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Derivation of computational formulas for Changhee polynomials and their functional and differential equations

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This paper is dedicated to Professor Gradimir V. Milovanović on the Occasion of his 70th Anniversary

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Abstract

The goal of this paper is to demonstrate many explicit computational formulas and relations involving the Changhee polynomials and numbers and their differential equations with the help of functional equations and partial derivative equations for generating functions of these polynomials and numbers. These formulas also include the Euler polynomials, the Stirling numbers, the Bernoulli numbers and polynomials of the second kind, the Changhee polynomials of higher order, and the Daehee polynomials of higher order, which are among the well known polynomial families. By using PDEs of these generating functions, not only some recurrence relations for derivative formulas of the Changhee polynomials of higher order, but also two open problems for partial derivative equations for generating functions are given. Moreover, by using functional equations of the generating functions, two inequalities including combinatorial sums, the Changhee numbers of negative order, and the Stirling numbers of the second kind are provided. Finally, further remarks and observations for the results of this paper are given.

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1 Introduction

Using generating functions, many kinds of partial differential equations (PDEs), ordinary differential equations (ODEs), and stochastic differential equation (SDEs), including boundary-value problems, initial-value problem, and discrete boundary-value problems have been studied and investigated. By using these equations many properties of the generating functions have been investigated. Recently, generating functions, their functional equations and their PDEs including special numbers and polynomials have been studied in many different areas. Because generating functions have many applications in mathematics, in physics, and in engineering (*cf.* [1–38]). In this paper, by using generating functions with their PDEs and functional functions, we investigate and study many new formulas and relations involving the Bernoulli numbers and polynomials of the second kind, the Euler numbers and polynomials, the Stirling numbers, the Peters polynomials, the Boole polynomials and numbers, the Daehee numbers, and also the Changhee polynomials.

The following notations and definitions are used in this paper:

Let $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ denote sets of integer numbers, real numbers, and complex numbers, respectively.

The falling and rising factorials functions, often used in the theory of the hypergeometric functions and partition theory, are defined as follows:

$$(y)_c = y(y-1)(y-2)\cdots(y-c+1),$$

$(y)_0 = 1$,

$$(y)_c = c!\binom{y}{c},$$

and

$$(y)^c = (-1)^c(-y)_c = y(y+1)\cdots(y+c-1)$$

where $c \in \mathbb{N}_0$ (*cf.* [1–38]).

We give some generating functions for some special polynomials and numbers as follows:

The Euler polynomials of order k are defined by means of the following generating function:

$$F_E(t, x; k) = \left(\frac{2}{e^t + 1}\right)^k e^{tx} = \sum_{n=0}^{\infty} E_n^{(k)}(x)\frac{t^n}{n!} \tag{1}$$

(*cf.* [1–38]). Substituting $k = 1$ into (1), we have the Euler polynomials:

$$E_n(x) = E_n^{(1)}(x).$$

The derivatives of the Euler polynomials is given as follows:

$$\frac{d^n}{dx^n}\left\{E_n(x)\right\} = nE_{n-1}(x)$$

(*cf.* [1–38]).

He and Ricci [7] gave the following differential equation for the Euler polynomials:

$$\begin{aligned} 0 = {} & \frac{e_{n-1}}{(n-1)!}\frac{d^n}{dx^n}\left\{E_n(x)\right\} + \frac{e_{n-2}}{(n-2)!}\frac{d^{n-1}}{dx^{n-1}}\left\{E_n(x)\right\} \\ & + \cdots + \frac{e_1}{1!}\frac{d^2}{dx^2}\left\{E_n(x)\right\} + \left(x - \frac{1}{2}\right)\frac{d}{dx}\left\{E_n(x)\right\} - ny, \end{aligned}$$

where

$$e_n = -\sum_{j=0}^{n} \binom{n}{j} 2^{-j} E_{n-j}\left(\frac{1}{2}\right)$$

and $n \in \mathbb{N}$.

We note that Lu and Luo [23] gave a differential equation for the generalized Apostol–Euler polynomials.

The Stirling numbers of the first kind are defined by means of the following generating function:

$$F_{S_1}(t,k) = \frac{(\log(1+t))^k}{k!} = \sum_{n=0}^{\infty} S_1(n,k)\frac{t^n}{n!} \tag{2}$$

and

$$(x)_j = \sum_{l=0}^{n} S_1(n,l)x^l \tag{3}$$

(*cf.* [3–38]).

The Stirling numbers of the second kind are defined by means of the following generating function:

$$F_{S_2}(t,k) = \frac{(e^t-1)^k}{k!} = \sum_{n=0}^{\infty} S_2(n,k)\frac{t^n}{n!} \tag{4}$$

(*cf.* [3–38]).

The Peters polynomials are defined by means of the following generating function:

$$F_P(t,x;\lambda,\mu) = \frac{(1+t)^x}{(1+(1+t)^{\lambda})^{\mu}} = \sum_{n=0}^{\infty} s_n(x;\lambda,\mu)\frac{t^n}{n!}, \tag{5}$$

where $x, t \in \mathbb{C}$ (*cf.* [1–38]).

Considering the work of the second author [32], we give the following relation between the Euler polynomials and Peters polynomials:

Substituting $t = e^z - 1$ into (5), we have the following functional equation:

$$F_P\left(t, e^z-1;\lambda,\mu\right) = \frac{1}{2^{\mu}} F_E\left(\lambda t, \frac{x}{\lambda};\mu\right).$$

Combining this functional equation with (1) and (5), we arrive at the following well-known result:

$$s_n(x;\lambda,\mu) = \frac{\lambda^n}{2^{\mu}} E_n^{(\mu)}\left(\frac{x}{\lambda}\right). \tag{6}$$

Peters polynomials are well known to have generalizations of the following polynomials and numbers: The Boole polynomials and numbers, the Bernoulli polynomials and numbers, the Euler polynomials and numbers, the Stirling numbers, the Changhee polynomials and numbers, and other well-known combinatorial polynomials and numbers (*cf.* [19, 20, 27–36]). For instance, substituting $\mu = 1$ into (5), we have the Boole polynomials: $\xi_n(x) = s_n(x;\lambda,1)$ (*cf.* [8], [26, pp. 113–117]).

The Bernoulli polynomials of the second kind are defined by means of the following generating functions:

$$F_{b_2}(t,x) = \frac{t}{\log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!} \tag{7}$$

and

$$b_n(x) = \int_x^{x+1} (z)_n \, dz \tag{8}$$

(*cf.* [26, pp. 113–117]). Substituting $x = 0$ into (7) or (8), we have the Bernoulli numbers of the second kind: $b_n = b_n(0)$ (*cf.* [3, 4, 8], [26, pp. 113–117]).

