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Existence of solution for first-order coupled system with nonlinear coupled boundary conditions

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Abstract

In this article, the existence of solution for the first-order nonlinear coupled system of ordinary differential equations with nonlinear coupled boundary condition (CBC for short) is studied using a coupled lower and upper solution approach. Our method for a nonlinear coupled system with nonlinear CBC is new and it unifies the treatment of many different first-order problems. Examples are included to ensure the validity of the results.

Keywords: lower and upper solutions; coupled nonlinear system; coupled nonlinear boundary condition; Arzela-Ascoli theorem; Schauder theorem

1 Introduction

In this article, we consider the following nonlinear coupled system of ordinary differential equations (ODEs for short):

$$\begin{aligned} u'(t) &= f\big(t, v(t)\big), \quad t \in [0,1], \\ v'(t) &= g\big(t, u(t)\big), \quad t \in [0,1], \end{aligned} \tag{1}$$

subject to the nonlinear CBC

$$h\big(u(0), v(0), u(1), v(1)\big) = (0,0), \tag{2}$$

where the nonlinear functions $f, g : [0,1] \times R \to R$ and $h : R^4 \to R^2$ are continuous.

A significant motivation factor for the study of the above system has been the applications of the nonlinear differential equations to the areas of mechanics; population dynamics; optimal control; ecology; biotechnology; harvesting; and physics [1–3]. Moreover, while dealing with nonlinear ordinary differential systems (ODSs for short) mostly authors only focus attention on the differential systems with uncoupled boundary conditions [4–6]. But, on the other hand, very little research work is available where the differential systems are coupled not only in the differential systems but also through the boundary conditions [7, 8]. Our system (1)-(2) deals with the latter case.

The other productive aspect of the article is the generalization of the classical concepts that had been discussed in [9–11]. We mean to say if $h(x,y,z,w) = (x - z, y - w)$, then (2) implies the periodic boundary conditions (BCs for short). Also if $h(x,y,z,w) = (x+z, y+w)$, then (2) implies the anti-periodic BCs. Definitely, in order to obtain a solution satisfying some initial or BCs and lying between a subsolution and a supersolution, we need additional conditions. For example, in the periodic case it suffices that

$$\begin{aligned} &\alpha_1(0) \le \alpha_1(1), \qquad \alpha_2(0) \le \alpha_2(1), \\ &\beta_1(0) \ge \beta_1(1), \qquad \beta_2(0) \ge \beta_2(1), \end{aligned} \tag{3}$$

and in the anti-periodic case it suffices that

$$\begin{aligned} &\alpha_1(0) \le -\beta_1(1), \qquad \alpha_2(0) \le -\beta_2(1), \\ &\beta_1(0) \ge -\alpha_1(1), \qquad \beta_2(0) \ge -\alpha_2(1), \end{aligned} \tag{4}$$

so to generalize the classical results (3) and (4) the concept of coupled lower and upper solution is defined, which allows us to obtain a solution in the sector $[\alpha_1, \beta_1] \times [\alpha_2, \beta_2]$ or $[\beta_1, \alpha_1] \times [\beta_2, \alpha_2]$ and also the inequalities (22)-(23) imply (3) and (4).

Definition 1.1 We say that a function $(\alpha_1, \alpha_2) \in C^1[0,1] \times C^1[0,1]$ is a subsolution of (1) if

$$\begin{aligned} &\alpha_1'(t) \le f\bigl(t, \alpha_2(t)\bigr), \quad t \in [0,1], \\ &\alpha_2'(t) \le g\bigl(t, \alpha_1(t)\bigr), \quad t \in [0,1]. \end{aligned} \tag{5}$$

In the same way, a supersolution is a function $(\beta_1, \beta_2) \in C^1[0,1] \times C^1[0,1]$ that satisfies the reversed inequalities in (5). In what follows we shall assume that $(\alpha_1, \alpha_2) \preceq (\beta_1, \beta_2)$, if $\alpha_1(t) \le \beta_1(t)$ and $\alpha_2(t) \le \beta_2(t)$, for all $t \in [0,1]$ or $(\alpha_1, \alpha_2) \succeq (\beta_1, \beta_2)$, if $\alpha_1(t) \ge \beta_1(t)$ and $\alpha_2(t) \ge \beta_2(t)$, for all $t \in [0,1]$.

