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Generalizations of Ostrowski type inequalities via Hermite polynomials

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Abstract

We present new generalizations of the weighted Montgomery identity constructed by using the Hermite interpolating polynomial. The obtained identities are used to establish new generalizations of weighted Ostrowski type inequalities for differentiable functions of class C^n. Also, we consider new bounds for the remainder of the obtained identities by using the Chebyshev functional and certain Grüss type inequalities for this functional. By applying those results we derive inequalities for the class of n-convex functions.

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1 Introduction

In 1938, A.M. Ostrowski [13] pointed out the following inequality which gives an approximation of the integral $\frac{1}{b-a}\int_a^b f(t)\,dt$:

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \left[\frac{1}{4}+\frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right](b-a)\|f'\|_\infty$$

for all $x \in [a,b]$, where $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b) with a bounded derivative. Since the Ostrowski inequality can be proved by using the Montgomery identity

$$f(x)=\frac{1}{b-a}\int_a^b f(t)\,dt+\frac{1}{b-a}\left(\int_a^x (t-a)f'(t)\,dt+\int_x^b (t-b)f'(t)\,dt\right),$$

in this paper we use the weighted Montgomery identity to obtain certain generalizations of Ostrowski type inequalities. The weighted Montgomery identity (see [14]) is defined by

$$f(x)=\int_a^b w(t)f(t)\,dt+\int_a^b P_w(x,t)f'(t)\,dt, \tag{1}$$

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where

$$P_w(x,t) = \begin{cases} \int_a^t w(u)\,du, & a \le t \le x, \\ \int_a^t w(u)\,du - 1, & x < t \le b, \end{cases} \tag{2}$$

is the weighted Peano kernel, $f : [a,b] \to \mathbb{R}$ is differentiable on $[a,b]$, $f' : [a,b] \to \mathbb{R}$ is integrable on $[a,b]$, and $w : [a,b] \to [0,\infty)$ is a normalized weighted function, i.e., an integrable function satisfying

$$\int_a^b w(s)\,ds = 1.$$

Over the last decades, Ostrowski type inequalities have been largely investigated in the literature since they are very useful in numerical analysis and probability theory. Aglić Aljinović et al. considered some weighted Ostrowski type inequalities via the Montgomery identity and the Taylor formula, and applications in numerical integration (see [2, 3] and the references cited therein). Certain Ostrowski type bounds for the Chebyshev functional and applications to the quadrature formulae can be found in papers [4, 5, 9, 10], and [16]. In [12] and [15], Ostrowski type inequalities for continuous functions with one point of nondifferentiability and applications in numerical integration are presented. Some other Ostrowski type inequalities can be found in [6, 7], and [8].

Throughout the paper, the symbol $C^n[a,b]$, $n \in \mathbb{N}$, denotes the set of n times continuously differentiable functions on the interval $[a,b]$. It is well known that the function f is called n times continuously differentiable iff it is n times differentiable and its nth order derivative $f^{(n)}$ is continuous.

The main purpose of this note is to consider new generalizations of weighted Ostrowski type inequalities for functions presented by a Hermite interpolating polynomial. Since a special case of the Hermite interpolating polynomial is the two-point Taylor polynomial, in this way we generalized results from paper [3], where Ostrowski type inequalities are established by using the Taylor formula. For this purpose, let us introduce notations and terminology used in relation to the Hermite interpolating polynomial (see [1, p. 62]).

Let $-\infty < a < b < \infty$ and $a \le a_1 < a_2 \cdots < a_r \le b$, $r \ge 2$, be the given points. Hermite interpolation of the function $f \in C^n[a,b]$, $n \ge r$, is of the form

$$f(t) = P_H(t) + e_H(t),$$

where P_H is a unique polynomial of degree $(n-1)$ satisfying any of the following Hermite conditions:

$$P_H^{(i)}(a_j) = f^{(i)}(a_j); \quad 0 \le i \le k_j, 1 \le j \le r, \sum_{j=1}^{r} k_j + r = n. \tag{3}$$

The polynomial P_H is known in literature as a Hermite interpolating polynomial of the function f. Further, the error $e_H(t)$ can be represented in terms of the Green function $G_{H,n}(t,s)$. Let K be the square $a \le t, s \le b$; the same square with straight lines of the form $s = a_j$ rejected be K_0 and K_0 with rejected diagonal $t = s$ be K_1. Then the Green function has the following fundamental property:

$$z^{(n)}(t) = 0,$$
$$z^{(i)}(a_j) = 0, \quad 0 \le i \le k_j, 1 \le j \le r,$$

in K_1.

