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Poly-Dedekind sums associated with poly-Bernoulli functions

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Abstract

Apostol considered generalized Dedekind sums by replacing the first Bernoulli function appearing in Dedekind sums by any Bernoulli functions and derived a reciprocity relation for them. Recently, poly-Dedekind sums were introduced by replacing the first Bernoulli function appearing in Dedekind sums by any type 2 poly-Bernoulli functions of arbitrary indices and were shown to satisfy a reciprocity relation. In this paper, we consider other poly-Dedekind sums that are obtained by replacing the first Bernoulli function appearing in Dedekind sums by any poly-Bernoulli functions of arbitrary indices. We derive a reciprocity relation for these poly-Dedekind sums.

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Keywords: Poly-Dedekind sum; Polylogarithmic function; Poly-Bernoulli polynomial

1 Introduction

The sawtooth function, denoted by $((x))$, is defined by

$$((x)) = \begin{cases} x - [x] - \frac{1}{2}, & \text{if } x \notin \mathbb{Z}, \\ 0, & \text{if } x \in \mathbb{Z}, \end{cases} \quad \text{(see [1–5])}, \tag{1}$$

where $[x]$ denotes the greatest integer function not exceeding x.

The Dedekind sums are defined by

$$\begin{aligned} S(h,m) &= \sum_{\mu=1}^{m-1} \left(\left(\frac{\mu}{m}\right)\right)\left(\left(\frac{h\mu}{m}\right)\right) \\ &= \sum_{\mu=1}^{m-1} \left(\frac{\mu}{m} - \frac{1}{2}\right)\left(\left(\frac{h\mu}{m}\right)\right) \\ &= \sum_{\mu=1}^{m-1} \frac{\mu}{m}\left(\left(\frac{h\mu}{m}\right)\right), \end{aligned} \tag{2}$$

where h is any integer and m is a positive integer (see [9–11, 17, 19, 20]).

It is well known that the Bernoulli polynomials are defined by

$$\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}\quad\bigl(|t|<2\pi\bigr),\text{(see [1–13, 17, 19, 20])}. \tag{3}$$

When $x=0$, $B_n=B_n(0)$, $(n\geq 0)$ are called the Bernoulli numbers.

From (3), we note that

$$B_n(x)=\sum_{l=0}^{n}\binom{n}{l}B_{n-l}x^l\quad(n\geq 0),\text{(see [7–13])}. \tag{4}$$

By (3), we easily get

$$\sum_{l=0}^{n-1}l^m=\frac{1}{m+1}\bigl(B_{m+1}(n)-B_{m+1}\bigr),\quad(n\in\mathbb{N},m\geq 0),\text{(see [13])}, \tag{5}$$

and

$$d^{n-1}\sum_{l=0}^{d-1}B_n\biggl(\frac{x+i}{d}\biggr)=B_n(x),\quad(n\geq 0,d\in\mathbb{N}),\text{(see [10, 13])}. \tag{6}$$

The modified Hardy's polyexponential function of index k is defined by

$$\mathrm{Ei}_k(x)=\sum_{n=1}^{\infty}\frac{x^n}{n^k(n-1)!}\quad(k\in\mathbb{Z}),\text{(see [7])}. \tag{7}$$

Note that $\mathrm{Ei}_1(x)=e^x-1$.

Recently, the type 2 poly-Bernoulli polynomials of index k are defined by

$$\frac{\mathrm{Ei}_k(\log(1+t))}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n^{(k)}(x)\frac{t^n}{n!}\quad(k\in\mathbb{Z}). \tag{8}$$

When $x=0$, $B_n^{(k)}=B_n^{(k)}(0)$, $(n\geq 0)$ are called the type 2 poly-Bernoulli numbers of index k. Note that $B_n^{(1)}(x)=B_n(x)$, $(n\geq 0)$.

It is well known that the polylogarithmic function of index k is defined by

$$\mathrm{Li}_k(x)=\sum_{n=1}^{\infty}\frac{x^n}{n^k},\quad(k\in\mathbb{Z}),|x|<1,\text{(see [6, 9, 12])}. \tag{9}$$

Note that $\mathrm{Li}_1(x)=-\log(1-x)$.

In [6, 7, 12], the poly-Bernoulli polynomials of index k are defined by the generating function

$$\frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}\beta_n^{(k)}(x)\frac{t^n}{n!}. \tag{10}$$

When $x=0$, $\beta_n^{(k)}=\beta_n^{(k)}(0)$ are called the poly-Bernoulli numbers of index k.