For $u, v \in C^1[0,1]$, we define the set

$$[u, v] = \bigl\{w \in C^1[0,1] : u(t) \le w(t) \le v(t), t \in [0,1]\bigr\}.$$

The following lemma is important for our work.

Lemma 1.2 *Let $L : C[0,1] \times C[0,1] \to C_0[0,1] \times C_0[0,1] \times R^2$ be defined by*

$$\begin{aligned} \bigl[L(u,v)\bigr](t) = \biggl(& u(t) - u(0) + \lambda \int_0^t u(s)\,\mathrm{d}s, v(t) - v(0) + \lambda \int_0^t v(s)\,\mathrm{d}s, \\ &\bigl(au(0) + bu(1), cv(0) + dv(1)\bigr)\biggr), \end{aligned} \tag{6}$$

where λ, a, b, c, and d are real constants such that

$$\bigl(a + be^{-\lambda}\bigr)\bigl(c + de^{-\lambda}\bigr) \neq 0$$

and

$$C_0[0,1] = \bigl\{w \in C^1[0,1] : w(0) = 0\bigr\}.$$

Then L^{-1} exists and is continuous and defined by

$$\begin{aligned}\left[L^{-1}(y,z,\gamma,\delta)\right] = \biggl(& e^{-\lambda t}A + y(t) - \lambda \int_0^t e^{\lambda(s-t)} y(s)\,\mathrm{d}s, \\ & e^{-\lambda t}B + z(t) - \lambda \int_0^t e^{\lambda(s-t)} z(s)\,\mathrm{d}s\biggr),\end{aligned} \tag{7}$$

with

$$\begin{aligned} A &= \frac{\gamma + \lambda b \int_0^1 e^{\lambda(s-1)} y(s)\,\mathrm{d}s - by(1)}{a + be^{-\lambda}}, \\ B &= \frac{\delta + \lambda d \int_0^1 e^{\lambda(s-1)} z(s)\,\mathrm{d}s - dz(1)}{c + de^{-\lambda}}. \end{aligned}$$

Proof Choose

$$y(t) = u(t) - u(0) + \lambda \int_0^t u(s)\,\mathrm{d}s, \tag{8}$$

$$z(t) = v(t) - v(0) + \lambda \int_0^t v(s)\,\mathrm{d}s, \tag{9}$$

$$\gamma = au(0) + bu(1) \tag{10}$$

and

$$\delta = cv(0) + dv(1). \tag{11}$$

In the light of (8)-(11), (6) can also be written as

$$\left[L(u,v)\right](t) = \bigl(y(t), z(t), (\gamma,\delta)\bigr). \tag{12}$$

Differentiating (8) w.r.t. t, we have

$$y'(t) = u'(t) + \lambda u(t). \tag{13}$$

Multiplying (13) with integrating factor $e^{\lambda t}$, we have

$$e^{\lambda t} y'(t) = \bigl(u(t)e^{\lambda t}\bigr)', \tag{14}$$

then after integrating and taking the limits of integration from 0 to t, (14) becomes

$$u(t) = u(0)e^{-\lambda t} + y(t) - \lambda \int_0^t e^{\lambda(s-t)} y(s)\,\mathrm{d}s, \tag{15}$$

$u(0)$ can easily be determined with the help of (10) as

$$\gamma = \bigl(a + be^{-\lambda}\bigr)u(0) + by(1) - b\lambda \int_0^1 e^{\lambda(s-1)} y(s)\,\mathrm{d}s,$$

then

$$u(0) = \frac{\gamma + b\lambda \int_0^1 e^{\lambda(s-1)} y(s)\,\mathrm{d}s - by(1)}{a + be^{-\lambda}}, \quad a + be^{-\lambda} \neq 0, \tag{16}$$

for simplicity of notation, let

$$A = \frac{\gamma + b\lambda \int_0^1 e^{\lambda(s-1)} y(s)\,\mathrm{d}s - by(1)}{a + be^{-\lambda}}, \quad a + be^{-\lambda} \neq 0. \tag{17}$$

Using (17) in (15), we have

$$u(t) = Ae^{-\lambda t} + y(t) - \lambda \int_0^t e^{\lambda(s-t)} y(s)\,\mathrm{d}s. \tag{18}$$