Theorem 1 (cf. [1, pp. 73–74]) *Let $f \in C^n[a,b]$, and let P_H be its Hermite interpolating polynomial. Then*

$$\begin{aligned} f(t) &= P_H(t) + e_H(t) \\ &= \sum_{j=1}^{r} \sum_{i=0}^{k_j} H_{ij}(t) f^{(i)}(a_j) + \int_a^b G_{H,n}(t,s) f^{(n)}(s)\, ds, \end{aligned} \tag{4}$$

where H_{ij} are the fundamental polynomials of the Hermite basis defined by

$$H_{ij}(t) = \frac{1}{i!} \frac{\omega(t)}{(t-a_j)^{k_j+1-i}} \sum_{k=0}^{k_j-i} \frac{1}{k!} \frac{d^k}{dt^k} \left(\frac{(t-a_j)^{k_j+1}}{\omega(t)} \right) \Bigg|_{t=a_j} (t-a_j)^k, \tag{5}$$

where

$$\omega(t) = \prod_{j=1}^{r} (t-a_j)^{k_j+1}, \tag{6}$$

and $G_{H,n}$ is the Green function defined by

$$G_{H,n}(t,s) = \begin{cases} \sum_{j=1}^{l} \sum_{i=0}^{k_j} \frac{(a_j-s)^{n-i-1}}{(n-i-1)!} H_{ij}(t), & s \le t \\ -\sum_{j=l+1}^{r} \sum_{i=0}^{k_j} \frac{(a_j-s)^{n-i-1}}{(n-i-1)!} H_{ij}(t), & s \ge t, \end{cases} \tag{7}$$

for all $a_l \le s \le a_{l+1}$, $l = 0, \dots, r$, with $a_0 = a$ and $a_{r+1} = b$.

Hermite conditions (3) in particular include the following $(m, n-m)$ type conditions ($r = 2$, $a_1 = a$, $a_2 = b$, $1 \le m \le n-1$, $k_1 = m-1$, $k_2 = n-m-1$):

$$P_{mn}^{(i)}(a) = f^{(i)}(a), \quad 0 \le i \le m-1,$$
$$P_{mn}^{(i)}(b) = f^{(i)}(b), \quad 0 \le i \le n-m-1.$$

In this case,

$$f(t) = \sum_{l=0}^{m-1} \eta_l(t) f^{(l)}(a) + \sum_{l=0}^{n-m-1} \rho_l(t) f^{(l)}(b) + \int_a^b G_{m,n}(t,s) f^{(n)}(s)\, ds, \tag{8}$$

where

$$\eta_l(t) = \frac{1}{l!}(t-a)^l \left(\frac{t-b}{a-b} \right)^{n-m} \sum_{k=0}^{m-1-l} \binom{n-m+k-1}{k} \left(\frac{t-a}{b-a} \right)^k, \tag{9}$$

$$\rho_l(t) = \frac{1}{l!}(t-b)^l\left(\frac{t-a}{b-a}\right)^m \sum_{k=0}^{n-m-1-l}\binom{m+k-1}{k}\left(\frac{t-b}{a-b}\right)^k, \tag{10}$$

and the Green function $G_{m,n}$ is of the form

$$G_{m,n}(t,s) = \begin{cases} \sum_{j=0}^{m-1}[\sum_{p=0}^{m-1-j}\binom{n-m+p-1}{p}(\frac{t-a}{b-a})^p] \\ \quad \times \frac{(t-a)^j(a-s)^{n-j-1}}{j!(n-j-1)!}(\frac{b-t}{b-a})^{n-m}, & s \le t, \\ -\sum_{i=0}^{n-m-1}[\sum_{q=0}^{n-m-1-i}\binom{m+q-1}{q}(\frac{b-t}{b-a})^q] \\ \quad \times \frac{(t-b)^i(b-s)^{n-i-1}}{i!(n-i-1)!}(\frac{t-a}{b-a})^m, & s \ge t. \end{cases} \tag{11}$$

Since we deal with an n-convex function, let us recall the definition of the divided difference (see [17, p. 15]).

Definition 1 Let f be a real-valued function defined on the segment $[a,b]$. The divided difference of order n of the function f at distinct points $x_0,\ldots,x_n \in [a,b]$ is defined recursively by

$$f[x_i] = f(x_i), \quad (i = 0,\ldots,n)$$

and

$$f[x_0,\ldots,x_n] = \frac{f[x_1,\ldots,x_n] - f[x_0,\ldots,x_{n-1}]}{x_n - x_0}.$$

The value $f[x_0,\ldots,x_n]$ is independent of the order of the points $x_0,\ldots,x_n$.

