho(t)} + (\alpha+1)\delta^{1/\alpha}\bigl(\tau(t)\bigr)\tau'(t)\right)^{\alpha+1} \\
= {}& -2^{1-\alpha}k\rho(t)Q(t)\frac{(z(\sigma(t)))^\alpha}{(z'(t))^\alpha} + G(t),
\end{aligned}
\tag{3.23}
$$

where G is defined as in (3.2). By virtue of $(r(t)(z''(t))^\alpha)' \le 0$, we have

$$
\begin{aligned}
z'(t) &= z'(t_2) + \int_{t_2}^{t} z''(s)\,\mathrm{d}s \\
&= z'(t_2) + \int_{t_2}^{t} \frac{(r(s)(z''(s))^\alpha)^{1/\alpha}}{r^{1/\alpha}(s)}\,\mathrm{d}s \\
&\ge r^{1/\alpha}(t)z''(t)\int_{t_2}^{t} r^{-1/\alpha}(s)\,\mathrm{d}s.
\end{aligned}
\tag{3.24}
$$

That is,

$$
z'(t)r^{-1/\alpha}(t) - z''(t)\int_{t_2}^{t} r^{-1/\alpha}(s)\,\mathrm{d}s \ge 0,
$$

which yields

$$
\left(\frac{z'(t)}{\int_{t_2}^{t} r^{-1/\alpha}(s)\,\mathrm{d}s}\right)' \le 0.
\tag{3.25}
$$

It follows from $\sigma(t) \le t$ that

$$
\frac{z'(\sigma(t))}{\int_{t_2}^{\sigma(t)} r^{-1/\alpha}(s)\,\mathrm{d}s} \ge \frac{z'(t)}{\int_{t_2}^{t} r^{-1/\alpha}(s)\,\mathrm{d}s},
$$

and so

$$
\frac{z'(\sigma(t))}{z'(t)} \ge \frac{\int_{t_2}^{\sigma(t)} r^{-1/\alpha}(s)\,\mathrm{d}s}{\int_{t_2}^{t} r^{-1/\alpha}(s)\,\mathrm{d}s}.
\tag{3.26}
$$

Using (3.25), we obtain

$$
\begin{aligned}
z(t) &= z(t_3) + \int_{t_3}^{t} z'(s)\,\mathrm{d}s \\
&= z(t_3) + \int_{t_3}^{t} \frac{z'(s)}{\int_{t_2}^{s} r^{-1/\alpha}(u)\,\mathrm{d}u}\int_{t_2}^{s} r^{-1/\alpha}(u)\,\mathrm{d}u\,\mathrm{d}s \\
&\ge \frac{z'(t)}{\int_{t_2}^{t} r^{-1/\alpha}(u)\,\mathrm{d}u}\int_{t_3}^{t}\int_{t_2}^{s} r^{-1/\alpha}(u)\,\mathrm{d}u\,\mathrm{d}s
\end{aligned}
$$

for $t \ge t_3 > t_2$, which implies that

$$
\frac{z(t)}{z'(t)} \ge \frac{\int_{t_3}^{t}\int_{t_2}^{s} r^{-1/\alpha}(u)\,\mathrm{d}u\,\mathrm{d}s}{\int_{t_2}^{t} r^{-1/\alpha}(u)\,\mathrm{d}u}.
\tag{3.27}
$$

It follows now from (3.26) and (3.27) that

$$\frac{(z(\sigma(t)))^\alpha}{(z'(t))^\alpha} = \left(\frac{z'(\sigma(t))}{z'(t)}\frac{z(\sigma(t))}{z'(\sigma(t))}\right)^\alpha \geq \left(\frac{\int_{t_3}^{\sigma(t)}\int_{t_2}^{s} r^{-1/\alpha}(u)\,\mathrm{d}u\,\mathrm{d}s}{\int_{t_2}^{t} r^{-1/\alpha}(u)\,\mathrm{d}u}\right)^\alpha. \tag{3.28}$$