Similarly along the same lines, it can easily be shown that

$$v(t) = Be^{-\lambda t} + z(t) - \lambda \int_0^t e^{\lambda(s-t)} z(s)\,\mathrm{d}s, \tag{19}$$

with

$$B = \frac{\delta + d\lambda \int_0^1 e^{\lambda(s-1)} z(s)\,\mathrm{d}s - dz(1)}{c + de^{-\lambda}}, \quad c + de^{-\lambda} \neq 0; \tag{20}$$

(12) can also be written as

$$\bigl[L^{-1}\bigl(y(t), z(t), (\gamma, \delta)\bigr)\bigr] = \bigl(u(t), v(t)\bigr). \tag{21}$$

Hence, (17)-(20) prove the result. □

2 Coupled lower and upper solutions

The following definition is very helpful to construct the statement of the main result (2.2), and also it covers different possibilities for the nonlinear function h.

Definition 2.1 We say that $(\alpha_1, \alpha_2), (\beta_1, \beta_2) \in C^1[0,1] \times C^1[0,1]$ are coupled lower and upper solutions for the problem (1) and (2) if (α_1, α_2) is a subsolution and (β_1, β_2) a supersolution for the system (1) such that

$$h\bigl(\alpha_1(0), \alpha_2(0), \alpha_1(1), \alpha_2(1)\bigr) \preceq (0,0) \preceq h\bigl(\beta_1(0), \beta_2(0), \beta_1(1), \beta_2(1)\bigr) \tag{22}$$

and

$$h\bigl(\alpha_1(0), \alpha_2(0), \beta_1(1), \beta_2(1)\bigr) \preceq (0,0) \preceq h\bigl(\beta_1(0), \beta_2(0), \alpha_1(1), \alpha_2(1)\bigr). \tag{23}$$

Theorem 2.2 *Assume that* (α_1, α_2), (β_1, β_2) *are coupled lower and upper solutions for the system* (1)-(2). *In addition, suppose that the functions*

$$h_{(\alpha_1,\alpha_2)}(x, y) := h\bigl(\alpha_1(0), \alpha_2(0), x, y\bigr),$$
$$h_{(\beta_1,\beta_2)}(x, y) := h\bigl(\beta_1(0), \beta_2(0), x, y\bigr),$$

are monotone on $[\alpha_1(1),\beta_1(1)]\times[\alpha_2(1),\beta_2(1)]$, *then the system* (1)-(2) *has at least one solution* $(u,v)\in[\alpha_1,\beta_1]\times[\alpha_2,\beta_2]$.

Proof Let $\lambda>0$ and consider the modified system

$$\begin{aligned}
&u'(t)+\lambda u(t)=F^*\bigl(t,u(t),v(t)\bigr),\quad t\in[0,1],\\
&v'(t)+\lambda v(t)=G^*\bigl(t,u(t),v(t)\bigr),\quad t\in[0,1],\\
&h^*\bigl(u(0),v(0),u(1),v(1)\bigr)=\bigl(u(0),v(0)\bigr),
\end{aligned}\tag{24}$$

with

$$F^*\bigl(t,u(t),v(t)\bigr)=\begin{cases}
f(t,\beta_2(t))+\lambda\beta_1(t) & \text{if } v(t)>\beta_2(t),u(t)>\beta_1(t),\\
f(t,v(t))+\lambda\beta_1(t) & \text{if } \alpha_2(t)\le v(t)\le\beta_2(t),u(t)>\beta_1(t),\\
f(t,\alpha_2(t))+\lambda\beta_1(t) & \text{if } v(t)<\alpha_2(t),u(t)>\beta_1(t),\\
f(t,\beta_2(t))+\lambda u(t) & \text{if } v(t)>\beta_2(t),\alpha_1(t)\le u(t)\le\beta_1(t),\\
f(t,v(t))+\lambda u(t) & \text{if } \alpha_2(t)\le v(t)\le\beta_2(t),\\
 & \quad \alpha_1(t)\le u(t)\le\beta_1(t),\\
f(t,\alpha_2(t))+\lambda u(t) & \text{if } v(t)<\alpha_2(t),\alpha_1(t)\le u(t)\le\beta_1(t),\\
f(t,\beta_2(t))+\lambda\alpha_1(t) & \text{if } v(t)>\beta_2(t),u(t)<\alpha_1(t),\\
f(t,v(t))+\lambda\alpha_1(t) & \text{if } \alpha_2(t)\le v(t)\le\beta_2(t),u(t)<\alpha_1(t),\\
f(t,\alpha_2(t))+\lambda\alpha_1(t) & \text{if } v(t)<\alpha_2(t),u(t)<\alpha_1(t)
\end{cases}$$