The definition may be extended to include the case that some (or all) of the points coincide. Assuming that $f^{(j-1)}(x)$ exists, we define

$$f[\underbrace{x,\ldots,x}_{j\text{-times}}] = \frac{f^{(j-1)}(x)}{(j-1)!}.$$

Also, the divided difference of order n of the function f can be represented as

$$f[x_0,\ldots,x_n] = \sum_{i=0}^{n}\frac{f(x_i)}{\nu(x_i)},$$

where $\nu(x_i) = \prod_{j=0,j\neq i}^{n}(x_i - x_j)$. With these observations in mind, Popoviciu defined n-convex function as follows (see [18]).

Definition 2 A function $f : [a,b] \to \mathbb{R}$ is said to be n-convex on $[a,b]$, $n \ge 0$, if for all choices of $(n+1)$ distinct points $x_0,\ldots,x_n \in [a,b]$, the nth order divided difference of f satisfies

$$f[x_0,\ldots,x_n] \ge 0.$$

If $n = 0$, then a convex function f of order 0 is a nonnegative function, a 1-convex function is a nondecreasing function, while the class of 2-convex functions coincides with the class of convex functions. It is well known that if the nth order derivative $f^{(n)}$ exists, then the function f is n-convex if and only if $f^{(n)} \geq 0$ (see for example [17, p. 16 and p. 293]).

The paper is organized as follows. After this introduction, in Sect. 2, we establish weighted generalizations of the Montgomery identity constructed by using the Hermite interpolating polynomial and the Green function. In Sect. 3, we derive Ostrowski type inequalities for differentiable functions of class C^n. As a special case, we consider results for $(m, n-m)$ interpolating polynomial. Further, in Sect. 4, we give some new bounds for the remainder of identities previously obtained by using the Chebyshev functional and certain Grüss type inequalities for this functional. Finally, in Sect. 5, applying the properties of n-convex functions and generalizations of the weighted Montgomery identity, we obtain inequalities for the class of n-convex functions.

Throughout the paper, it is assumed that all integrals under consideration exist and that they are finite.

2 Generalizations of the weighted Montgomery identity

In this section, applying the weighted Montgomery identity (1) and the Hermite interpolation polynomial of the n times continuously differentiable function f, (4), we derive new generalizations of the weighted Montgomery identity.

Theorem 2 *Suppose that* $f \in C^n[a,b]$, $w : [a,b] \to [0,\infty)$ *is some normalized weight function and* H_{lj} *is defined by* (5). *Then, for* $-\infty < a \leq a_1 < a_2 \cdots < a_r \leq b < \infty$, $r \geq 2$, $\sum_{j=1}^{r} k_j + r = n-1$, *the following identity holds*:

$$f(x) = \int_a^b w(t)f(t)\,dt + \sum_{j=1}^{r}\sum_{l=0}^{k_j} f^{(l+1)}(a_j)\int_a^b P_w(x,t)H_{lj}(t)\,dt + \int_a^b\left(\int_a^b P_w(x,t)G_{H,n-1}(t,s)\,dt\right)f^{(n)}(s)\,ds. \tag{12}$$

Proof By applying (4) with $f' \in C^{(n)}[a,b]$ instead of f, we obtain

$$f'(t) = \sum_{j=1}^{r}\sum_{l=0}^{k_j} H_{lj}(t)f^{(l+1)}(a_j) + \int_a^b G_{H,n-1}(t,s)f^{(n)}(s)\,ds. \tag{13}$$

By inserting (13) into the weighted Montgomery identity (1), we derive (12). □

Theorem 3 *Let* $f \in C^n[a,b]$, $w : [a,b] \to [0,\infty)$ *be some normalized weight function, and let* H_{lj} *be defined as* (5). *Then, for* $-\infty < a \leq a_1 < a_2 \cdots < a_r \leq b < \infty$, $r \geq 2$, $\sum_{j=1}^{r} k_j + r = n$, *the following identity holds*:

$$f(x) = \int_a^b w(t)f(t)\,dt + \sum_{j=1}^{r}\sum_{l=0}^{k_j} f^{(l)}(a_j)\int_a^b P_w(x,t)H'_{lj}(t)\,dt + \int_a^b\left(\int_a^b P_w(x,t)\frac{\partial}{\partial t}G_{H,n}(t,s)\,dt\right)f^{(n)}(s)\,ds. \tag{14}$$