The Apostol-type Daehee numbers of higher order defined by means of the following generating function:

$$F_D(t,k) = \frac{(\log(1+\lambda t))^k}{(\lambda t)^k} = \sum_{n=0}^{\infty} D_n^{(k)}(\lambda)\frac{t^n}{n!} \tag{9}$$

(*cf.* [29], [30, Eq-(10a)]).

By combining (2) and (9), we have

$$\sum_{n=0}^{\infty} \lambda^k D_n^{(k)}(\lambda)\frac{t^{n+k}}{n!} = k!\sum_{n=0}^{\infty} S_1(n,k)\frac{(\lambda t)^n}{n!}.$$

After comparing the coefficients of $\frac{t^n}{n!}$ on the both sides of the above equation, a relation between the Stirling numbers of the first kind and Apostol-type Daehee numbers of higher order is given as follows:

$$S_1(n,k) = \binom{n}{k}\lambda^{k-n} D_{n-k}^{(k)}(\lambda). \tag{10}$$

Assuming that d is a positive integer. The Changhee polynomials of order d are defined by means of the following generating function:

$$F(t,x,d) = \frac{2^d(1+t)^x}{(2+t)^d} = \sum_{n=0}^{\infty} \mathrm{Ch}_n^{(d)}(x)\frac{t^n}{n!} \tag{11}$$

(*cf.* [15, 16]).

Substituting $t = e^z - 1$ into Eq. (11), we get the following functional equation:

$$F\big(e^z - 1, x, d\big) = F_E(z,x;d).$$

By using this equation, we get

$$\sum_{n=0}^{\infty} \mathrm{Ch}_n^{(d)}(x) F_{S_2}(z,n) = \sum_{m=0}^{\infty} E_m^{(d)}\frac{z^m}{m!}.$$

Combining the above equation with (4), we arrive at the following well-known identity:

$$E_m^{(d)} = \sum_{j=0}^{m} \mathrm{Ch}_n^{(d)} S_2(m,n) \tag{12}$$

(*cf.* [16, Theorem 2.3]).

Combining (6) and (12), we give a relation among the Peters polynomials, Changhee numbers of order d and the Stirling numbers of the second kind:

$$s_m(0;1,d) = \frac{1}{2^d} \sum_{j=0}^{m} \mathrm{Ch}_n^{(d)} S_2(m,n).$$

Substituting $d = 1$ into (11), we have the Changhee polynomials $\mathrm{Ch}_n(x) = \mathrm{Ch}_n^{(1)}(x)$. When $x = 0$, we have the Changhee numbers of order d: $\mathrm{Ch}_n^{(d)} = \mathrm{Ch}_n^{(d)}(0)$ (*cf.* [15, 16]).

Using Eq. (11), we have the following theorem.

Theorem 1 (*cf.* [16])

$$\mathrm{Ch}_n^{(d)}(x) = \sum_{j=0}^{n} \binom{n}{j} (x)_j \, \mathrm{Ch}_{n-j}^{(d)}. \tag{13}$$

Remark 1 Substituting $\lambda = \mu = 1$ into (5), we have

$$\mathrm{Ch}_n(x) = 2s_n(x;1,1)$$

(*cf.* [1, 3, 8, 15, 26]).

The Changhee polynomials of negative order are defined by means of the following generating function:

$$H(t,x,-k) = \frac{(1+t)^x(2+t)^k}{2^k} = \sum_{n=0}^{\infty} \mathrm{Ch}_n^{(-k)}(x)\frac{t^n}{n!}, \tag{14}$$

where k is a positive integer (*cf.* [9]).

Substituting $x = 0$ into (14), we have the Changhee numbers of negative order:

$$\mathrm{Ch}_n^{(-k)} = \mathrm{Ch}_n^{(-k)}(0).$$

By using Eq. (14), we have the following theorem.

Theorem 2 (*cf.* [9])

$$\mathrm{Ch}_n^{(-k)}(x) = \frac{1}{2^k} \sum_{j=0}^{k} \sum_{l=0}^{n} \binom{k}{j} \binom{n}{l} (j)_{n-l}(x)_l. \tag{15}$$

We summarize the results of this paper as follows:

In Sect. 2, we gave computation formulas and combinatorial sums for the Changhee numbers and polynomials of negative order. By using these formulas, some numerical values of the Changhee numbers and polynomials of negative order are given.

In Sect. 3, we give partial derivative equations for generating functions of the Changhee polynomials of order d. By using these equations, we derive some derivative formulas, identities and recurrence relations including the Changhee polynomial, the Daehee numbers, the Stirling numbers and also two open problems.

In Sect. 4, we give integral formulas for the Changhee polynomials and the Bernoulli numbers of the second kind. By using these formulas, we give finite combinatorial sums.

In Sect. 5, by using generating functions and their derivative formulas, we derive some identities and relations for the Changhee polynomials.

Finally, this paper is completed with the Conclusion.

2 Computation formulas and combinatorial sums for Changhee numbers and polynomials of negative order

By applying umbral calculus methods to the theory of polynomial sequences of binomial type polynomials and the Sheffer polynomials, involving the falling and rising factorial functions, various interesting and novel identities and relations for the Peters type polynomials, which are a member of the family of the Sheffer polynomials, have recently been given (*cf.* [1–38]). By using the Chu–Vandermonde identity and the falling factorial functions, we give some computation formulas and combinatorial sums for the Changhee numbers and polynomials of negative order.

Theorem 3 *Let $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Then we have*

$$\mathrm{Ch}_n^{(-k)} = \frac{(k)_n}{2^n}. \tag{16}$$

Proof We set

$$\sum_{n=0}^{\infty} \mathrm{Ch}_n^{(-k)} \frac{t^n}{n!} = \sum_{n=0}^{\infty} (k)_n \frac{(\frac{t}{2})^n}{n!}.$$

By combining the binomial series, the Taylor series for the function $(1+t)^z$, where $z \in \mathbb{C}$ and $|t| < 1$, with the above equation, we get

$$\sum_{n=0}^{\infty} \mathrm{Ch}_n^{(-k)} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \binom{k}{n} \left(\frac{t}{2}\right)^n = \left(1 + \frac{t}{2}\right)^k.$$

Combining the final equation with (14), we have

$$H(t, 0, -k) = \left(1 + \frac{t}{2}\right)^k.$$

Thus, the proof of the theorem is completed. □

By combining (16) with the following well-known Chu–Vandermonde identity:

$$(x+y)_n = \sum_{j=0}^{n} \binom{n}{j} (x)_j (y)_{n-j}$$

we get the following relation:

$$\mathrm{Ch}_n^{(-k-l)} = \frac{1}{2^n} \sum_{j=0}^{n} \binom{n}{j} (k)_j (l)_{n-j}.$$

After some calculations as in the previous equation, we arrive at the following theorem.