and

$$G^*\bigl(t,u(t),v(t)\bigr)=\begin{cases}
g(t,\beta_1(t))+\lambda\beta_2(t) & \text{if } v(t)>\beta_2(t),u(t)>\beta_1(t),\\
g(t,u(t))+\lambda\beta_2(t) & \text{if } \alpha_1(t)\le u(t)\le\beta_1(t),v(t)>\beta_2(t),\\
g(t,\alpha_1(t))+\lambda\beta_2(t) & \text{if } u(t)<\alpha_1(t),v(t)>\beta_2(t),\\
g(t,\beta_1(t))+\lambda v(t) & \text{if } u(t)>\beta_1(t),\alpha_2(t)\le v(t)\le\beta_2(t),\\
g(t,u(t))+\lambda v(t) & \text{if } \alpha_1(t)\le u(t)\le\beta_1(t),\\
 & \quad \alpha_2(t)\le v(t)\le\beta_2(t),\\
g(t,\alpha_1(t))+\lambda v(t) & \text{if } u(t)<\alpha_1(t),\alpha_2(t)\le v(t)\le\beta_2(t),\\
g(t,\beta_1(t))+\lambda\alpha_2(t) & \text{if } u(t)>\beta_1(t),v(t)<\alpha_2(t),\\
g(t,u(t))+\lambda\alpha_2(t) & \text{if } \alpha_1(t)\le u(t)\le\beta_1(t),v(t)<\alpha_2(t),\\
g(t,\alpha_1(t))+\lambda\alpha_2(t) & \text{if } u(t)<\alpha_1(t),v(t)<\alpha_2(t),
\end{cases}$$

$$h^*(x,y,z,w)=p\bigl(0,(x,y)\bigr)-h(x,y,z,w)$$

and

$$p\bigl(t,(x,y)\bigr)=\begin{cases}
(\beta_1(t),\beta_2(t)) & \text{if } (x,y)\not\preceq(\beta_1(t),\beta_2(t)),\\
(x,y) & \text{if } (\alpha_1(t),\alpha_2(t))\preceq(x,y)\preceq(\beta_1(t),\beta_2(t)),\\
(\alpha_1(t),\alpha_2(t)) & \text{if } (x,y)\not\succeq(\alpha_1(t),\alpha_2(t)).
\end{cases}$$

Note that if $(u,v) \in [\alpha_1,\beta_1] \times [\alpha_2,\beta_2]$ is a solution of the system (24), then (u,v) is a solution of the system (1)-(2).

For the sake of simplicity we divide the proof into three steps.

Step 1: We define the mappings

$$L,N : C[0,1] \times C[0,1] \to C_0[0,1] \times C_0[0,1] \times \mathbb{R}^2,$$

by

$$\begin{aligned}\big[L(u,v)\big](t) = \bigg(&u(t) - u(0) + \lambda \int_0^t u(s)\,\mathrm{d}s, v(t) - v(0) + \lambda \int_0^t v(s)\,\mathrm{d}s, \\ &\big(u(0), v(0)\big)\bigg)\end{aligned}$$

and

$$\begin{aligned}\big[N(u,v)\big](t) = \bigg(&\int_0^t F^*\big(s,u(s),v(s)\big)\,\mathrm{d}s, \int_0^t G^*\big(s,u(s),v(s)\big)\,\mathrm{d}s, \\ &h^*\big(u(0),v(0),u(1),v(1)\big)\bigg).\end{aligned}$$

Clearly, N is continuous and compact by the direct application of the Arzela-Ascoli theorem. Also from Lemma 1.2 with $a = 1$, $b = 0$, $c = 1$, and $d = 0$, L^{-1} exists and is continuous.

On the other hand, solving (24) is equivalent to finding a fixed point of

$$L^{-1}N : C[0,1] \times C[0,1] \to C[0,1] \times C[0,1].$$

Now, the Schauder fixed point theorem guarantees the existence of at least a fixed point since $L^{-1}N$ is continuous and compact.

Step 2: It remains to show that $(u,v) \in [\alpha_1,\beta_1] \times [\alpha_2,\beta_2]$.