From (10), we note that

$$\sum_{l=0}^{n}\binom{n}{l}\beta_{n-l}^{(k)}x^{l}=\beta_{n}^{(k)}(x),\quad (n\geq 0),\text{(see [6, 7, 12])}. \tag{11}$$

The fractional part of x is defined by

$$\langle x\rangle = x-[x].$$

The Bernoulli functions are defined by

$$\overline{B}_n(x)=B_n\big(\langle x\rangle\big),\quad (n\geq 0),\text{(see [1, 2])}.$$

From (2), we have

$$\begin{aligned} S(h,m)&=\sum_{\mu=1}^{m-1}\frac{\mu}{m}\left(\frac{h\mu}{m}-\left[\frac{h\mu}{m}\right]-\frac{1}{2}\right) \\ &=\sum_{\mu=1}^{m-1}\left(\frac{\mu}{m}-\frac{1}{2}\right)\left(\frac{h\mu}{m}-\left[\frac{h\mu}{m}\right]-\frac{1}{2}\right) \\ &=\sum_{\mu=1}^{m-1}\overline{B}_1\left(\frac{\mu}{m}\right)\overline{B}_1\left(\frac{h\mu}{m}\right), \end{aligned} \tag{12}$$

where h, m are relatively prime positive integers.

Apostol considered the generalized Dedekind sums, which are given by

$$S_p(h,m)=\sum_{\mu=1}^{m-1}\frac{\mu}{m}\overline{B}_p\left(\frac{h\mu}{m}\right), \tag{13}$$

and showed in [1, 2] that they satisfy the reciprocity relation

$$(p+1)\big(hm^pS_p(h,m)+mh^pS_p(m,h)\big)=pB_{p+1}+\sum_{s=0}^{p+1}\binom{p+1}{s}(-1)^sB_sB_{p+1-s}h^sm^{p+1-s}.$$

As one generalization of Apostol's generalized Dedekind sums, the poly-Dedekind sums associated with the type 2 poly-Bernoulli functions of index k

$$S_p^{(k)}(h,m)=\sum_{\mu=1}^{m-1}\frac{\mu}{m}\overline{B}_p^{(k)}\left(\frac{h\mu}{m}\right) \tag{14}$$

were recently introduced (see [13]) and, among other things, a reciprocity relation for them was derived.

In this paper, as another generalization of Apostol's generalized Dedekind sums, we consider the poly-Dedekind sums defined by

$$T_p^{(k)}(h,m)=\sum_{\mu=1}^{m-1}\frac{\mu}{m}\overline{\beta}_p^{(k)}\left(\frac{h\mu}{m}\right),$$

where $\overline{\beta}_p^{(k)}(x) = \beta_p^{(k)}(\langle x\rangle)$ are the poly-Bernoulli functions of index k (see (10)). Note here that $T_p^{(1)}(h,m) = S_p(h,m)$. We show the following reciprocity relation for the poly-Dedekind sums given by (see Theorem 7)

$$
\begin{aligned}
&hm^p T_p^{(k)}(h,m) + mh^p T_p^{(k)}(m,h) \\
&\quad = \sum_{\mu=0}^{m-1}\sum_{j=0}^{p}\sum_{\nu=0}^{h-1}\sum_{l=1}^{p-j+1} \frac{(mh)^{j-1} l! S_2(p-j+1,l)}{(p-j+1)l^k} \\
&\qquad \times \binom{p}{j}(-1)^{p-j+1-l}\big((\mu h)m^{p-j} + (m\nu)h^{p-j}\big)\overline{B}_j\left(\frac{\nu}{h}+\frac{\mu}{m}\right).
\end{aligned}
$$

For $k = 1$, this reciprocity relation for the poly-Dedekind sums reduces to that for Apostol's generalized Dedekind sums given by (see Corollary 8)

$$
\begin{aligned}
&hm^p S_p(h,m) + mh^p S_p(m,h) \\
&\quad = \sum_{\mu=0}^{m-1}\sum_{\nu=0}^{h-1}(mh)^{p-1}(\mu h + m\nu)\overline{B}_p\left(\frac{\nu}{h}+\frac{\mu}{m}\right).
\end{aligned}
$$

We recommend the readers to look at the articles [15, 16, 18, 21] and the more recent one [14], which are related to the present paper. In Sect. 2, we derive various facts about the poly-Bernoulli polynomials that will be needed in the next section. In Sect. 3, we define the poly-Dedekind sums associated with the poly-Bernoulli functions and demonstrate a reciprocity relation for them.