Proof Multiplying identity (4) by $w(t)$ and integrating with respect to t from a to b, we obtain the following identity:

$$\int_a^b w(t)f(t)\,dt = \sum_{j=1}^{r}\sum_{l=0}^{k_j} f^{(l)}(a_j)\int_a^b w(t)H_{lj}(t)\,dt + \int_a^b\int_a^b w(t)G_{H,n}(t,s)f^{(n)}(s)\,ds\,dt. \tag{15}$$

If we subtract (15) from identity (4) stated for the variable x instead of t, we get

$$f(x) - \int_a^b w(t)f(t)\,dt = \sum_{j=1}^{r}\sum_{l=0}^{k_j} f^{(l)}(a_j)\left(H_{lj}(x) - \int_a^b w(t)H_{lj}(t)\,dt\right) + \int_a^b \left(G_{H,n}(x,s) - \int_a^b w(t)G_{H,n}(t,s)\,dt\right)f^{(n)}(s)\,ds. \tag{16}$$

By applying the weighted Montgomery identity (1) for $H_{lj}(x)$ and $G_{H,n}(x,s)$, we obtain the following identities:

$$H_{lj}(x) = \int_a^b w(t)H_{lj}(t)\,dt + \int_a^b P_w(x,t)H'_{lj}(t)\,dt \tag{17}$$

and

$$G_{H,n}(x,s) = \int_a^b w(t)G_{H,n}(t,s)\,dt + \int_a^b P_w(x,t)\frac{\partial}{\partial t}G_{H,n}(t,s)\,dt. \tag{18}$$

Finally, inserting (17) and (18) into (16), we obtain (14). □

3 Ostrowski type inequalities

In this section, we use identity (12), identity (14), and Hölder's inequality to prove some sharp and best possible inequalities for the functions whose higher order derivatives belong to L_p spaces, $1 \le p \le \infty$. As a special case, we discuss results for $(m, n-m)$ interpolating polynomial.

In what follows, (p,q) is a pair of conjugate exponents if $1 \le p,q \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, with the convention $\frac{1}{\infty} = 0$ and $\frac{1}{0} = \infty$. The symbol $L_p[a,b]$, $1 \le p < \infty$, denotes the space of p-power integrable functions on the interval $[a,b]$ equipped with the norm $\|f\|_p = (\int_a^b |f(t)|^p\,dt)^{1/p}$, and $L_\infty[a,b]$ stands for the space of all essentially bounded functions on the interval $[a,b]$ with the norm $\|f\|_\infty = \operatorname{ess\,sup}_{t\in[a,b]} |f(t)|$.

Further, we denote

$$\Lambda_w(s) = \int_a^b P_w(x,t)G_{H,n-1}(t,s)\,dt, \quad s \in [a,b] \tag{19}$$

and

$$\Omega_w(s) = \int_a^b P_w(x,t)\frac{\partial}{\partial t}G_{H,n}(t,s)\,dt, \quad s \in [a,b], \tag{20}$$

where the Green function $G_{H,n}$ is as defined in (7).

Theorem 4 *Suppose that all the assumptions of Theorem 2 hold. Additionally, assume that (p,q) is a pair of conjugate exponents $1 \leq p,q \leq \infty$ and $f^{(n)} \in L_p[a,b]$. Then the following inequality holds*:

$$\begin{aligned} &\left| f(x) - \int_a^b w(t)f(t)\,dt - \sum_{j=1}^{r}\sum_{l=0}^{k_j} f^{(l+1)}(a_j)\int_a^b P_w(x,t)H_{lj}(t)\,dt \right| \\ &\quad \leq \|\varLambda_w\|_q \bigl\| f^{(n)} \bigr\|_p, \end{aligned} \tag{21}$$

where $\varLambda_w$ is defined by (19). The constant on the right-hand side of (21) is sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.

Proof By applying Hölder's inequality to (12), we obtain (21). For the proof of the sharpness of the constant $\|\varLambda_w\|_q$, let us find a function f for which the equality in (21) is obtained.

For $1 < p < \infty$, take f to be such that

$$f^{(n)}(s) = \operatorname{sgn} \varLambda_w(s) \bigl|\varLambda_w(s)\bigr|^{\frac{1}{p-1}}.$$

For $p = \infty$, take $f^{(n)}(s) = \operatorname{sgn} \varLambda_w(s)$.