We claim that $(u,v) \preceq (\beta_1,\beta_2)$. If $(u,v) \npreceq (\beta_1,\beta_2)$, then $u \npreceq \beta_1$ and/or $v \npreceq \beta_2$. If $u \npreceq \beta_1$, then there exist some $r_0 \in [0,1]$, such that $u - \beta_1$ attains a positive maximum at $r_0 \in [0,1]$. We shall consider three cases.

Case 1. $r_0 \in (0,1]$. Then there exists $\xi \in (0,r_0)$, such that $0 < u(t) - \beta_1(t) < u(r_0) - \beta_1(r_0)$, for all $t \in [\xi, r_0)$. This yields a contradiction, since

$$\begin{aligned}\beta_1(r_0) - \beta_1(\xi) &< u(r_0) - u(\xi) \\ &= \int_\xi^{r_0} \Big(f\big(s,\beta_2(s)\big) - \lambda\big(u(s) - \beta_1(s)\big)\Big)\,\mathrm{d}s \\ &< \int_\xi^{r_0} f\big(s,\beta_2(s)\big)\,\mathrm{d}s = \int_\xi^{r_0} \beta_1'(s)\,\mathrm{d}s = \beta_1(r_0) - \beta_1(\xi).\end{aligned}$$

Case 2. $r_0 = 0$ and h_β is monotone nonincreasing. Then $u(0) - \beta_1(0) > 0$ or $v(0) - \beta_2(0) > 0$, and in view of (22), we have

$$\begin{aligned}\big(u(0),v(0)\big) &= h^*\big(u(0),v(0),u(1),v(1)\big) \\ &= \big(\beta_1(0),\beta_2(0)\big) - h\big(\beta_1(0),\beta_2(0),u(1),v(1)\big)\end{aligned}$$

$$
\begin{aligned}
&\preceq \bigl(\beta_1(0),\beta_2(0)\bigr) - h\bigl(\beta_1(0),\beta_2(0),\beta_1(1),\beta_2(1)\bigr) \\
&\preceq \bigl(\beta_1(0),\beta_2(0)\bigr),
\end{aligned} \tag{25}
$$

a contradiction.

Case 3. Similarly h_β is monotone nondecreasing. We shall change the inequality (25) by $(u(0),v(0)) \preceq (\beta_1(0),\beta_2(0)) - h(\beta_1(0),\beta_2(0),\alpha_1(1),\alpha_2(1))$ and again we get a contradiction. Consequently, $(u,v) \preceq (\beta_1,\beta_2)$, for all $t \in [0,1]$. Similarly, we can show that $(u,v) \succeq (\alpha_1,\alpha_2)$, for all $t \in [0,1]$.

Step 3: Now, it remains to show that (u,v) satisfies the boundary condition (2).

For this, we claim that

$$
\bigl(\alpha_1(0),\alpha_2(0)\bigr) \preceq \bigl(u(0),v(0)\bigr) - h\bigl(u(0),v(0),u(1),v(1)\bigr) \preceq \bigl(\beta_1(0),\beta_2(0)\bigr). \tag{26}
$$

If $(u(0),v(0)) - h(u(0),v(0),u(1),v(1)) \npreceq (\beta_1(0),\beta_2(0))$, then

$$
\begin{aligned}
\bigl(u(0),v(0)\bigr) &= h^*\bigl(u(0),v(0),u(1),v(1)\bigr) \\
&= p\bigl(0,\bigl(u(0),v(0)\bigr)\bigr) - h\bigl(u(0),v(0),u(1),v(1)\bigr) \\
&= \bigl(\beta_1(0),\beta_2(0)\bigr).
\end{aligned}
$$

If $h_\beta(x,y)$ is monotone nonincreasing, then we have

$$
\begin{aligned}
&\bigl(u(0),v(0)\bigr) - h\bigl(u(0),v(0),u(1),v(1)\bigr) \\
&\quad = \bigl(\beta_1(0),\beta_2(0)\bigr) - h\bigl(\beta_1(0),\beta_2(0),u(1),v(1)\bigr) \\
&\quad = \bigl(\beta_1(0),\beta_2(0)\bigr) - h_\beta\bigl(u(1),v(1)\bigr) \\
&\quad \preceq \bigl(\beta_1(0),\beta_2(0)\bigr) - h_\beta\bigl(\beta_1(1),\beta_2(1)\bigr) \\
&\quad = \bigl(\beta_1(0),\beta_2(0)\bigr) - h\bigl(\beta_1(0),\beta_2(0),\beta_1(1),\beta_2(1)\bigr) \\
&\quad \preceq \bigl(\beta_1(0),\beta_2(0)\bigr),
\end{aligned} \tag{27}
$$

a contradiction. Similarly if $h_\beta(x,y)$ is monotone nondecreasing, then we get the same contradiction. Consequently, (26) holds. Hence the system of BVPs (1)-(2) has a solution $(u,v) \in [\alpha_1,\beta_1] \times [\alpha_2,\beta_2]$. □