2 Poly-Dedekind sums associated with poly-Bernoulli functions

Let n be a nonnegative integer. Then the Stirling numbers of the second kind are defined by

$$
x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k, \quad (n \geq 0), (\text{see } [1\text{–}14, 17, 19]),
$$

where $(x)_0 = 1$, $(x)_n = x(x-1)\cdots(x-n+1)$, $(n \geq 1)$.

From (9) and (10), we note that

$$
\frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1} = \sum_{n=0}^{\infty}\beta_n^{(k)}\frac{t^n}{n!}. \tag{15}
$$

Thus, by (15), we get

$$
\begin{aligned}
\mathrm{Li}_k\big(1-e^{-t}\big) &= \left(\sum_{l=0}^{\infty}\beta_l^{(k)}\frac{t^l}{l!}\right)\big(e^t-1\big) \\
&= \sum_{n=0}^{\infty}\big(\beta_n^{(k)}(1)-\beta_n^{(k)}\big)\frac{t^n}{n!}.
\end{aligned} \tag{16}
$$

On the other hand,

$$\begin{aligned}\mathrm{Li}_k\left(1-e^{-t}\right) &= \sum_{m=1}^{\infty}\frac{1}{m^k}\left(1-e^{-t}\right)^m = \sum_{m=1}^{\infty}\frac{(-1)^m m!}{m^k}\frac{1}{m!}\left(e^{-t}-1\right)^m \\ &= \sum_{m=1}^{\infty}\frac{(-1)^m m!}{m^k}\sum_{n=m}^{\infty}S_2(n,m)(-1)^n\frac{t^n}{n!} \\ &= \sum_{m=1}^{\infty}\left(\sum_{m=1}^{n}\frac{(-1)^{n-m}m!}{m^k}S_2(n,m)\right)\frac{t^n}{n!}.\end{aligned} \tag{17}$$

Therefore, by (16) and (17), we obtain the following theorem.

Theorem 1 *For $n \in \mathbb{N}$, we have*

$$\beta_n^{(k)}(1) - \beta_n^{(k)} = \sum_{m=1}^{n}\frac{(-1)^{n-m}m!}{m^k}S_2(n,m).$$

From Theorem 1, we note that

$$\beta_0^{(k)} = 1, \qquad \beta_1^{(k)} = -1 + \frac{1}{2^k}, \qquad \beta_2^{(k)} = 1 - \frac{3}{2^k} + \frac{2}{3^k}, \ldots.$$

Taking $k = 1$ in Theorem 1 gives us the following corollary.

Corollary 2 *For $n \in \mathbb{N}$, we have*

$$\sum_{m=1}^{n}(-1)^{n-m}(m-1)!S_2(n,m) = \delta_{n,1},$$

where $\delta_{n,k}$ is the Kronecker symbol.

The three identities in the following lemma can be shown just as in Theorem 3, Corollary 4, and Theorem 5 of [13], and hence their proofs are left to the reader as exercises.

Lemma 3 *For $s, p \in \mathbb{N}$, we have*

$$\sum_{\nu=0}^{p}\binom{p}{\nu}\frac{\beta_\nu^{(k)}}{p-\nu+2} = \binom{p+1}{s}\frac{\beta_{p-s+1}^{(k)}(1)}{p+1} + \frac{s-1}{p+1}\binom{p+2}{s}\frac{\beta_{p-s+2}^{(k)}(1)}{p+2},$$

$$\begin{aligned}&\sum_{\nu=0}^{p-s+1}\binom{p}{\nu}\binom{p-\nu+2}{s}\frac{\beta_\nu^{(k)}}{p-\nu+2} \\ &\quad = \binom{p+1}{s}\frac{\beta_{p-s+1}^{(k)}(1)}{p+1} + \frac{s-1}{p+1}\binom{p+2}{s}\frac{\beta_{p-s+2}^{(k)}(1)}{p+2} - \frac{1}{s}\binom{p}{s-2}\beta_{p-s+2}^{(k)},\end{aligned}$$

and

$$\sum_{s=0}^{p}\binom{p}{s}\beta_s^{(k)}\frac{1}{p+2-s} = \frac{\beta_{p+1}^{(k)}(1)}{p+1} - \frac{\beta_{p+2}^{(k)}(1)}{(p+1)(p+2)} + \frac{\beta_{p+2}^{(k)}}{(p+1)(p+2)}.$$