Theorem 4 *Let $n \in \mathbb{N}_0$ and $k, l \in \mathbb{N}$. Then we have*

$$\mathrm{Ch}_n^{(-k-l)} = \sum_{j=0}^{n} \binom{n}{j} \mathrm{Ch}_j^{(-k)} \mathrm{Ch}_{n-j}^{(-l)}. \tag{17}$$

Note that the proof of assertion (17) of Theorem 4 is also given by Eq. (14). That is, for $x = 0$, we have the following functional equation:

$$H(t, 0, -k-l) = H(t, 0, -k) H(t, 0, -l), \tag{18}$$

which leads us to the assertion (17) of Theorem 4. Let us briefly give a few steps of the second proof of assertion (17) of Theorem 4. Combining Eq. (18) with Eq. (14), we obtain

$$\sum_{n=0}^{\infty} \mathrm{Ch}_n^{(-k-l)}(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \mathrm{Ch}_n^{(-k)}(x) \frac{t^n}{n!} \sum_{n=0}^{\infty} \mathrm{Ch}_n^{(-l)}(x) \frac{t^n}{n!}.$$

Using the Cauchy rule for the product of series in the previous equation, the following relation is obtained:

$$\sum_{n=0}^{\infty} \mathrm{Ch}_n^{(-k-l)}(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} \mathrm{Ch}_j^{(-k)} \mathrm{Ch}_{n-j}^{(-l)} \frac{t^n}{n!}.$$

If the coefficients of $\frac{t^n}{n!}$ on both sides of this last equation are equalized, we arrive at the proof of assertion (17) of Theorem 4.

By combining (16) with the following well-known identity for the falling factorial functions:

$$(x)_n (x)_m = \sum_{j=0}^{n} \binom{n}{j} \binom{m}{j} j! (x)_{m+n-j} \tag{19}$$

we get the following relation:

$$\mathrm{Ch}_n^{(-k)} \mathrm{Ch}_m^{(-k)} = \frac{1}{2^{n+m}} \sum_{j=0}^{n} \binom{n}{j} \binom{m}{j} j! (k)_{m+n-j}.$$

After some calculations in the previous equation, we arrive at the following theorem:

Theorem 5 *Let $m, n \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Then we have*

$$\mathrm{Ch}_n^{(-k)} \mathrm{Ch}_m^{(-k)} = \sum_{j=0}^{n} \binom{n}{j} \binom{m}{j} \frac{j!}{2^j} \mathrm{Ch}_{m+n-j}^{(-k)}.$$

By using (16), we get the following combinatorial sum:

$$\sum_{n=0}^{k} \binom{k}{n} 2^n \mathrm{Ch}_n^{(-k)} = \sum_{n=0}^{k} \frac{((k)_n)^2}{n!}. \tag{20}$$

Combining (19) with (20), we obtain

$$\sum_{n=0}^{k} \binom{k}{n} 2^n \mathrm{Ch}_n^{(-k)} = \sum_{n=0}^{k} \sum_{j=0}^{n} \binom{n}{j}^2 \frac{j!}{n!} (k)_{2n-j}.$$

Combining the previous equation with (16), we arrive at the following theorem:

Theorem 6 *Let $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Then we have*

$$\sum_{n=0}^{k} \binom{k}{n} 2^n \mathrm{Ch}_n^{(-k)} = \sum_{n=0}^{k} \sum_{j=0}^{n} \binom{n}{j}^2 \frac{j! 2^{2n-j}}{n!} \mathrm{Ch}_{2n-j}^{(-k)}.$$

Combining (20) with (3), we arrive at the following corollary:

Corollary 1 *Let $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Then we have*

$$\sum_{n=0}^{k} \binom{k}{n} 2^n \mathrm{Ch}_n^{(-k)} = \sum_{n=0}^{k} \sum_{j=0}^{n} \binom{k}{n} S_1(n,j) k^j,$$

$$\sum_{n=0}^{k} \binom{k}{n} 2^n \mathrm{Ch}_n^{(-k)} = \sum_{n=0}^{k} \binom{k}{n} (k)_n,$$

and

$$\sum_{n=0}^{k} \binom{k}{n} (k)_n = \sum_{n=0}^{k} \sum_{j=0}^{n} \binom{k}{n} S_1(n,j) k^j.$$

By using (16), some values of the numbers $\mathrm{Ch}_n^{(-k)}$ are given as follows:

$$\mathrm{Ch}_0^{(-k)} = 1,$$

$$\mathrm{Ch}_1^{(-k)} = \frac{k}{2},$$

$$\mathrm{Ch}_2^{(-k)} = \frac{k^2 - k}{4},$$

$$\mathrm{Ch}_3^{(-k)} = \frac{k^3 - 3k^2 + 2k}{8},$$

$$\mathrm{Ch}_4^{(-k)} = \frac{k^4 - 6k^3 + 11k^2 - 6k}{16}, \qquad \ldots.$$

See also (*cf.* [9]).

For some special values of k, we have

$$\begin{aligned}
&\mathrm{Ch}_0^{(-1)} = 1, \quad \mathrm{Ch}_1^{(-1)} = \frac{1}{2}, \quad \mathrm{Ch}_2^{(-1)} = 0, \quad \mathrm{Ch}_3^{(-1)} = 0, \quad \mathrm{Ch}_4^{(-1)} = 0, \quad \ldots, \\
&\mathrm{Ch}_0^{(-2)} = 1, \quad \mathrm{Ch}_1^{(-2)} = 1, \quad \mathrm{Ch}_2^{(-2)} = \frac{1}{2}, \quad \mathrm{Ch}_3^{(-2)} = 0, \quad \mathrm{Ch}_4^{(-2)} = 0, \quad \ldots,
\end{aligned}$$

and

$$\mathrm{Ch}_0^{(-3)} = 1, \quad \mathrm{Ch}_1^{(-3)} = \frac{3}{2}, \quad \mathrm{Ch}_2^{(-3)} = \frac{3}{2}, \quad \mathrm{Ch}_3^{(-3)} = \frac{3}{4}, \quad \mathrm{Ch}_4^{(-3)} = 0, \quad \ldots.$$