Remark 2.3 If $(\alpha_1,\alpha_2) \succeq (\beta_1,\beta_2)$, then (23) is replaced by

$$
h\bigl(\beta_1(0),\beta_2(0),\alpha_1(1),\alpha_2(1)\bigr) \preceq (0,0) \preceq h\bigl(\alpha_1(0),\alpha_2(0),\beta_1(1),\beta_2(1)\bigr). \tag{28}
$$

Theorem 2.4 *Assume that* (α_1,α_2), (β_1,β_2) *are coupled lower and upper solutions in reverse order for the system* (1)-(2). *In addition, suppose that the functions*

$$
\begin{aligned}
&h_{(\alpha_1,\alpha_2)}(x,y) := h\bigl(x,y,\alpha_1(1),\alpha_2(1)\bigr), \\
&h_{(\beta_1,\beta_2)}(x,y) := h\bigl(x,y,\beta_1(1),\beta_2(1)\bigr),
\end{aligned}
$$

are monotone in $[\beta_1(0),\alpha_1(0)] \times [\beta_2(0),\alpha_2(0)]$, *then the system* (1)-(2) *has at least one solution* $(u,v) \in [\beta_1,\alpha_1] \times [\beta_2,\alpha_2]$.

Proof The proof of Theorem 2.4 is analogous to the proof of Theorem 2.2. □

3 Examples

Example 3.1 Let

$$\begin{aligned} &f\bigl(t,v(t)\bigr) = -2v(t) + \gamma \sin(\omega t), \quad t \in [0,1], \\ &g\bigl(t,u(t)\bigr) = -2u^3(t) + \gamma \cos(\omega t), \quad t \in [0,1], \\ &h(x,y,z,w) = \bigl(x^3 - y^3, z^3 - w^3\bigr). \end{aligned}$$

Let $\alpha_1(t) = -2\gamma$, $\alpha_2(t) = -\gamma$, and $\beta_1(t) = 2\gamma$, $\beta_2(t) = \gamma$. It is easy to show that (α_1,α_2), (β_1,β_2) are a subsolution and a supersolution of the system (1), respectively. Further, (α_1,α_2), (β_1,β_2) satisfy (22)-(23). Hence by Theorem 2.2, the system of BVPs (1)-(2) has at least one solution $(u,v) \in [\alpha_1,\beta_1] \times [\alpha_2,\beta_2]$.

Example 3.2 Let

$$\begin{aligned} &f\bigl(t,v(t)\bigr) = 4v(t) + \gamma \sin(\omega t), \quad t \in [0,1], \\ &g\bigl(t,u(t)\bigr) = 4u^5(t) + \gamma \cos(\omega t), \quad t \in [0,1], \\ &h(x,y,z,w) = (yw - xz, y + xz). \end{aligned}$$

Choose $\alpha_1(t) = 3\gamma$, $\alpha_2(t) = 2\gamma$, $\beta_1(t) = -3\gamma$, $\beta_2(t) = -2\gamma$. We can show that (α_1,α_2), (β_1,β_2) are a subsolution and a supersolution of the system (1), respectively. Further, (α_1,α_2), (β_1,β_2) satisfy (22) and (28). Hence by Theorem 2.4, the system of BVPs (1)-(2) has at least one solution $(u,v) \in [\beta_1,\alpha_1] \times [\beta_2,\alpha_2]$.

4 Conclusion

The new existence results are established for a nonlinear ordinary coupled system with nonlinear CBCs. The developed result unifies the treatment of many first-order problems [12–15]. Examples are included to verify the theoretical results. The existence results are also discussed when the lower and upper solutions are in reverse order $(\alpha_1,\alpha_2) \succeq (\beta_1,\beta_2)$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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