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On quasi-periodic solutions to a higher-order Emden-Fowler type differential equation

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In honor of Professor IT Kiguradze

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Abstract
The paper is devoted to the existence of oscillatory and non-oscillatory quasi-periodic, in some sense, solutions to a higher-order Emden-Fowler type differential equation.

Keywords: Emden-Fowler type equation; quasi-periodic solutions; oscillatory and non-oscillatory solutions

1 Introduction

The paper is devoted to the existence of oscillatory and non-oscillatory quasi-periodic, in some sense, solutions to the higher-order Emden-Fowler type differential equation

$$y^{(n)} + p_0|y|^k \operatorname{sgn} y = 0, \quad n > 2, k \in \mathbb{R}, k > 1, p_0 \neq 0. \tag{1}$$

The fact of the existence of such solutions answers the two questions posed by IT Kiguradze:

Question 1 Can we describe more precisely qualitative properties of oscillatory solutions to (1)?

Question 2 Do all blow-up solutions to this equation (and similarly all Kneser solutions) have the power asymptotic behavior?

A lot of results on the asymptotic behavior of solutions to (1) are described in detail in [1]. In particular (see Ch. IV, §15), the existence of oscillatory solutions to a generalization of this equation was proved (see also [2] Ch. I, §6.1). In [3] a result was formulated on non-extensibility of oscillatory solutions to (1) with odd n and $p_0 > 0$. In the cases $n = 3$ and $n = 4$ the asymptotic behavior of all oscillatory solutions is described in [4–6]. Some results on the existence of blow-up solutions are in [1] (Ch. IV, §16), [2] (Ch. I, §5), [7, 8]. Some results on the existence of some special solutions to this equation are in [2, 4, 5, 7, 9–13].

2 On existence of quasi-periodic oscillatory solutions

In this section some results will be obtained on the existence of special oscillatory solutions. The main results of this section were formulated in [14].

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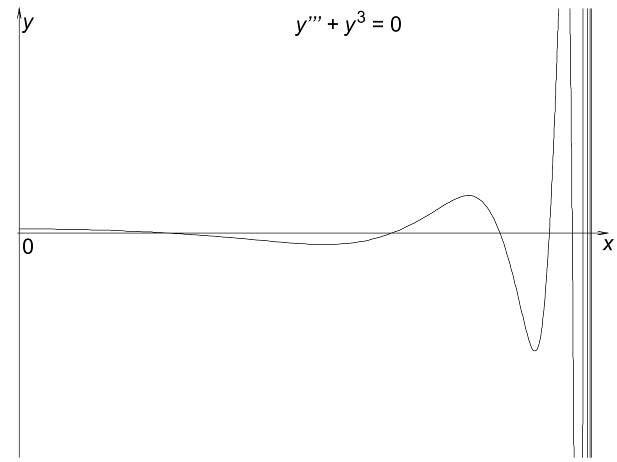

Figure 1 A quasi-periodic solution for the equation $y''' + y^3 = 0$.

Theorem 1 *For any integer $n > 2$ and real $k > 1$ there exists a periodic oscillatory function h such that for any $p_0 > 0$ and $x^* \in \mathbb{R}$ the function*

$$y(x) = p_0^{\frac{1}{k-1}} \left(x^* - x\right)^{-\alpha} h\left(\log\left(x^* - x\right)\right), \quad -\infty < x < x^*, \tag{2}$$

with $\alpha = \frac{n}{k-1}$ is a solution to (1). (*See Figure* 1.)

Proof For any $q = (q_0, \dots, q_{n-1}) \in \mathbb{R}^n$ let $y_q(x)$ be the maximally extended solution to the equation

$$y^{(n)}(x) + \left|y(x)\right|^k = 0 \tag{3}$$

satisfying the initial conditions $y^{(j)}(0) = q_j$ with $j = 0, \dots, n-1$.

For $0 \le j < n$ put $B_j = \frac{nk}{n+j(k-1)} > 1$ and $\beta_j = \frac{1}{B_j}$.