It follows from the above computations that

$$\mathrm{Ch}_n^{(-k)} = 0 \quad \text{if } n > k.$$

By using (14), Equation (27) in [9], and (16), we get the following theorem:

Theorem 7 *Let $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Let $x \in \mathbb{R}$. Then we have*

$$\mathrm{Ch}_n^{(-k)}(x) = \sum_{j=0}^{n} \binom{n}{j} \frac{(k)_{n-j}}{2^{n-j}} (x)_j. \tag{21}$$

By using (21), some values of the numbers $\mathrm{Ch}_n^{(-k)}(x)$ are given as follows:

$$\begin{aligned}
\mathrm{Ch}_0^{(-k)}(x) &= 1, \\
\mathrm{Ch}_1^{(-k)}(x) &= x + \frac{k}{2}, \\
\mathrm{Ch}_2^{(-k)}(x) &= x^2 + (k-1)x + \frac{k^2 - k}{4}, \\
\mathrm{Ch}_3^{(-k)}(x) &= x^3 + \frac{3}{2}(k-2)x^2 + \frac{3k^2 - 9k + 8}{4}x + \frac{k^3 - 3k^2 + 2k}{8}, \\
\mathrm{Ch}_4^{(-k)}(x) &= x^4 + (2k-6)x^3 + \frac{3k^2 - 15k + 22}{2}x^2 \\
&\quad + \frac{k^3 - 6k^2 + 13k}{2}x + \frac{k^4 - 6k^3 + 11k^2 - 6k}{16}, \qquad \ldots
\end{aligned}$$

(*cf.* [9]).

For some special values of k, we have

$$\begin{aligned}
\mathrm{Ch}_0^{(-1)}(x) &= 1, \\
\mathrm{Ch}_1^{(-1)}(x) &= x + \frac{1}{2}, \\
\mathrm{Ch}_2^{(-1)}(x) &= x^2,
\end{aligned}$$

$$\mathrm{Ch}_3^{(-1)}(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x,$$
$$\mathrm{Ch}_4^{(-1)}(x) = x^4 - 4x^3 + 5x^2 + 4x, \qquad \ldots,$$
$$\mathrm{Ch}_0^{(-2)}(x) = 1,$$
$$\mathrm{Ch}_1^{(-2)}(x) = x + 1,$$
$$\mathrm{Ch}_2^{(-2)}(x) = x^2 + x + \frac{1}{2},$$
$$\mathrm{Ch}_3^{(-2)}(x) = x^3 + \frac{1}{2}x,$$
$$\mathrm{Ch}_4^{(-2)}(x) = x^4 - 2x^3 + 2x^2 + 5x, \qquad \ldots,$$

and

$$\mathrm{Ch}_0^{(-3)}(x) = 1,$$
$$\mathrm{Ch}_1^{(-3)}(x) = x + \frac{3}{2},$$
$$\mathrm{Ch}_2^{(-3)}(x) = x^2 + 2x + \frac{3}{2},$$
$$\mathrm{Ch}_3^{(-3)}(x) = x^3 + \frac{3}{2}x^2 + 2x + \frac{3}{4},$$
$$\mathrm{Ch}_4^{(-3)}(x) = x^4 + 2x^2 + 6x, \qquad \ldots.$$

3 Partial derivative equations of generating functions for the Changhee polynomials of higher order

In this section, we give partial derivative equations of Eq. (11). By using these partial derivative equations, we give derivative formulas, recurrence relations and combinatorial sums for the Changhee polynomials of higher order. These formulas and relations include the Changhee polynomials of higher order, the Daehee numbers, the Bernoulli polynomials and numbers of the second kind, and the Stirling numbers of the first kind. We also give two open problems related to the partial derivative equations and the Changhee polynomials.

3.1 Derivative formulas for the Changhee polynomials of higher order

Here, partial derivative equations of generating functions for the Changhee polynomials of higher order are given.

Differentiating both sides of Eq. (11) with respect to x, we get the following partial differential equations:

$$\frac{\partial^m}{\partial x^m}\left\{F(t,x,k)\right\} = m!F(t,x,k)F_{S_1}(t,m) \tag{22}$$

and

$$\frac{\partial^m}{\partial x^m}\left\{F(t,x,k)\right\} = t^m F(t,x,k)F_D(t,m). \tag{23}$$

Theorem 8 *Let $m, n \in \mathbb{N}_0$ with $n > m$. Then we have*

$$\sum_{j=0}^{n} \binom{n}{j} S_1(n-j,m)\,\mathrm{Ch}_j^{(k)}(x) = \binom{n}{m} \sum_{j=0}^{n-m} \binom{n-m}{j} D_{n-m-j}^{(m)}(1)\,\mathrm{Ch}_j^{(k)}(x)$$

Proof By combining (11) and (2) with (22), using the above equation, we get

$$\sum_{n=0}^{\infty} \frac{\partial^m}{\partial x^m}\left\{\mathrm{Ch}_n^{(k)}(x)\right\}\frac{t^n}{n!} = m! \sum_{n=0}^{\infty} S_1(n,m)\frac{t^n}{n!} \sum_{n=0}^{\infty} \mathrm{Ch}_n^{(k)}(x)\frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} \frac{\partial^m}{\partial x^m}\left\{\mathrm{Ch}_n^{(k)}(x)\right\}\frac{t^n}{n!} = m! \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} S_1(n-j,m)\,\mathrm{Ch}_j^{(k)}(x)\frac{t^n}{n!}.$$

After some elementary calculations, comparing the coefficients of $\frac{t^n}{n!}$ on the both sides of the above equation, we have

$$\frac{\partial^m}{\partial x^m}\left\{\mathrm{Ch}_n^{(k)}(x)\right\} = m! \sum_{j=0}^{n} \binom{n}{j} S_1(n-j,m)\,\mathrm{Ch}_j^{(k)}(x). \tag{24}$$

Similarly, by combining (11) and (9) with (23), we get

$$\sum_{n=0}^{\infty} \frac{\partial^m}{\partial x^m}\left\{\mathrm{Ch}_n^{(k)}(x)\right\}\frac{t^n}{n!} = t^m \sum_{n=0}^{\infty} D_n^{(m)}(1)\frac{t^n}{n!} \sum_{n=0}^{\infty} \mathrm{Ch}_n^{(k)}(x)\frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} \frac{\partial^m}{\partial x^m}\left\{\mathrm{Ch}_n^{(k)}(x)\right\}\frac{t^n}{n!} = \sum_{n=0}^{\infty} (n)_m \sum_{j=0}^{n-m} \binom{n-m}{j} D_{n-k-j}^{(m)}(1)\,\mathrm{Ch}_n^{(k)}(x)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on the both sides of the above equation, we get the following derivative formula for the Changhee polynomials:

$$\frac{\partial^m}{\partial x^m}\left\{\mathrm{Ch}_n^{(k)}(x)\right\} = (n)_m \sum_{j=0}^{n-m} \binom{n-m}{j} D_{n-m-j}^{(m)}(1)\,\mathrm{Ch}_j^{(k)}(x). \tag{25}$$

Combining (24) and (25), we arrive at the desired result. □

3.2 Recurrence relations for the Changhee polynomials of higher order and partial derivative equations for generating functions

Here, we give partial derivative equations of generating functions for the Changhee polynomials of higher order. By using these equations, we also give some new formulas of the Changhee polynomials of higher order.