Substitution of (3.28) into (3.23) yields

$$\omega'(t) + \frac{p_0^\alpha}{\tau_0}\nu'(t) \leq -2^{1-\alpha}k\rho(t)Q(t)\left(\frac{\int_{t_3}^{\sigma(t)}\int_{t_2}^{s} r^{-1/\alpha}(u)\,\mathrm{d}u\,\mathrm{d}s}{\int_{t_2}^{t} r^{-1/\alpha}(u)\,\mathrm{d}u}\right)^\alpha + G(t). \tag{3.29}$$

Integrating (3.29) from t_4 ($t_4 > t_3$) to t, we have

$$\int_{t_4}^{t}\left[2^{1-\alpha}k\rho(v)Q(v)\left(\frac{\int_{t_3}^{\sigma(v)}\int_{t_2}^{s} r^{-1/\alpha}(u)\,\mathrm{d}u\,\mathrm{d}s}{\int_{t_2}^{v} r^{-1/\alpha}(u)\,\mathrm{d}u}\right)^\alpha - G(v)\right]\mathrm{d}v \leq \omega(t_4) + \frac{p_0^\alpha}{\tau_0}\nu(t_4), \tag{3.30}$$

which contradicts (3.1).

Assume now that case (II) holds. By virtue of Lemma 2.2, $\lim_{t\to\infty} x(t) = 0$. This completes the proof. □

Remark 3.1 With an appropriate choice of the functions ρ and δ, one can derive from Theorem 3.1 a number of oscillation criteria for equation (1.1). The details are left to the reader.

4 Examples and discussion

We give the following examples to illustrate applications of Theorem 3.1.

Example 4.1 For $t \geq 1$, consider a third-order neutral delay differential equation

$$\left(x(t) + 2x\left(\frac{t}{2}\right)\right)''' + \frac{\gamma}{t^3}x\left(\frac{t}{3}\right) = 0, \tag{4.1}$$

where $\gamma > 0$ is a constant. Let $\alpha = k = 1$, $r(t) = 1$, $p(t) = p_0 = 2$, $\tau(t) = t/2$, $\tau_0 = 1/2$, $q(t) = \gamma/t^3$, $f(u) = u$, $\sigma(t) = t/3$, $\sigma_0 = 3$, $\rho(t) = t$, and $\delta(t) = 0$. It is not difficult to verify that conditions (A_1)-(A_5) are satisfied and $G(t) = 9/(4t)$. Noticing that $R(t) = \max\{r(t), r(\tau(t))\} = 1$ and $Q(t) = \min\{q(t), q(\tau(t))\} = \min\{\gamma/t^3, 8\gamma/t^3\} = \gamma/t^3$, we have

$$\int_{t_0}^{\infty}\xi\left(\frac{1}{R(\xi)}\int_{\xi}^{\infty}Q(s)\,\mathrm{d}s\right)^{1/\alpha}\mathrm{d}\xi = \int_{1}^{\infty}\xi\int_{\xi}^{\infty}\frac{\gamma}{s^3}\,\mathrm{d}s\,\mathrm{d}\xi = \frac{\gamma}{2}\int_{1}^{\infty}\xi^{-1}\,\mathrm{d}\xi = \infty.$$

Denote the left hand side of (3.1) by $\psi(t_*)$. Then

$$\begin{aligned}
\psi(t_*) &= \int_{t_*}^{\infty}\left(\frac{\gamma}{t^2}\frac{\int_{t_2}^{t/3}(s-t_1)\,\mathrm{d}s}{\int_{t_1}^{t}\mathrm{d}u} - \frac{9}{4t}\right)\mathrm{d}t \\
&= \int_{t_*}^{\infty}\left[\frac{\gamma}{t^2(t-t_1)}\left(\frac{t^2}{18} - \frac{1}{2}t_2^2 - \frac{1}{3}t_1t + t_1t_2\right) - \frac{9}{4t}\right]\mathrm{d}t \\
&= \int_{t_*}^{\infty}\left(\frac{\gamma}{18(t-t_1)} - \frac{9}{4t} - \frac{\gamma t_1}{3t(t-t_1)} + \frac{\gamma(t_1t_2 - 0.5t_2^2)}{t^2(t-t_1)}\right)\mathrm{d}t = \infty,
\end{aligned}$$