Consider the function $N : \mathbb{R}^n \to \mathbb{R}$ defined by the formula

$$N(q_0, \dots, q_{n-1}) = \sum_{j=0}^{n-1} |q_j|^{B_j} \tag{4}$$

and the mapping $\tilde{N} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ defined by the formula

$$\tilde{N}(q)_j = N(q)^{-\beta_j} q_j, \quad j = 0, \dots, n-1,$$

and satisfying the equality $N(\tilde{N}(q)) = 1$ for all $q \in \mathbb{R}^n \setminus \{0\}$.

Next, consider the subset $Q \subset \mathbb{R}^n$ consisting of all $q = (q_0, \dots, q_{n-1}) \in \mathbb{R}^n$ satisfying the following conditions:

(1) $q_0 = 0$,
(2) $q_j \ge 0$ for all $j \in \{1, \dots, n-1\}$,
(3) $N(q) = 1$.

The restriction of the projection $(q_0,\dots,q_{n-1}) \mapsto (q_1,\dots,q_{n-2})$ to the set Q is a homeomorphism of Q onto the convex compact subset

$$\left\{(q_1,\dots,q_{n-2}) : \sum_{j=1}^{n-2} |q_j|^{B_j} \le 1 \text{ and } q_j \ge 0, j = 1,\dots,n-2\right\} \subset \mathbb{R}^{n-2}.$$

Lemma 1 *For any $q \in Q$ there exists $a_q > 0$ such that $y_q(a_q) = 0$ and $y_q^{(j)}(a_q) < 0$ for all $j \in \{1,\dots,n-1\}$.*

Proof Put $J = \max\{j \in \mathbb{Z} : 0 \le j < n, q_j > 0\}$. This J exists and is positive due to the definition of Q. On some interval $(0;\varepsilon)$ all derivatives $y_q^{(j)}(x)$ with $0 \le j \le J$ are positive. Those with $J < j \le n$, due to (3), are negative on the same interval.

While keeping this sign combination, the function y_q and its derivatives are bounded, which provides extensibility of $y_q(x)$ as the solution to (3) outside the interval $(0;\varepsilon)$.

On the other hand, this sign combination cannot take place up to $+\infty$. Indeed, in that case $y_q(x)$ would increase providing $y_q^{(n)}(x) < -y_q(\varepsilon)^k < 0$ for all $x > \varepsilon$, which is impossible for any positive function on the unbounded interval $(0;+\infty)$.

So, $y_q(x)$ must change the sign combination of its derivatives. The only possible combination to be the next one corresponds to the positive derivatives $y_q^{(j)}(x)$ with $0 \le j \le J-1$ and the negative ones with $J \le j \le n$.

The same arguments show that the new sign combination must also change and finally, after J changes, we arrive at the case with $y_q(x) > 0$ and $y_q^{(j)}(x) < 0$ with $1 \le j \le n$. Now, contrary to the previous cases, the function $y_q(x)$ does not increase, but its first derivative is negative and decreases (recall that $n > 2$). Hence this sign combination also cannot take place on an unbounded interval and therefore it must change to the case with all negative $y_q^{(j)}(x)$, $0 \le j \le n$. By the way, the function $y_q(x)$ must vanish at some point $a_q > 0$, which completes the proof of Lemma 1. □

Note that a_q is not only the first positive zero of $y_q(x)$, but the only positive one. Indeed, all $y_q^{(j)}(x)$ with $0 < j < n$ are negative at a_q, whence, according to (3), all $y_q^{(j)}(x)$ with $0 \le j < n$ decrease and are negative for all $x > a_q$ in the domain of $y_q(x)$.