Differentiating both sides of Eq. (11) with respect to t, we get the following partial differential equations:

$$\frac{\partial}{\partial t}\left\{F(t,x,k)\right\} = -\frac{(k)^1}{2}F(t,x,k+1) + xF(t,x-1,k), \tag{26}$$

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\{F(t,x,k)\} &= \frac{(k)^2}{2^2}F(t,x,k+2) - (k)^1(x)_1F(t,x-1,k+1)\\&\quad + (x)_2(t,x-2,k),\end{aligned} \tag{27}$$

$$\begin{aligned}\frac{\partial^3}{\partial t^3}\{F(t,x,k)\} &= -\frac{(k)^3}{2^3}F(t,x,k+3) + \frac{3}{2^2}(k)^2(x)_1F(t,x-1,k+2)\\&\quad - \frac{3}{2}(k)^1(x)_2F(t,x-2,k+1)\\&\quad + (x)_3F(t,x-3,k),\end{aligned} \tag{28}$$

and

$$\begin{aligned}\frac{\partial^4}{\partial t^4}\{F(t,x,k)\} &= \frac{(k)^4}{2^4}F(t,x,k+4) - \frac{1}{2}(k)^3(x)_1F(t,x-1,k+3)\\&\quad + \frac{3}{2}(k)^2(x)_2F(t,x-2,k+2)\\&\quad - 2(k)^1(x)_3F(t,x-3,k+1) + (x)_4F(t,x-4,k).\end{aligned} \tag{29}$$

If we continue to take m times derivative similar to the above way, we come up with the following problem.

Problem 1 Let $k \in \mathbb{N}_0$. Let $\alpha_j \in \mathbb{Q}$ numbers with $j \in \{1,2,3,\dots,m\}$. Then, $\frac{\partial^m}{\partial t^m}\{F(t,x,k)\}$ has the possible following forms:

If m is an even positive integer, we have

$$\frac{\partial^m}{\partial t^m}\{F(t,x,k)\} = \sum_{j=0}^{m}(-1)^j\alpha_j(k)^{m-j}(x)_jF(t,x-j,k+m-j). \tag{30}$$

If m is an odd positive integer, we have

$$\frac{\partial^m}{\partial t^m}\{F(t,x,k)\} = \sum_{j=0}^{m}(-1)^{j+1}\alpha_j(k)^{m-j}(x)_jF(t,x-j,k+m-j). \tag{31}$$

The previous question also arises how to find the coefficients α_j in the given equations (30) and (31).

Partial differential equations in the problem 1 have many important applications. By using special values of these equations, we obtain the explicit computational formulas involving recurrence relations for the Changhee polynomials of higher order. Therefore, we present the following claim:

Using (30) and (31), we obtain the following formulas:

If m is an even positive integer, we have

$$\mathrm{Ch}_{n+m}(x) = \sum_{j=0}^{m}(-1)^j\alpha_j(k)^{m-j}(x)_j\,\mathrm{Ch}_n^{(k+m-j)}(x-j), \tag{32}$$

and if m is an odd positive integer, we have

$$\mathrm{Ch}_{n+m}^{(k)}(x) = \sum_{j=0}^{m}(-1)^{j+1}\alpha_j(k)^{m-j}(x)_j\,\mathrm{Ch}_n^{(k+m-j)}(x-j), \tag{33}$$

where $\mathrm{Ch}_n^{(a+b)}(x)$ means that

$$\mathrm{Ch}_n^{(a+b)}(x) = \sum_{j=0}^{n} \binom{n}{j} \mathrm{Ch}_{n-j}^{(b)} \mathrm{Ch}_j^{(a)}(x), \tag{34}$$

Note that Eq. (34), easily obtained from Eq. (11), is a well-known formula for the Changhee polynomials of order $a+b$.

3.3 Some special values for Problem 1

Here, we give some special values of the partial differential equations and recurrence relations for the Changhee polynomials of higher order.

For $k = 1$, (26)–(29) reduce to the following partial differential equations, respectively:

$$\frac{\partial}{\partial t}\{F(t,x,1)\} = -\frac{1}{2}F(t,x,2) + xF(t,x-1,1), \tag{35}$$

$$\frac{\partial^2}{\partial t^2}\{F(t,x,1)\} = \frac{1}{2}F(t,x,3) - xF(t,x-1,2) + x(x-1)F(t,x-2,1), \tag{36}$$

$$\begin{aligned} \frac{\partial^3}{\partial t^3}\{F(t,x,1)\} = {} & -\frac{3}{4}F(t,x,4) + \frac{3}{2}xF(t,x-1,3) \\ & - \frac{3}{2}x(x-1)F(t,x-2,2) + x(x-1)(x-2)F(t,x-3,1), \end{aligned} \tag{37}$$

and

$$\begin{aligned} \frac{\partial^4}{\partial t^4}\{F(t,x,1)\} = {} & \frac{3}{2}F(t,x,5) - 3xF(t,x-1,4) + 3x(x-1)F(t,x-2,3) \\ & - 2x(x-1)(x-2)F(t,x-3,2) \\ & + x(x-1)(x-2)(x-3)F(t,x-4,1). \end{aligned} \tag{38}$$

Therefore, by using Problem 1, we get the following results.

Problem 2 Let $m \in \mathbb{N}_0$. Let $\alpha_j \in \mathbb{Q}$ with $j \in \{1,2,3,\dots,m\}$. Then $\frac{\partial^m}{\partial t^m}\{F(t,x)\}$ has the possible following forms:

If m is an even positive integer, we have

$$\frac{\partial^m}{\partial t^m}\{F(t,x,1)\} = \sum_{j=0}^{m}(-1)^j \alpha_j (m-j)!(x)_j F(t,x-j,m+1-j). \tag{39}$$

If m is an odd positive integer, we have

$$\frac{\partial^k}{\partial t^k}\{F(t,x,1)\} = \sum_{j=0}^{k}(-1)^{j+1} \alpha_j (m-j)!(x)_j F(t,x-j,k+1-j). \tag{40}$$

As with Problem 1, the following similar results can be given for Problem 2:

Problem 2 also arises how to find the coefficients α_j in the given equations (39) and (40).