provided that $\gamma > 40.5$. Hence, by Theorem 3.1, every solution x of (4.1) is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$ for any $\gamma > 40.5$.

On the other hand, $\bar{Q}(t) = \min\{q(\sigma^{-1}(t)), q(\sigma^{-1}(\tau(t)))\} = \min\{\gamma/(27t^3), 8\gamma/(27t^3)\} = \gamma/(27t^3)$, and so

$$\liminf_{t\to\infty} \int_{\tau^{-1}(\sigma(t))}^{t} \frac{s^2\bar{Q}(s)}{r(s)}\,\mathrm{d}s = \liminf_{t\to\infty} \int_{\frac{2}{3}t}^{t} \frac{\gamma}{27s}\,\mathrm{d}s > \frac{5}{1.5\mathrm{e}},$$

provided that $\gamma > 90/(\mathrm{e}\ln 1.5) \approx 81.7$. An application of Theorem 1.1 implies that every solution x of (4.1) is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$ for all $\gamma > 81.7$. Therefore, Theorem 3.1 improves Theorem 1.1. Observe that results reported in Baculíková and Džurina [1, 2], Candan [7], Džurina *et al.* [9], Jiang and Li [10], and Li *et al.* [14] cannot be applied to (4.1) due to the fact that $p(t) = 2 > 1$.

Example 4.2 For $t \geq 1$, consider a third-order neutral delay differential equation

$$\left((2+\sin t)\left[\left(x(t)+2x(t-k_1)\right)''\right]^3\right)' + \frac{\gamma}{t^7}x^3(t-k_2) = 0, \tag{4.2}$$

where γ, k_1, and k_2 are positive constants. Let $\alpha = 3$, $k = 1$, $r(t) = 2 + \sin t$, $p(t) = p_0 = 2$, $\tau(t) = t - k_1$, $\tau_0 = 1$, $q(t) = \gamma/t^7$, $f(u) = u^3$, $\sigma(t) = t - k_2$, $\rho(t) = t^3$, and $\delta(t) = 0$. By virtue of $1 \leq r(t) \leq 3$, $\int_1^\infty r^{-1/\alpha}(s)\,\mathrm{d}s = \infty$. It is not hard to see that assumptions (A_1)-(A_5) hold. Noticing that $1 \leq R(t) = \max\{r(t), r(\tau(t))\} = \max\{2+\sin t, 2+\sin(t-k_1)\} \leq 3$ and $Q(t) = \min\{q(t), q(\tau(t))\} = \min\{\gamma/t^7, \gamma/(t-k_1)^7\} = \gamma/t^7$, we obtain

$$\begin{aligned}\int_{t_0}^\infty \xi\left(\frac{1}{R(\xi)}\int_\xi^\infty Q(s)\,\mathrm{d}s\right)^{1/\alpha}\mathrm{d}\xi &= \int_1^\infty \xi\left(\frac{1}{R(\xi)}\right)^{1/3}\left(\int_\xi^\infty \frac{\gamma}{s^7}\,\mathrm{d}s\right)^{1/3}\mathrm{d}\xi \\ &\geq \left(\frac{\gamma}{18}\right)^{1/3}\int_1^\infty \xi^{-1}\,\mathrm{d}\xi \\ &= \infty.\end{aligned}$$

Moreover,

$$G(t) = \frac{t^3(2+\sin t)}{4^4}\left(\frac{3t^2}{t^3}\right)^4 + \frac{8t^3(2+\sin(t-k_1))}{4^4}\left(\frac{3t^2}{t^3}\right)^4 \leq \frac{3^7}{4^4 t}.$$