To continue the proof of Theorem 1, consider the function $\xi : q \mapsto a_q$ taking each $q \in Q$ to the first positive zero of the function y_q. To prove its continuity, we apply the implicit function theorem. The function $\xi(q)$ can be considered as a local solution $X(q)$ to the equation $S_0(q,X) = 0$, where

$$S : (q,x) \mapsto \big(S_0(q,x), S_1(q,x),\dots,S_{n-1}(q,x)\big) = \big(y_q(x), y_q'(x),\dots,y_q^{(n-1)}(x)\big)$$

is the C^1 'solution' mapping defined on a domain including $\mathbb{R}^n \times \{0\}$. The necessary for the implicit function theorem condition $\frac{\partial S_0}{\partial X}(q_0,\dots,q_{n-1},a_q) \ne 0$ is satisfied since the left-hand side of the last inequality is equal to $y_q'(a_q) < 0$. Besides, any function $X(q)$ implicitly defined near a point (q_0, a_{q_0}) must be positive in some its neighborhood. Hence locally $X(q)$ must be equal to $\xi(q)$, but neither to a non-positive zero of $y_q(x)$ nor to a non-first positive one, which does not exist. Hence the function $\xi(q)$ is continuous as well as $X(q)$.

Now we can consider the mapping $\tilde{S} : q \mapsto \tilde{N}(-S(q,\xi(q)))$, which maps Q into itself. Since $\tilde{S}$ is continuous and Q is homeomorphic to a convex compact subset of $\mathbb{R}^{n-2}$, the Brouwer fixed-point theorem can be applied. Thus, there exists $\hat{q} \in Q$ such that $\tilde{S}(\hat{q}) = \hat{q}$.

According to the definitions of the functions $\tilde{N}$, S, and ξ, this yields the result that there exists a non-negative solution $\hat{y}(x) = y_{\hat{q}}(x)$ to (3) defined on a segment $[0;a_1]$ with $a_1 = a_{\hat{q}}$, positive on the open interval $(0;a_1)$, and such that

$$\lambda^{-\beta_j}\hat{y}^{(j)}(a_1) = -\hat{y}^{(j)}(0), \quad j = 0,\dots,n-1, \tag{5}$$

with

$$\lambda = N\big(S\big(\hat{q},\xi(\hat{q})\big)\big) = \sum_{j=0}^{n-1}\big|\hat{y}^{(j)}(a_1)\big|^{B_j} > 0. \tag{6}$$

Since $\hat{y}(x)$ is non-negative, it is also a solution to the equation

$$y^{(n)}(x) + \big|y(x)\big|^k \operatorname{sgn} y(x) = 0. \tag{7}$$

Note that for any solution $y_1(x)$ to (7) the function $y_2(x) = -b^\alpha y_1(bx+c)$ with arbitrary constants $b > 0$ and c is also a solution to (7). Indeed, we have $\alpha + n = k\alpha$ and $y_2^{(j)}(x) = -b^{\alpha+j}y_1^{(j)}(bx+c)$ for all $j = 0,\dots,n$, whence

$$\begin{aligned} &y_2^{(n)}(x) + \big|y_2(x)\big|^k \operatorname{sgn} y_2(x) \\ &\quad = -b^{\alpha+n}y_1^{(n)}(bx+c) - b^{k\alpha}\big|y_1(bx+c)\big|^k \operatorname{sgn} y_1(bx+c) \\ &\quad = -b^{k\alpha}\big(y_1^{(n)}(bx+c) + \big|y_1(bx+c)\big|^k \operatorname{sgn} y_1(bx+c)\big) = 0. \end{aligned}$$

So, the function $z(x) = -b^\alpha\hat{y}(bx - a_1 b)$ is a solution to (7) and is defined on the segment $[a_1;a_2]$ with $a_2 = a_1 + \frac{a_1}{b}$.