As a further generalization of Apostol's Dedekind sums, we study poly-Dedekind sums associated with poly-Bernoulli functions of index k, which are given by

$$T_p^{(k)}(h,m)=\sum_{\mu=1}^{m-1}\frac{\mu}{m}\overline{\beta}_p^{(k)}\left(\frac{h\mu}{m}\right), \tag{18}$$

where $h,m,p\in\mathbb{N}$, $k\in\mathbb{Z}$, and $\overline{\beta}_p^{(k)}(x)=\beta_p^{(k)}(\langle x\rangle)$ are the poly-Bernoulli functions of index k.

Note that

$$T_p^{(1)}(h,m)=\sum_{\mu=1}^{m-1}\frac{\mu}{m}\overline{B}_p\left(\frac{h\mu}{m}\right)=S_p(h,m).$$

The two identities in Lemma 4 can be proved in the same way as in Proposition 6 and Theorem 7 in [13], while the identity in Lemma 5 can be shown just as in Theorem 8 in [13]. Therefore their proofs are left to the reader.

Lemma 4 *Let p be an odd positive integer ≥ 3, and $m\in\mathbb{N}$. Then we have*

$$\begin{aligned}&m^pT_p^{(k)}(1,m)\\&=\sum_{\nu=0}^{p}\binom{p}{\nu}\frac{\beta_\nu^{(k)}}{p+2-\nu}m^{p+1}+\sum_{i=1}^{p-1}\sum_{\nu=0}^{p+1-i}\binom{p}{\nu}\binom{p+2-\nu}{i}\frac{\beta_\nu^{(k)}}{p+2-\nu}B_im^{p+1-i}+B_{p+1}\end{aligned}$$

and

$$\begin{aligned}&(p+1)m^pT_p^{(k)}(1,m)\\&=\sum_{i=0}^{p+1}\binom{p+1}{i}B_im^{p+1-i}\beta_{p+1-i}^{(k)}(1)\\&\quad+\frac{1}{p+2}\sum_{i=0}^{p+1}\binom{p+2}{i}(i-1)B_im^{p+1-i}\left(\beta_{p+2-i}^{(k)}(1)-\beta_{p+2-i}^{(k)}\right).\end{aligned}$$

Lemma 5 *For $m,n,h\in\mathbb{N}$ with $(h,m)=1$, and p any positive odd integer ≥ 3, we have*

$$\begin{aligned}&\sum_{s=0}^{p+1}\binom{p+1}{s}B_s\beta_{p+1-s}^{(k)}(1)(mh)^{p+1-s}\\&=m^p\sum_{\mu=0}^{m-1}\sum_{s=0}^{p+1}\binom{p+1}{s}h^s\beta_s^{(k)}\left(\frac{\mu}{m}\right)B_{p+1-s}\left(h-\left[\frac{h\mu}{m}\right]\right).\end{aligned}$$

For $d\in\mathbb{N}$, we observe that

$$\begin{aligned}\sum_{n=0}^{\infty}\beta_n^{(k)}(x)\frac{t^n}{n!}&=\frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1}e^{xt}=\frac{\mathrm{Li}_k(1-e^{-t})}{e^{dt}-1}\sum_{i=0}^{d-1}e^{(i+x)t}\\&=\frac{1}{dt}\mathrm{Li}_k\left(1-e^{-t}\right)\sum_{i=0}^{d-1}\frac{dt}{e^{dt}-1}e^{(\frac{i+x}{d})dt}\end{aligned} \tag{19}$$