For $p = 1$, we prove that

$$\left| \int_a^b \varLambda_w(s) f^{(n)}(s)\,ds \right| \leq \max_{s\in[a,b]} \bigl|\varLambda_w(s)\bigr| \left(\int_a^b \bigl| f^{(n)}(s) \bigr|\,ds \right) \tag{22}$$

is the best possible inequality. Suppose that $|\varLambda_w(s)|$ attains its maximum at $s_0 \in [a,b]$. First, we assume that $\varLambda_w(s_0) > 0$. For ε small enough, we define $f_\varepsilon(s)$ by

$$f_\varepsilon(s) = \begin{cases} 0, & a \leq s \leq s_0, \\ \frac{1}{\varepsilon n!}(s-s_0)^n, & s_0 \leq s \leq s_0+\varepsilon, \\ \frac{1}{(n-1)!}(s-s_0)^{n-1}, & s_0+\varepsilon \leq s \leq b. \end{cases}$$

Then, for ε small enough,

$$\left| \int_a^b \varLambda_w(s) f^{(n)}(s)\,ds \right| = \left| \int_{s_0}^{s_0+\varepsilon} \varLambda_w(s)\frac{1}{\varepsilon}\,ds \right| = \frac{1}{\varepsilon}\int_{s_0}^{s_0+\varepsilon} \varLambda_w(s)\,ds.$$

Now, from inequality (22) we have

$$\frac{1}{\varepsilon}\int_{s_0}^{s_0+\varepsilon} \varLambda_w(s)\,ds \leq \varLambda_w(s_0)\int_{s_0}^{s_0+\varepsilon} \frac{1}{\varepsilon}\,ds = \varLambda_w(s_0).$$

Since

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{s_0}^{s_0+\varepsilon} \varLambda_w(s)\,ds = \varLambda_w(s_0),$$

the statement follows. In the case $\Lambda_w(s_0) < 0$, we define $f_\varepsilon(s)$ by

$$f_\varepsilon(s) = \begin{cases} \frac{1}{(n-1)!}(s-s_0-\varepsilon)^{n-1}, & a \le s \le s_0, \\ -\frac{1}{\varepsilon n!}(s-s_0-\varepsilon)^n, & s_0 \le s \le s_0+\varepsilon, \\ 0, & s_0+\varepsilon \le s \le b, \end{cases}$$

and the rest of the proof is the same as above. □

Theorem 5 *Suppose that all the assumptions of Theorem 3 hold. Additionally, assume that (p,q) is a pair of conjugate exponents $1 \le p,q \le \infty$ and $f^{(n)} \in L_p[a,b]$. Then the following inequality holds*:

$$\left| f(x) - \int_a^b w(t)f(t)\,dt - \sum_{j=1}^{r}\sum_{l=0}^{k_j} f^{(l)}(a_j)\int_a^b P_w(x,t)H'_{lj}(t)\,dt \right| \le \|\Omega_w\|_q \left\| f^{(n)} \right\|_p, \tag{23}$$

where Ω_w is defined by (20). The constant on the right-hand side of (23) is sharp for $1 < p \le \infty$ and the best possible for $p = 1$.

Proof By applying Hölder's inequality to (14), we obtain (23). The proof of the sharpness of the constant $\|\Omega_w\|_q$ is analogous to the proof of Theorem 4. □

By using $(m, n-m)$ type conditions, we obtain the following generalizations of Ostrowski type inequalities as special cases of Theorem 4 and Theorem 5, respectively.

Theorem 6 *Let $w : [a,b] \to [0,\infty)$ be some normalized weight function, $f \in C^n[a,b]$, and (p,q) be a pair of conjugate exponents. Let η_l, ρ_l, and $G_{m,n}$ be given by (9), (10), and (11), respectively. Then the following inequality holds*:

$$\left| f(x) - \int_a^b w(t)f(t)\,dt - \sum_{l=0}^{m-1} f^{(l)}(a)\int_a^b P_w(x,t)\eta'_l(t)\,dt - \sum_{l=0}^{n-m-1} f^{(l)}(b)\int_a^b P_w(x,t)\rho'_l(t)\,dt \right| \le \|K_w\|_q \left\| f^{(n)} \right\|_p, \tag{24}$$

where

$$K_w(s) = \int_a^b P_w(x,t)\frac{\partial}{\partial t}G_{m,n}(t,s)\,dt.$$

Proof This is a special case of Theorem 5 for $r = 2$, $a_1 = a$, $a_2 = b$, $1 \le m \le n-1$, $k_1 = m-1$, $k_2 = n-m-1$. □

Corollary 1 *Let $w : [a,b] \to [0,\infty)$ be some normalized weight function, $f \in C^2[a,b]$, and (p,q) be a pair of conjugate exponents. Then the following inequality holds*:

$$\left| f(x) - \int_a^b w(t)f(t)\,dt + \frac{f(a)-f(b)}{b-a}\left(\int_a^b P_w(x,t)\,dt\right) \right| \le \|K_w\|_q \left\| f'' \right\|_p,$$

where

$$K_w(s) = \int_a^b P_w(x,t)\frac{\partial}{\partial t}G_{1,2}(t,s)\,dt$$

and

$$G_{1,2}(t,s) = \begin{cases} \frac{(t-b)(s-a)}{b-a}, & s \le t, \\ \frac{(s-b)(t-a)}{b-a}, & s \ge t. \end{cases}$$