Put $b = \lambda^{\frac{k-1}{nk}}$ with λ defined by (6). Then

$$b^{\alpha+j} = \lambda^{\frac{k-1}{nk}\cdot(\frac{n}{k-1}+j)} = \lambda^{\frac{n+(k-1)j}{nk}} = \lambda^{\beta_j},$$

whence, taking into account (5), we obtain $z^{(j)}(a_1) = -\lambda^{\beta_j}\hat{y}^{(j)}(0) = \hat{y}^{(j)}(a_1)$. Thus, $z(x)$ can be used to extend the solution $\hat{y}(x)$ on $[0;a_2]$. Since $z(x)$ satisfies the conditions similar to (5), namely,

$$\lambda^{-\beta_j}z^{(j)}(a_2) = -\lambda^{-\beta_j}b^{\alpha+j}\hat{y}^{(j)}(a_1) = -z^{(j)}(a_1),$$

the procedure of extension can be repeated on $[0;a_3]$, $[0;a_4]$, and so on with $a_{s+1} = a_s + \frac{a_s - a_{s-1}}{b}$. In the same way the solution $\hat{y}(x)$ can be extended to the left. Its restrictions to the neighboring segments satisfy the following equality:

$$\hat{y}(x) = -b^\alpha\hat{y}\big(b(x-a_s) + a_{s-1}\big), \tag{8}$$

where $x \in [a_s;a_{s+1}]$ and hence $b(x-a_s) + a_{s-1} \in [a_{s-1};a_s]$.

Now we will investigate whether b is greater or less than 1.

Let $a_{j,s}$ be the zero of the derivative $\hat{y}^{(j)}(x)$ belonging to the interval $(a_{s-1};a_s)$. Note that according to the above consideration on changing the sign combinations, we have

$$a_{j+1,s} < a_{j,s} < \cdots < a_{0,s} = a_s < a_{n-1,s+1} < a_{n-2,s+1} < \cdots.$$

Lemma 2 *In the above notation the solution $y(x) = \hat{y}(x)$ satisfies the following inequalities*:

$$\left|y(a_{1,s})\right| < \left|y(a_{n-1,s+1})\right|, \tag{9}$$

$$\left|y(a_{j+1,s})\right| < \left|y(a_{j,s})\right|, \quad 0 < j < n-1. \tag{10}$$

Proof Indeed,

$$\begin{aligned}
&\frac{1}{k+1}\left(\left|y(a_{n-1,s+1})\right|^{k+1} - \left|y(a_{1,s})\right|^{k+1}\right) \\
&\quad = \int_{a_{1,s}}^{a_{n-1,s+1}} y'(x)\left|y(x)\right|^k \operatorname{sgn} y(x)\,dx = -\int_{a_{1,s}}^{a_{n-1,s+1}} y'(x)y^{(n)}(x)\,dx \\
&\quad = -y'(x)y^{(n-1)}(x)\big|_{a_{1,s}}^{a_{n-1,s+1}} + \int_{a_{1,s}}^{a_{n-1,s+1}} y''(x)y^{(n-1)}(x)\,dx > 0
\end{aligned}$$

since $y'(a_{1,s}) = y^{(n-1)}(a_{n-1,s+1}) = 0$ and $y''(x)y^{(n-1)}(x) > 0$ on the interval $(a_{1,s}; a_{n-1,s+1})$, where only $y(x)$ itself changes its sign, while all other $y^{(j)}(x)$ with $0 < j < n$ keep the same one. Recall that $n > 2$, which makes $y''(x)$ to be one of these others. Inequality (9) is proved.

Inequality (10) follows from $y(x)y'(x) > 0$ on the interval $(a_{j+1,s}, a_{j,s})$, where the derivatives $y^{(j)}(x)$ and $y^{(j+1)}(x)$ with $0 < j < n-1$ keep different signs, while all lower-order derivatives keep the same sign as $y^{(j)}(x)$. □

From the lemma proved it follows that $|\hat{y}(a_{1,s})| < |\hat{y}(a_{1,s+1})| = b^\alpha|\hat{y}(a_{1,s})|$, whence it follows that $b > 1$ and $a_s - a_{s-1} = b(a_{s+1} - a_s) > a_{s+1} - a_s$.