By using Problem 2, we deduce the following results:

Using (39) and (40), we obtain the following relations:

If m is an even positive integer, we have

$$\mathrm{Ch}_{n+m}(x) = \sum_{j=0}^{m}(-1)^{j}\alpha_j(m-j)!(x)_j\,\mathrm{Ch}_n^{(m+1-j)}(x-j). \tag{41}$$

If m is an odd positive integer, we have

$$\mathrm{Ch}_{n+m}(x) = \sum_{j=0}^{m}(-1)^{j+1}\alpha_j(m-j)!(x)_j\,\mathrm{Ch}_n^{(m+1-j)}(x-j). \tag{42}$$

We set

$$\beta_j = \alpha_j(m-j)!,$$

where $j \in \{0,1,2,\dots,m\}$.

We compute few values of β_j, $j \in \{0,1,2,\dots,m\}$ as follows:

Substituting $m = 1$ into (40), we have

$$\frac{\partial}{\partial t}\bigl\{F(t,x,1)\bigr\} = -\beta_0 F(t,x,2) + \beta_1 x F(t,x-1,1).$$

By using the above equation, we get

$$\sum_{n=0}^{\infty}\mathrm{Ch}_{n+1}(x)\frac{t^n}{n!} = -\beta_0\sum_{n=0}^{\infty}\mathrm{Ch}_n^{(2)}(x)\frac{t^n}{n!} + \beta_1 x\sum_{n=0}^{\infty}\mathrm{Ch}_n(x-1)\frac{t^n}{n!}.$$

Comparing coefficients of the above equation with (35) and (42), since $\beta_0 = \frac{1}{2}$ and $\beta_1 = 1$, we obtain the following recurrence formula:

Theorem 9 *Let $n \in \mathbb{N}_0$. Then we have*

$$\mathrm{Ch}_{n+1}(x) = -\frac{1}{2}\,\mathrm{Ch}_n^{(2)}(x) + xCh_n(x-1).$$

Substituting $m = 2$ into (39), we have

$$\frac{\partial^2}{\partial t^2}\bigl\{F(t,x,1)\bigr\} = \beta_0 F(t,x,3) - \beta_1 x F(t,x-1,2) + \beta_2 x(x-1)F(t,x-2,1).$$

By using the above equation, we get

$$\begin{aligned}\sum_{n=0}^{\infty}\mathrm{Ch}_{n+2}(x)\frac{t^n}{n!} &= \beta_0\sum_{n=0}^{\infty}\mathrm{Ch}_n^{(3)}(x)\frac{t^n}{n!} - \beta_1 x\sum_{n=0}^{\infty}\mathrm{Ch}_n^{(2)}(x-1)\frac{t^n}{n!}\\ &\quad + \beta_2 x(x-1)\sum_{n=0}^{\infty}\mathrm{Ch}_n(x-2)\frac{t^n}{n!}.\end{aligned}$$

Comparing coefficients of the above equation with (36) and (41), since $\beta_0 = \frac{1}{2}$, $\beta_1 = 1$ and $\beta_2 = 1$, we obtain the following recurrence formula:

Theorem 10 *Let $n \in \mathbb{N}_0$. Then we have*

$$\mathrm{Ch}_{n+2}(x) = \frac{1}{2}\mathrm{Ch}_n^{(3)}(x) - xCh_n^{(2)}(x-1) + x(x-1)\mathrm{Ch}_n(x-2).$$

Substituting $m = 3$ into (40), we have

$$\frac{\partial^3}{\partial t^3}\{F(t,x,1)\} = -\beta_0 F(t,x,4) + \beta_1 x F(t,x-1,3) - \beta_2 x(x-1)F(t,x-2,2) + \beta_3 x(x-1)(x-2)F(t,x-3,1).$$

By using the above equation, we get

$$\begin{aligned}\sum_{n=0}^{\infty}\mathrm{Ch}_{n+3}(x)\frac{t^n}{n!} &= -\beta_0\sum_{n=0}^{\infty}\mathrm{Ch}_n^{(4)}(x)\frac{t^n}{n!} + \beta_1 x\sum_{n=0}^{\infty}\mathrm{Ch}_n^{(3)}(x-1)\frac{t^n}{n!} \\ &\quad - \beta_2 x(x-1)\sum_{n=0}^{\infty}\mathrm{Ch}_n^{(2)}(x-2)\frac{t^n}{n!} \\ &\quad + \beta_3 x(x-1)(x-2)\sum_{n=0}^{\infty}\mathrm{Ch}_n(x-3)\frac{t^n}{n!}.\end{aligned}$$

Comparing coefficients of the above equation with (37) and (42), since $\beta_0 = \frac{3}{4}$, $\beta_1 = \frac{3}{2}$, $\beta_2 = \frac{3}{2}$ and $\beta_3 = 1$, we obtain the following recurrence formula:

Theorem 11 *Let $n \in \mathbb{N}_0$. Then we have*

$$\begin{aligned}\mathrm{Ch}_{n+3}(x) &= -\frac{3}{4}\mathrm{Ch}_n^{(4)}(x) + \frac{3}{2}xCh_n^{(3)}(x-1) \\ &\quad - \frac{3}{2}x(x-1)\mathrm{Ch}_n^{(2)}(x-2) + x(x-1)(x-2)\mathrm{Ch}_n(x-3).\end{aligned}$$

Substituting $m = 4$ into (39), we get

$$\begin{aligned}\frac{\partial^4}{\partial t^4}\{F(t,x,1)\} &= \beta_0 F(t,x,5) - \beta_1 x F(t,x-1,4) \\ &\quad + \beta_2 x(x-1)F(t,x-2,3) - \beta_3 x(x-1)(x-2)F(t,x-3,2) \\ &\quad + \beta_4 x(x-1)(x-2)(x-3)F(t,x-4,1).\end{aligned}$$

Comparing coefficients of the above equation with (38) and (41), since $\beta_0 = \frac{3}{2}$, $\beta_1 = 3$, $\beta_2 = 3$, $\beta_3 = 2$ and $\beta_4 = 1$, we obtain the following recurrence formula.