Proof This is a special case of Theorem 6 for $n = 2$. □

Remark 1 By applying Corollary 1 to the uniform weight function $w(t) = \frac{1}{b-a}$, $t \in [a,b]$, we deduce

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{f(a)-f(b)}{b-a}\left(x - \frac{a+b}{2}\right)\right| \le \|K\|_q \|f''\|_p,$$

where

$$K(s) = \begin{cases} \frac{(s-a)(2x-b-s)}{2(b-a)}, & s \le x, \\ \frac{(s-b)(2x-a-s)}{2(b-a)}, & x \le s. \end{cases}$$

Corollary 2 *Let $w : [a,b] \to [0,\infty)$ be some normalized weight function, $f \in C^3[a,b]$, and (p,q) be a pair of conjugate exponents. Then*

$$\begin{aligned} &\left| f(x) - \int_a^b w(t)f(t)\,dt - \frac{1}{b-a}\left[f'(a)\int_a^b (b-t)P_w(x,t)\,dt \right.\right. \\ &\quad \left.\left. + f'(b)\int_a^b (t-a)P_w(x,t)\,dt\right]\right| \le \|V_w\|_q \|f'''\|_p, \end{aligned}$$

where

$$V_w(s) = \int_a^b P_w(x,t)G_{1,2}(t,s)\,dt.$$

Proof This is a special case of Theorem 4 for $n = 3$, $r = 2$, $a_1 = a$, and $a_2 = b$. □

Remark 2 By applying Corollary 2 to the uniform weight function $w(t) = \frac{1}{b-a}$, $t \in [a,b]$, we obtain

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt - f'(a)\left(\frac{1}{6}(b-a) - \frac{(b-x)^2}{2(b-a)}\right) \right. \\ &\quad \left. - f'(b)\left(\frac{1}{6}(b-a) - \frac{(x-a)^2}{2(b-a)}\right)\right| \le \|V\|_q \|f'''\|_p, \end{aligned}$$

where

$$V(s)=\begin{cases}\frac{(s-a)}{(b-a)^2}[(s-b)\frac{(s-a)^2}{3} \\ \quad +[\frac{x^3-s^3}{3}-(a+b)\frac{x^2-s^2}{2}+ab(x-s)]-\frac{(x-b)^3}{3}], & s\le x,\\ \frac{(s-b)}{(b-a)^2}[\frac{(x-a)^3}{3} \\ \quad +[\frac{s^3-x^3}{3}-(a+b)\frac{s^2-x^2}{2}+ab(s-x)]-(s-a)\frac{(s-b)^2}{3}], & s\ge x.\end{cases}$$

4 Grüss type inequalities

We start this section by observation about the Chebyshev functional and certain inequalities for the Chebyshev functional. These inequalities are very useful in numerical integration, some recent results can be found in papers [9, 10], and [16]. For that reason, we consider some new bounds for the remainder of identities (12) and (14) by using the Chebyshev functional and Grüss type inequalities for this functional.

For two real functions $f,h:[a,b]\to\mathbb{R}$ such that $f,h,f\cdot h\in L_1[a,b]$, Chebyshev functional [11] is defined by

$$S(f,h)=\frac{1}{b-a}\int_a^b f(t)h(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b h(t)\,dt. \tag{25}$$

In [5], Cerone and Dragomir established the following inequalities for the Chebyshev functional.

Theorem 7 (cf. [5, Th. 1]) *Let $f:[a,b]\to\mathbb{R}$ be an integrable function, $h:[a,b]\to\mathbb{R}$ be an absolutely continuous function, and $g:[a,b]\to\mathbb{R}$, defined by $g(t)=(t-a)(b-t)[h'(t)]^2$, such that $g\in L_1[a,b]$. Then the following inequality holds*:

$$\left|S(f,h)\right|\le\frac{1}{\sqrt{2}}\left[\frac{1}{b-a}S(f,f)\int_a^b(t-a)(b-t)\bigl(h'(t)\bigr)^2\,dt\right]^{\frac{1}{2}}. \tag{26}$$

Remark 3 The constant $\frac{1}{\sqrt{2}}$ in (26) is the best possible.