$$
\begin{aligned}
&= \sum_{j=0}^{\infty} d^{j-1} \sum_{i=0}^{d-1} B_j\left(\frac{x+i}{d}\right) \frac{t^j}{j!} \frac{1}{t} \sum_{l=1}^{\infty} \frac{l!}{l^k} \frac{1}{l!}\left(1-e^{-t}\right)^l \\
&= \sum_{j=0}^{\infty} d^{j-1} \sum_{i=0}^{d-1} B_j\left(\frac{x+i}{d}\right) \frac{t^j}{j!} \frac{1}{t} \sum_{l=1}^{\infty} \frac{(-1)^l l!}{l^k} \sum_{m=l}^{\infty} S_2(m,l) \frac{(-t)^m}{m!} \\
&= \sum_{j=0}^{\infty} d^{j-1} \sum_{i=0}^{d-1} B_j\left(\frac{x+i}{d}\right) \frac{t^j}{j!} \sum_{m=0}^{\infty} \frac{1}{m+1} \sum_{l=1}^{m+1} \frac{l!(-1)^{l+m-1}}{l^k} S_2(m+1,l) \frac{t^m}{m!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} \sum_{i=0}^{d-1} \sum_{l=1}^{n-j+1} \binom{n}{j} d^{j-1} B_j\left(\frac{x+i}{d}\right) \frac{l!(-1)^{n-j+1-l}}{(n-j+1)l^k} S_2(n-j+1,l)\right) \frac{t^n}{n!}.
\end{aligned}
$$

Therefore, by (19), we obtain the following theorem.

Theorem 6 *For $k \in \mathbb{Z}$, $d \in \mathbb{N}$, and $n \geq 0$, we have*

$$
\beta_n^{(k)}(x) = \sum_{j=0}^{n} \sum_{i=0}^{d-1} \sum_{l=1}^{n-j+1} \binom{n}{j} d^{j-1} B_j\left(\frac{x+i}{d}\right) \frac{l!(-1)^{n-j+1-l}}{(n-j+1)l^k} S_2(n-j+1,l).
$$

By (18), Lemmas 3–5, and Theorem 6, we get

$$
\begin{aligned}
&hm^p T_p^{(k)}(h,m) + mh^p T_p^{(k)}(m,h) \qquad (20) \\
&= hm^p \sum_{\mu=0}^{m-1} \frac{\mu}{m} \overline{\beta}_p^{(k)}\left(\frac{h\mu}{m}\right) + mh^p \sum_{\nu=0}^{h-1} \left(\frac{\mu}{h}\right) \overline{\beta}_p^{(k)}\left(\frac{m\nu}{h}\right) \\
&= hm^p \sum_{\mu=0}^{m-1} \frac{\mu}{m} \sum_{j=0}^{p} h^{j-1} \binom{p}{j} \sum_{\nu=0}^{h-1} \sum_{l=1}^{p-j+1} \frac{l!(-1)^{p-j+1-l}}{(p-j+1)l^k} S_2(p-j+1,l) \overline{B}_j\left(\frac{\mu}{m}+\frac{\nu}{h}\right) \\
&\quad + mh^p \sum_{\nu=0}^{h-1} \frac{\nu}{h} \sum_{j=0}^{p} m^{j-1} \binom{p}{j} \sum_{\mu=0}^{m-1} \sum_{l=1}^{p-j+1} \frac{l!(-1)^{p-j+1-l}}{(p-j+1)l^k} S_2(p-j+1,l) \overline{B}_j\left(\frac{\nu}{h}+\frac{\mu}{m}\right) \\
&= \sum_{\mu=0}^{m-1} \frac{\mu}{m} \sum_{j=0}^{p} m^{p-j}(mh)^j \binom{p}{j} \sum_{\nu=0}^{h-1} \sum_{l=1}^{p-j+1} \overline{B}_j\left(\frac{\mu}{m}+\frac{\nu}{h}\right) \frac{l! S_2(p-j+1,l)}{(p-j+1)l^k} (-1)^{p-j+1-l} \\
&\quad + \sum_{\nu=0}^{h-1} \frac{\nu}{h} \sum_{j=0}^{p} h^{p-j}(mh)^j \binom{p}{j} \sum_{\mu=0}^{m-1} \sum_{l=1}^{p-j+1} \overline{B}_j\left(\frac{\nu}{h}+\frac{\mu}{m}\right) \frac{l! S_2(p-j+1,l)}{(p-j+1)l^k} (-1)^{p-j+1-l} \\
&= \sum_{\mu=0}^{m-1} \sum_{j=0}^{p} \sum_{\nu=0}^{h-1} \sum_{l=1}^{p-j+1} (\mu h)(mh)^{-1} m^{p-j}(mh)^j \binom{p}{j} \\
&\quad \times \overline{B}_j\left(\frac{\mu}{m}+\frac{\nu}{h}\right) \frac{l! S_2(p-j+1,l)}{(p-j+1)l^k} (-1)^{p-j+1-l} \\
&\quad + \sum_{\mu=0}^{m-1} \sum_{j=0}^{p} \sum_{\nu=0}^{h-1} \sum_{l=1}^{p-j+1} (m\nu)(mh)^{-1} h^{p-j}(mh)^j \binom{p}{j} \\
&\quad \times \overline{B}_j\left(\frac{\nu}{h}+\frac{\mu}{m}\right) \frac{l! S_2(p-j+1,l)}{(p-j+1)l^k} (-1)^{p-j+1-l}
\end{aligned}
$$