Theorem 12 *Let $n \in \mathbb{N}_0$. Then we have*

$$\begin{aligned}\mathrm{Ch}_{n+4}(x) &= \frac{3}{2}\mathrm{Ch}_n^{(5)}(x) - 3xCh_n^{(4)}(x-1) + 3x(x-1)\mathrm{Ch}_n^{(3)}(x-2) \\ &\quad - 2x(x-1)(x-2)\mathrm{Ch}_n^{(2)}(x-3) \\ &\quad + x(x-1)(x-2)(x-3)\mathrm{Ch}_n(x-4).\end{aligned}$$

4 Integrals formulas for the Changhee polynomials

In this section, we give integral equations and integral formulas of Eq. (11) and the Changhee polynomials of higher order. By using these formulas, we derive and combinatorial sums including the Changhee polynomials, the Daehee numbers, the Bernoulli polynomials and numbers of the second kind and the Stirling numbers of the first kind.

Integrating Eq. (11) from z to $z+1$ with respect to x, we have

$$\int_z^{z+1} F(t,x,1)\,dx = \frac{F(t,z+1,1)-F(t,z,1)}{t} F_{b_2}(t,0).$$

By using the above integral equation, we get

$$\sum_{n=0}^{\infty}\int_z^{z+1} \mathrm{Ch}_n(x)\frac{t^n}{n!}\,dx = \sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}\left(\frac{\mathrm{Ch}_j(z+1)-\mathrm{Ch}_j(z)}{j+1}\right)b_{n-j}(0)\frac{t^n}{n!}.$$

By comparing the coefficients of $\frac{t^n}{n!}$ on the both sides of the above equation, we arrive at the following theorem.

Theorem 13 *Let $n \in \mathbb{N}_0$. Then we have*

$$\int_z^{z+1} \mathrm{Ch}_n(x)\,dx = \sum_{j=0}^{n}\binom{n}{j}\left(\frac{\mathrm{Ch}_j(z+1)-\mathrm{Ch}_j(z)}{j+1}\right)b_{n-j}(0). \tag{43}$$

Theorem 14 *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{j=0}^{n}\binom{n}{j}\mathrm{Ch}_{n-j}^{(d)}\,b_j(z) = \sum_{j=0}^{n}\binom{n}{j}b_{n-j}(0)\left(\frac{\mathrm{Ch}_j(z+1)-\mathrm{Ch}_j(z)}{j+1}\right).$$

Proof Substituting $d=1$ into (11), after that integrating this equation from z to $z+1$ with respect to x, we get

$$\int_z^{z+1} \mathrm{Ch}_n(x)\,dx = \sum_{j=0}^{n}\binom{n}{j}\mathrm{Ch}_{n-j}\int_z^{z+1}(x)_j\,dx.$$

Combining the above equation with (8), we get

$$\int_z^{z+1} \mathrm{Ch}_n(x)\,dx = \sum_{j=0}^{n}\binom{n}{j}\mathrm{Ch}_{n-j}\,b_j(z). \tag{44}$$

Combining (44) with (43), we arrive at the desired result. □

5 Identities, formulas and combinatorial sums including special numbers and polynomials

In this section, by using Eq. (11) with its functional equations, we derive recurrence relations and combinatorial sums including the Changhee polynomials, the Daehee numbers, the Bernoulli polynomials and numbers of the second kind, and the Stirling numbers of the first kind.

By using (11), we get

$$F(t,x+1,d)=\sum_{n=0}^{\infty}\mathrm{Ch}_n^{(d)}(x+1)\frac{t^n}{n!}.$$

From the above equation, we have

$$\sum_{n=0}^{\infty}\mathrm{Ch}_n^{(d)}(x+1)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\binom{x+1}{n}t^n\sum_{n=0}^{\infty}\mathrm{Ch}_n^{(d)}\frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty}\mathrm{Ch}_n^{(d)}(x+1)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{x+1}{j}\mathrm{Ch}_{n-j}^{(d)}\frac{t^n}{(n-j)!}.$$

By comparing the coefficients of t^n on the both sides of the above equation, we obtain the following identity:

$$\mathrm{Ch}_n^{(d)}(x+1)=\sum_{j=0}^{n}\mathrm{Ch}_{n-j}^{(d)}\frac{n!}{(n-j)!}\binom{x+1}{j}. \tag{45}$$

By using (11), we also have

$$2^d\sum_{n=0}^{\infty}(x+1)_n\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{j=0}^{d}\binom{d}{j}2^{d-j}(n)_j\,\mathrm{Ch}_{n-j}^{(d)}(x+1)\frac{t^n}{n!}.$$

By comparing the coefficients of t^n on the both sides of the above equation, we obtain the following recurrence relation for the Changhee polynomials of order d:

Theorem 15 *Let $n\in\mathbb{N}_0$. Then we have*

$$(x+1)_n=\sum_{j=0}^{d}2^{-j}\binom{d}{j}(n)_j\,\mathrm{Ch}_{n-j}^{(d)}(x+1). \tag{46}$$

Substituting $d=1$ and $x=z-1$ into (46), we have the following recurrence relation for the Changhee polynomials:

$$\mathrm{Ch}_n(z)+\frac{n}{2}\mathrm{Ch}_{n-1}(z)=(z)_n \tag{47}$$

(*cf.* [8, 30]).

Combining (45) and (46) with the following well-known identity:

$$\binom{x+1}{n}=\binom{x}{n-1}+\binom{x}{n},$$

we arrive at the following theorem.

Theorem 16 *Let $n \in \mathbb{N}$. Then we have*

$$\sum_{j=0}^{d} 2^{-j} \binom{d}{j} (n)_j \operatorname{Ch}_{n-j}^{(d)}(x+1) = \sum_{j=0}^{d} 2^{-j} \binom{d}{j} \bigl((n-1)_j \operatorname{Ch}_{n-1-j}^{(d)}(x) + (n)_j \operatorname{Ch}_{n-j}^{(d)}(x)\bigr).$$

Theorem 17 *Let k be a positive integers. Then we have*

$$\operatorname{Ch}_n^{(d_1+d_2+\cdots+d_k)}(x_1+x_2+\cdots+x_k) = \sum_{n_1+n_2+\cdots+n_k=n} \binom{n}{n_1, n_2, \ldots n_{k-1}, n-n_1-n_2-\cdots-n_{k-1}} \prod_{j=1}^{k} \operatorname{Ch}_{n_j}^{(d_j)}(x_j), \tag{48}$$

where

$$\binom{n}{n_1, n_2, \ldots n_{k-1}, n-n_1-n_2-\cdots-n_{k-1}} = \frac{n!}{n_1!n_2!\cdots n_{k-1}!(n-n_1-n_2-\cdots-n_{k-1})!}$$

and

$$\sum_{n_1+n_2+\cdots+n_k=n} = \sum_{n_1=0}^{n} \sum_{n_2=0}^{k-n_1} \cdots \sum_{n_k=0}^{n-n_1-n_2-\cdots-n_{k-1}}.$$