Denote the left hand side of (3.1) by $\psi(t_*)$. Then

$$\begin{aligned}\psi(t_*) &= \int_{t_*}^\infty\left[\frac{\gamma}{4t^4}\left(\frac{\int_{t_2}^{t-k_2}\int_{t_1}^{s}(2+\sin u)^{-1/3}\,\mathrm{d}u\mathrm{d}s}{\int_{t_1}^{t}(2+\sin u)^{-1/3}\,\mathrm{d}u}\right)^3 - G(t)\right]\mathrm{d}t \\ &\geq \int_{t_*}^\infty\left[\frac{\gamma}{4t^4}\left(\frac{\int_{t_2}^{t-k_2}(s-t_1)\,\mathrm{d}s}{3^{-1/3}(t-t_1)}\right)^3 - \frac{3^7}{4^4t}\right]\mathrm{d}t \\ &= \int_{t_*}^\infty\left[\frac{\gamma}{4t^4}\left(\frac{(t-k_2)^2/2 - t_1t + \beta}{3^{-1/3}(t-t_1)}\right)^3 - \frac{3^7}{4^4t}\right]\mathrm{d}t \\ &= \infty,\end{aligned}$$

provided that $\gamma > 729/8 = 91.125$, where $\beta = -t_2^2/2 + t_1k_2 + t_1t_2$. Therefore, by Theorem 3.1, every solution x of (4.2) is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$ for any $\gamma > 91.125$.

Theorem 1.1 and results of Baculíková and Džurina [4] do not allow us to arrive at this conclusion due to the fact that $\alpha \neq 1$. Observe that results reported in Baculíková and Džurina [1, 2], Candan [7], Džurina *et al.* [9], Jiang and Li [10], and Li *et al.* [14] cannot be applied to (4.2) since $p(t) = 2 > 1$.

Remark 4.1 Without requiring assumptions (1.2) and (1.4), Theorem 3.1 is presented by using the double generalized Riccati substitutions (3.8) and (3.17). We stress that the study of oscillatory properties of equation (1.1) in the case $p(t) > 1$ brings about additional difficulties. In particular, as in the papers by Baculíková and Džurina [4] and Li and Rogovchenko [12], we have to impose additional assumptions (A_5). One of the principal difficulties one encounters lies in the fact that $x(t) \geq (1 - p(t))z(t)$ is not a valid estimate if $p(t) > 1$ and x is an eventually positive solution of (1.1). The question regarding the analysis of oscillatory behavior of solutions to (1.1) with other methods that do not require these assumptions remains open at the moment.

Remark 4.2 Note that Theorem 3.1 guarantees that every solution x of (1.1) is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$ and, unfortunately, this result cannot distinguish solutions with different behaviors. Since the sign of the derivative z' is not fixed, it is not easy to establish sufficient conditions which ensure that all solutions of (1.1) are just oscillatory and do not satisfy $\lim_{t\to\infty} x(t) = 0$. Neither is it possible to use the technique exploited in this paper for proving that all solutions of (1.1) satisfy $\lim_{t\to\infty} x(t) = 0$. Hence, these two interesting problems remain for future research.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All three authors contributed equally to this work. They all read and approved the final version of the manuscript.

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Acknowledgements
The authors are grateful to the editors and three anonymous referees for a very thorough reading of the manuscript and for pointing out several inaccuracies. This research is supported by NNSF of P.R. China (Grant Nos. 61503171, 61403061, and 11447005), CPSF (Grant No. 2015M582091), NSF of Shandong Province (Grant No. ZR2012FL06), DSRF of Linyi University (Grant No. LYDX2015BS001), and the AMEP of Linyi University, P.R. China.

Received: 26 August 2015 Accepted: 16 June 2016 Published online: 30 June 2016

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