$$= \sum_{\mu=0}^{m-1} \sum_{j=0}^{p} \sum_{\nu=0}^{h-1} \sum_{l=1}^{p-j+1} \frac{(mh)^{j-1} l! S_2(p-j+1,l)}{(p-j+1)l^k} \times \binom{p}{j} (-1)^{p-j+1-l} \big((\mu h) m^{p-j} + (m\nu) h^{p-j}\big) \overline{B}_j\left(\frac{\nu}{h} + \frac{\mu}{m}\right).$$

Therefore, by (20), we obtain the following reciprocity theorem for the poly-Dedekind sums associated with poly-Bernoulli functions with index k.

Theorem 7 *For $m, h, p \in \mathbb{N}$ and $k \in \mathbb{Z}$, we have*

$$\begin{aligned} &hm^p T_p^{(k)}(h,m) + mh^p T_p^{(k)}(m,h) \\ &= \sum_{\mu=0}^{m-1} \sum_{j=0}^{p} \sum_{\nu=0}^{h-1} \sum_{l=1}^{p-j+1} \frac{(mh)^{j-1} l! S_2(p-j+1,l)}{(p-j+1)l^k} \\ &\quad \times \binom{p}{j} (-1)^{p-j+1-l} \big((\mu h) m^{p-j} + (m\nu) h^{p-j}\big) \overline{B}_j\left(\frac{\nu}{h} + \frac{\mu}{m}\right). \end{aligned}$$

In case of $k = 1$, by making use of Corollary 2, we obtain the following reciprocity relation for the generalized Dedekind sums defined by Apostol.

Corollary 8 *For $m, h, p \in \mathbb{N}$, we have*

$$\begin{aligned} hm^p T_p^{(1)}(h,m) + mh^p T_p^{(1)}(m,h) &= mh^p S_p(h,m) + mh^p S_p(m,h) \\ &= \sum_{\mu=0}^{m-1} \sum_{\nu=0}^{h-1} (mh)^{p-1} (\mu h + m\nu) \overline{B}_p\left(\frac{\nu}{h} + \frac{\mu}{m}\right). \end{aligned}$$

3 Conclusion

The quantity called the Dedekind sum,

$$S(h,m) = \sum_{\mu=1}^{m-1} \frac{\mu}{m} \overline{B}_1\left(\frac{h\mu}{m}\right),$$

occurs in the transformation behavior of the logarithm of the Dedekind eta-function under substitutions from the modular group. It was shown by Dedekind that they satisfy the following reciprocity relation:

$$S(h,m) + S(m,h) = \frac{1}{12}\left(\frac{h}{m} + \frac{1}{hm} + \frac{m}{h}\right) - \frac{1}{4}$$

if h and m are relatively prime positive integers.

Apostol considered the generalized Dedekind sums

$$S_p(h,m) = \sum_{\mu=1}^{m-1} \frac{\mu}{m} \overline{B}_p\left(\frac{h\mu}{m}\right)$$

and derived a reciprocity relation for them. Recently, as one generalization of the generalized Dedekind sums, the poly-Dedekind sums

$$S_P^{(k)}(h,m) = \sum_{\mu=1}^{m-1} \frac{\mu}{m} \overline{B}_p^{(k)}\left(\frac{h\mu}{m}\right),$$

associated with the type 2 poly-Bernoulli functions of arbitrary indices, were introduced and were shown to satisfy a reciprocity relation. In this paper, as another generalization of the generalized Dedekind sums, we considered the poly-Dedekind sums

$$T_p^{(k)}(h,m) = \sum_{\mu=1}^{m-1} \frac{\mu}{m} \overline{\beta}_p^{(k)}\left(\frac{h\mu}{m}\right),$$

associated with the poly-Bernoulli functions of arbitrary indices, and derived a reciprocity relation for these poly-Dedekind sums.

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