Proof We set the following equation:

$$F(t, x_1+x_2+\cdots+x_k, d_1+d_2+\cdots+d_k) = \prod_{j=0}^{k} F(t, x_j, d_j).$$

Combining the above eq*uation with Eq. (11), we get

$$\sum_{n=0}^{\infty} \operatorname{Ch}_n^{(d_1+d_2+\cdots+d_k)}(x_1+x_2+\cdots+x_k) \frac{t^n}{n!} = \prod_{j=1}^{k} \sum_{n_j=0}^{\infty} \operatorname{Ch}_{n_j}^{(d_j)}(x_j) \frac{t^{n_j}}{n_j!}.$$

By using Cauchy product rule in the right side of the above equation, we get

$$\begin{aligned}\sum_{n=0}^{\infty} \operatorname{Ch}_n^{(d_1+d_2+\cdots+d_k)}(x_1+x_2+\cdots+x_k) \frac{t^n}{n!} &= \sum_{n=0}^{\infty} \sum_{n_1+n_2+\cdots+n_k=n} \binom{n}{n_1, n_2, \ldots n_{k-1}, n-n_1-n_2-\cdots-n_{k-1}} \\ &\quad \times \prod_{j=1}^{k} \operatorname{Ch}_{n_j}^{(d_j)}(x_j) \frac{t^n}{n!}.\end{aligned}$$

By comparing the coefficients of $\frac{t^n}{n!}$ on the both sides of the above equation, we get the desired result. □

Note that Substituting $k = 2$ into (48), we arrive at equation (34). That is, we also have

$$\mathrm{Ch}_n^{(d_1+d_2)}(x_1 + x_2) = \sum_{j=0}^{n} \binom{n}{j} (x_2)_{n-j} \mathrm{Ch}_j^{(d_1+d_2)}(x_1).$$

If $x_1 = 0$, the above equation reduces to Theorem 2.8 in the work of Kim et al. [16].

Substituting $x = 0$ into (48), we arrive at Eq. (2.6) of the work of Kim et al. [16].

6 Inequalities related to the Changhee numbers of negative order and the Stirling numbers of the second kind

In this section, by using functional equation of the generating functions, we give combinatorial sums and two inequalities involving the Changhee numbers of negative order and the Stirling numbers of the second kind.

The lower bound for the Stirling numbers of the second kind was given by Comtet [3], as follows:

$$S_2(m,k) \geq k^{m-k} \tag{49}$$

(see also [36, 39]).

The upper bound for the Stirling numbers of the second kind was given by Comtet as follows:

$$S_2(m,k) \leq k^{m-k} \binom{m-1}{k-1} \tag{50}$$

(see also [36, 39]).

Theorem 18 *Let $m, k \in \mathbb{N}$. Then we have*

$$\sum_{n=0}^{m} \sum_{j=0}^{n} (-1)^j \binom{n}{j} j! 2^{n+k-j} \mathrm{Ch}_n^{(-k)} S_2(m,j) \leq k! k^{m-k} \binom{m-1}{k-1}.$$

Proof We set the following functional equation:

$$k! H\left(e^t - 3, 0, -k\right) = 2^k \sum_{n=0}^{\infty} \sum_{j=0}^{n} (-1)^j \binom{n}{j} j! F_{S_2}(t,j).$$

Combining the above equation wit (4) and (14), we obtain

$$k! \sum_{n=0}^{\infty} S_2(m,k) \frac{t^m}{m!} \sum_{m=0}^{\infty} \sum_{n=0}^{m} \sum_{j=0}^{n} (-1)^j \binom{n}{j} j! 2^{n+k-j} \mathrm{Ch}_n^{(-k)} S_2(m,j) \frac{t^m}{m!}.$$

By comparing the coefficients of $\frac{t^m}{m!}$ on the both sides of the above equation, we obtain the following identity:

$$S_2(m,k) = \frac{1}{k!}\sum_{n=0}^{m}\sum_{j=0}^{n}(-1)^j\binom{n}{j}j!2^{n+k-j}\,\mathrm{Ch}_n^{(-k)}\,S_2(m,j). \tag{51}$$

Combining (50) with (51), we get

$$\frac{1}{k!}\sum_{n=0}^{m}\sum_{j=0}^{n}(-1)^j\binom{n}{j}j!2^{n+k-j}\,\mathrm{Ch}_n^{(-k)}\,S_2(m,j) \le k^{m-k}\binom{m-1}{k-1}.$$

Thus, the proof of the theorem is completed. □

Theorem 19 *Let $m,k \in \mathbb{N}$. Then we have*

$$\sum_{n=0}^{m}\sum_{j=0}^{n}(-1)^j\binom{n}{j}j!2^{n+k-j}\,\mathrm{Ch}_j^{(-k)}\,S_2(m,j) \ge k!k^{m-k}.$$

Proof Combining (49) with (51), we get

$$\frac{1}{k!}\sum_{n=0}^{m}\sum_{j=0}^{n}(-1)^j\binom{n}{j}j!2^{n+k-j}\,\mathrm{Ch}_n^{(-k)}\,S_2(m,j) \ge k^{m-k}.$$

Thus, the proof of the theorem is completed. □

7 Conclusions

As we emphasized in the introduction, generating functions for special numbers and polynomials with their derivative equations (PDEs, ODEs, and SDEs) and functional equations are the most important fields of quantum physics, mathematical physics, engineering and other applied sciences, especially mathematics. By using generating functions for special numbers and polynomials with their derivative equations (PDEs and ODEs) and functional equations, we give computations of many new formulas, relations, identities, and combinatorial sums involving the Bernoulli numbers and polynomials of the second kind, the Euler numbers and polynomials, the Stirling numbers, the Peters polynomials, the Boole polynomials and numbers, the Daehee numbers, and also the Changhee polynomials. By using partial derivative equations for generating functions of the Changhee polynomials, we give two open questions. Some special values of these questions are given. We also give some recurrence relations including the Changhee polynomial, the Daehee numbers, the Stirling numbers. Integral representations and integral formulas for the Changhee polynomials are given. Two inequalities including combinatorial sums, the Changhee numbers of negative order, and the Stirling numbers of the second kind are given. The interesting and new results given in this paper have the potential to be used in the main areas mentioned above, especially in mathematics.

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