Now we see that

$$\sum_{s=-\infty}^{0}(a_{s+1} - a_s) = a_1\sum_{s=0}^{\infty} b^s = \infty \quad \text{and} \quad \sum_{s=0}^{\infty}(a_{s+1} - a_s) = a_1\sum_{s=0}^{\infty} b^{-s} = a^* < \infty.$$

So, the solution $\hat{y}(x)$ is extended on the half-bounded interval $(-\infty; a^*)$ and cannot be extended outside it since

$$\limsup_{x\to a^*}\left|\hat{y}(x)\right| = \lim_{s\to+\infty}\left|\hat{y}(a_{1,s})\right| = \left|\hat{y}(a_{1,0})\right| \lim_{s\to+\infty} b^{s\alpha} = +\infty.$$

Now consider the function

$$h(t) = e^{t\alpha}\hat{y}\left(a^* - e^t\right), \tag{11}$$

which is periodic. Indeed, if $a_* - e^t \in [a_s; a_{s+1}]$ for some $s \in \mathbb{Z}$, then

$$h(t + \log b) = e^{t\alpha}b^\alpha\hat{y}\left(a^* - be^t\right)$$

and, according to (8),

$$h(t) = e^{\alpha t}\hat{y}\left(a^* - e^t\right) = -e^{\alpha t}b^\alpha\,\hat{y}\left(ba^* - be^t - ba_s + a_{s-1}\right).$$

The expression in the last parentheses is equal to

$$b\left(a^* - a_s\right) - be^t + a_{s-1} = b\cdot\frac{a_{s+1} - a_s}{1 - b^{-1}} - be^t + a_{s-1} = \frac{a_s - a_{s-1}}{1 - b^{-1}} + a_{s-1} - be^t = a^* - be^t.$$

So, $h(t + \log b) = -h(t)$ for all $t \in \mathbb{R}$ and hence the function $h(t)$ is periodic with period $2\log b$.

Now, according to (11), we can express the solution $\hat{y}(x)$ to (7) just as $\hat{y}(x) = (a^* - x)^{-\alpha} h(\log(a^* - x))$. Multiplying it by $p_0^{\frac{1}{k-1}}$ we obtain a solution to (3) having the form needed. It still will be a solution to (3) after replacing a^* by arbitrary $x^* \in \mathbb{R}$. $\square$

The substitution $x \mapsto -x$ produces the following.

Corollary 1 *For any integer $n > 2$ and real $k > 1$ there exists a periodic oscillatory function h such that for any $p_0 \in \mathbb{R}$ satisfying $(-1)^n p_0 > 0$ and any $x^* \in \mathbb{R}$ the function*

$$y(x) = |p_0|^{\frac{1}{k-1}} \bigl(x - x^*\bigr)^{-\alpha} h\bigl(\log\bigl(x - x^*\bigr)\bigr), \quad x^* < x < \infty,$$

is a solution to (1).

Note that the following theorem was earlier proved in [4, 5].

Theorem 2 *For $n = 3$, there exists a constant $B \in (0,1)$ such that any oscillatory solution $y(x)$ to* (1) *with $p_0 < 0$ satisfies the conditions*

$$(1)\quad \frac{x_{i+1} - x_i}{x_i - x_{i-1}} = B^{-1}, \quad i = 2, 3, \ldots,$$

$$(2)\quad \frac{y(x'_{i+1})}{y(x'_i)} = -B^{\alpha}, \quad i = 1, 2, 3, \ldots,$$

$$(3)\quad \frac{y'(x_{i+1})}{y'(x_i)} = -B^{\alpha+1}, \quad i = 1, 2, 3, \ldots,$$

$$(4)\quad \bigl|y\bigl(x'_i\bigr)\bigr| = M\bigl(x'_i - x_*\bigr)^{-\alpha}, \quad i = 1, 2, 3, \ldots,$$

for some $M > 0$ and x_*, where $x_1 < x_2 < \cdots < x_i < \cdots$ and $x'_1 < x'_2 < \cdots < x'_i < \cdots$ are sequences satisfying $y(x_j) = 0$, $y'(x'_j) = 0$, $y(x) \neq 0$ if $x \in (x_i, x_{i+1})$, $y'(x) \neq 0$ if $x \in (x'_i, x'_{i+1})$.