Theorem 8 (cf. [5, Th. 2]) *Suppose that $h:[a,b]\to\mathbb{R}$ is monotonically nondecreasing on $[a,b]$ and $f:[a,b]\to\mathbb{R}$ is absolutely continuous with $f'\in L_\infty[a,b]$. Then the following inequality holds*:

$$\left|S(f,h)\right|\le\frac{1}{2(b-a)}\|f'\|_\infty\int_a^b(t-a)(b-t)\,dh(t). \tag{27}$$

Remark 4 The constant $\frac{1}{2}$ in (27) is the best possible.

Now we use the above theorems and the results proved in the previous sections to obtain certain Grüss type inequalities.

Theorem 9 *Let $-\infty<a\le a_1<a_2\cdots<a_r\le b<\infty$, $r\ge2$, let $f:[a,b]\to\mathbb{R}$ be such that $f\in C^{n+1}[a,b]$, and let the functions H_{lj}, $l=0,\dots,k_j$, $j=1,\dots,r$, Λ_w, Ω_w and the functional S be given by* (5), (19), (20), *and* (25), *respectively.*

(i) *If* $\sum_{j=1}^{r} k_j + r = n-1$, *then*

$$\begin{aligned} f(x) - \int_a^b w(t)f(t)\,dt &= \sum_{j=1}^{r}\sum_{l=0}^{k_j} f^{(l+1)}(a_j)\int_a^b P_w(x,t)H_{lj}(t)\,dt \\ &\quad + \frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\int_a^b\int_a^b P_w(x,t)G_{H,n-1}(t,s)\,dt\,ds \\ &\quad + R_n^1(f;a,b), \end{aligned} \tag{28}$$

where the remainder $R_n^1(f;a,b)$ *satisfies the estimation*

$$\left|R_n^1(f;a,b)\right| \le \left[\frac{b-a}{2}S(\Lambda_w,\Lambda_w)\int_a^b (s-a)(b-s)\bigl(f^{(n+1)}(s)\bigr)^2\,ds\right]^{\frac{1}{2}}. \tag{29}$$

(ii) *If* $\sum_{j=1}^{r} k_j + r = n$, *then*

$$\begin{aligned} f(x) - \int_a^b w(t)f(t)\,dt &= \sum_{j=1}^{r}\sum_{l=0}^{k_j} f^{(l)}(a_j)\int_a^b P_w(x,t)H'_{lj}(t)\,dt \\ &\quad + \frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\int_a^b\int_a^b P_w(x,t)\frac{\partial}{\partial t}G_{H,n}(t,s)\,dt\,ds \\ &\quad + R_n^2(f;a,b), \end{aligned} \tag{30}$$

where the remainder $R_n^2(f;a,b)$ *satisfies the estimation*

$$\left|R_n^2(f;a,b)\right| \le \left[\frac{b-a}{2}S(\Omega_w,\Omega_w)\int_a^b (s-a)(b-s)\bigl(f^{(n+1)}(s)\bigr)^2\,ds\right]^{\frac{1}{2}}. \tag{31}$$

Proof

(i) By applying Theorem 7 to Λ_w in place of f and $f^{(n)}$ in place of h, we obtain the following:

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b \Lambda_w(s)f^{(n)}(s)\,ds - \frac{1}{b-a}\int_a^b \Lambda_w(s)\,ds\cdot\frac{1}{b-a}\int_a^b f^{(n)}(s)\,ds\right| \\ &\quad \le \frac{1}{\sqrt{2}}\left[\frac{1}{b-a}S(\Lambda_w,\Lambda_w)\int_a^b (s-a)(b-s)\bigl(f^{(n+1)}(s)\bigr)^2\,ds\right]^{\frac{1}{2}}. \end{aligned}$$

Since

$$\begin{aligned} &\int_a^b \Lambda_w(s)f^{(n)}(s)\,ds \\ &\quad = \frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\int_a^b \Lambda_w(s)\,ds + R_n^1(f;a,b), \end{aligned}$$

from identity (12) we obtain (28). Further, the remainder $R_n^1(f;a,b)$ satisfies estimation (29).