With the help of this theorem, another one can be proved, namely the following.

Theorem 3 *For $n = 3$ and any real $k > 1$ there exists a periodic oscillatory function h such that the functions $y(x) = p_0^{\frac{1}{k-1}} |x - x_*|^{-\alpha} h(\log|x - x_*|)$ with $\alpha = \frac{n}{k-1}$ and arbitrary x_* are solutions, respectively, to* (1) *with $p_0 < 0$ if defined on $(-\infty; x_*)$ and to* (1) *with $p_0 > 0$ if defined on $(x_*; +\infty)$.*

3 On existence of positive solutions with non-power asymptotic behavior

For (1) with $p_0 = -1$ it was proved [11] that for any N and $K > 1$ there exist an integer $n > N$ and $k \in \mathbf{R}$ such that $1 < k < K$ and (1) has a solution of the form

$$y = \bigl(x^* - x\bigr)^{-\alpha} h\bigl(\log\bigl(x^* - x\bigr)\bigr), \tag{12}$$

where $\alpha = \frac{n}{k-1}$ and h is a positive periodic non-constant function on $\mathbf{R}$.

A similar result was also proved [11] about Kneser solutions, *i.e.* those satisfying $y(x) \to 0$ as $x \to \infty$ and $(-1)^j y^{(j)}(x) > 0$ for $0 \le j < n$. Namely, if $p_0 = (-1)^{n-1}$, then for any N and $K > 1$ there exist an integer $n > N$ and $k \in \mathbf{R}$ such that $1 < k < K$ and (1) has a solution of the form

$$y(x) = (x - x_*)^{-\alpha} h\bigl(\log(x - x_*)\bigr),$$

where h is a positive periodic non-constant function on $\mathbf{R}$.

Still it was not clear how large n should be for the existence of that type of positive solutions.

Theorem 4 [13] *If* $12 \le n \le 14$, *then there exists* $k > 1$ *such that* (1) *with* $p_0 = -1$ *has a solution* $y(x)$ *such that*

$$y^{(j)}(x) = \bigl(x^* - x\bigr)^{-\alpha - j} h_j\bigl(\log\bigl(x^* - x\bigr)\bigr), \quad j = 0, 1, \ldots, n-1,$$

where $\alpha = \frac{n}{k-1}$ *and* h_j *are periodic positive non-constant functions on* $\mathbf{R}$.

Remark Computer calculations give approximate values of α. They are, with the corresponding values of k, as follows:

if $n = 12$, then $\alpha \approx 0.56$, $k \approx 22.4$;
if $n = 13$, then $\alpha \approx 1.44$, $k \approx 10.0$;
if $n = 14$, then $\alpha \approx 2.37$, $k \approx 6.9$.

Corollary 2 *If* $12 \le n \le 14$, *then there exists* $k > 1$ *such that* (1) *with* $p_0 = (-1)^{n-1}$ *has a Kneser solution* $y(x)$ *satisfying*

$$y^{(j)}(x) = (x - x_0)^{-\alpha - j} h_j\bigl(\log(x - x_0)\bigr), \quad j = 0, 1, \ldots, n-1,$$

with periodic positive non-constant functions h_j *on* $\mathbf{R}$.

4 Conclusions, concluding remarks, and open problems

1. So, we give the negative answer to Question 1 and prove the existence of oscillatory solutions with special qualitative properties for Question 2.
2. It would be interesting to know if positive solutions like (12) exist for $n \ge 15$ and for $5 \le n \le 11$.
3. If a positive solution like (12) exists for some $k_0 > 1$, does it follow, for the same n, that such solutions exist for all $k > k_0$?

Competing interests
The author declares that she has no competing interests.

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