(ii) Analogous to (i). □

Theorem 10 *Let $-\infty < a \leq a_1 < a_2 \cdots < a_r \leq b < \infty$, $r \geq 2$, let $f : [a,b] \to \mathbb{R}$ be such that $f \in C^{n+1}[a,b]$ with $f^{(n+1)} \geq 0$ on $[a,b]$, and let Λ_w, Ω_w be defined in (19) and (20). Then we have representations (28) and (30) and the remainders $R_n^i(f;a,b)$, $i = 1,2$, satisfy the bounds*

$$\left|R_n^1(f;a,b)\right| \leq \left\|\Lambda_w'\right\|_\infty \left[\frac{b-a}{2}\left(f^{(n-1)}(b) + f^{(n-1)}(a)\right) - f^{(n-2)}(b) + f^{(n-2)}(a)\right] \tag{32}$$

and

$$\left|R_n^2(f;a,b)\right| \leq \left\|\Omega_w'\right\|_\infty \left[\frac{b-a}{2}\left(f^{(n-1)}(b) + f^{(n-1)}(a)\right) - f^{(n-2)}(b) + f^{(n-2)}(a)\right]. \tag{33}$$

Proof By applying Theorem 8 to Λ_w in place of f and $f^{(n)}$ in place of h, we deduce

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b \Lambda_w(s)f^{(n)}(s)\,ds - \frac{1}{b-a}\int_a^b \Lambda_w(s)\,ds \cdot \frac{1}{b-a}\int_a^b f^{(n)}(s)\,ds\right| \\ &\quad\leq \frac{1}{2(b-a)}\left\|\Lambda_w'\right\|_\infty \int_a^b (s-a)(b-s)f^{(n+1)}(s)\,ds. \end{aligned} \tag{34}$$

Since

$$\begin{aligned} \int_a^b (s-a)(b-s)f^{(n+1)}(s)\,ds &= \int_a^b \left[2s-(a+b)\right]f^{(n)}(s)\,ds \\ &= (b-a)\left[f^{(n-1)}(b) + f^{(n-1)}(a)\right] - 2\left[f^{(n-2)}(b) - f^{(n-2)}(a)\right], \end{aligned}$$

using identity (12) and (34), we obtain (32). Similarly, from identity (14) we get inequality (33). □

5 Inequalities for *n*-convex functions

The aim of this section is to consider certain inequalities for n-convex functions. This will be done by using the properties of n-convex functions and generalizations of weighted Montgomery identity obtained in Sect. 2.

Theorem 11 *Let $-\infty < a \leq a_1 < a_2 \cdots < a_r \leq b < \infty$, $r \geq 2$, $\sum_{j=1}^r k_j + r = n-1$, and let the functions H_{lj}, $l = 0,\ldots,k_j$, $j = 1,\ldots,r$, and $G_{H,n-1}$ be defined as (5) and (7), respectively. If $f : [a,b] \to \mathbb{R}$ is n-convex and*

$$\int_a^b P_w(x,t)G_{H,n-1}(t,s)\,dt \geq 0 \quad \textit{for all } s \in [a,b], \tag{35}$$

then

$$f(x) - \int_a^b w(t)f(t)\,dt - \sum_{j=1}^r \sum_{l=0}^{k_j} f^{(l+1)}(a_j)\int_a^b P_w(x,t)H_{lj}(t)\,dt \geq 0. \tag{36}$$

If the inequality in (35) is reversed, then the inequality in (36) is reversed, too.

Proof Since the function f is n-convex, therefore, without loss of generality, we can assume that f is n-times differentiable and $f^{(n)}(t) \geq 0$, $t \in [a,b]$. Using this fact and assumption (35), by applying Theorem 2, we obtain (36). □

Theorem 12 *Let $-\infty < a \leq a_1 < a_2 \cdots < a_r \leq b < \infty$, $r \geq 2$, $\sum_{j=1}^{r} k_j + r = n$, and let the functions H_{lj}, $l = 0,\ldots,k_j$, $j = 1,\ldots,r$, and $G_{H,n}$ be defined as (5) and (7), respectively. If $f : [a,b] \to \mathbb{R}$ is n-convex and*

$$\int_a^b P_w(x,t) \frac{\partial}{\partial t} G_{H,n}(t,s)\, dt \geq 0 \quad \text{for all } s \in [a,b], \tag{37}$$

then

$$f(x) - \int_a^b w(t) f(t)\, dt - \sum_{j=1}^{r} \sum_{l=0}^{k_j} f^{(l)}(a_j) \int_a^b P_w(x,t) H'_{lj}(t)\, dt \geq 0. \tag{38}$$

If the inequality in (37) is reversed, then the inequality in (38) is reversed, too.

Proof The proof is similar to the proof of Theorem 11. □

6 Conclusion

In this paper, new generalizations of Ostrowski type inequalities are obtained. The methods used are based on the classical real analysis, application of the Hermite interpolating polynomials and the weighted Montgomery identity. The obtained results and the Chebyshev functional are then applied to establish new upper bounds for the remainder of generalized Montgomery identity. Also, certain inequalities for the class of n-convex functions are derived. In our future work, we will investigate some applications of the above results in numerical analysis and probability theory.

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