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On the norms of circulant and r-circulant matrices with the hyperharmonic Fibonacci numbers

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Abstract

In this paper, we study norms of circulant and r-circulant matrices involving harmonic Fibonacci and hyperharmonic Fibonacci numbers. We obtain inequalities by using matrix norms.

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Keywords: hyperharmonic Fibonacci number; r-Circulant matrix; matrix norm

1 Introduction

The circulant and r-circulant matrices have a connection to signal processing, probability, numerical analysis, coding theory, and many other areas. An $n \times n$ matrix C_r is called an r-circulant matrix defined as follows:

$$C_r = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ rc_{n-2} & rc_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ rc_1 & rc_2 & rc_3 & \cdots & rc_{n-1} & c_0 \end{pmatrix}.$$

Since the matrix C_r is determined by its row elements and r, we denote $C_r = \mathrm{Circ}(c_0, c_1, c_2, \ldots, c_{n-1})$. In particular for $r = 1$

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ c_1 & c_2 & c_3 & \cdots & c_{n-1} & c_0 \end{pmatrix}$$

is called a circulant matrix and we denote it for brevity by $C = \mathrm{Circ}(c_0, c_1, c_2, \ldots, c_{n-1})$. The eigenvalues of C are

$$\lambda_j = \sum_{i=0}^{n-1} c_i \left(w^j\right)^i, \tag{1}$$

where $w = e^{\frac{2\pi i}{n}}$ and $i = \sqrt{-1}$.

Many authors have investigated the norms of circulant and r-circulant matrices. In [1], Solak studied the lower and upper bounds for the spectral norms of circulant matrices with classical Fibonacci and Lucas numbers entries. In [2], Kocer *et al.* obtained norms of circulant and semicirculant matrices with Horadams numbers. In [3], Zhou *et al.* gave the spectral norms of circulant-type matrices involving binomial coefficients and harmonic numbers. In [4], Zhou calculated spectral norms for circulant matrices with binomial coefficients combined with Fibonacci and Lucas number entries. In [5], Shen and Cen have given upper and lower bounds for the spectral norms of r-circulant matrices with classical Fibonacci and Lucas number entries. In [6], Bahşı and Solak computed the spectral norms of circulant and r-circulant matrices with the hyper-Fibonacci and hyper-Lucas numbers. In [7], Jiang and Zhou studied spectral norms of even-order r-circulant matrices.

Motivated by the above papers, we compute the spectral norms and Euclidean norm of circulant and r-circulant matrices with the harmonic and hyperharmonic Fibonacci entries. The scheme of this paper is as follows. In Section 2, we present some definitions, preliminaries, and lemmas related to our study. In Section 3, we calculate spectral norms of circulant matrix with harmonic Fibonacci entries. Moreover, we obtain the Euclidean norms of r-circulant matrices and give lower and upper bounds for the spectral norms of r-circulant matrices with harmonic and hyperharmonic Fibonacci entries.

2 Preliminaries

The Fibonacci numbers F_n are defined by the following recurrence relation for $n \geq 1$:

$$F_{n+1} = F_n + F_{n-1},$$

where $F_0 = 0$, $F_1 = 1$. In [8], the authors investigated the finite sum of the reciprocals of Fibonacci numbers,

$$\mathbb{F}_n = \sum_{k=1}^{n} \frac{1}{F_k},$$

which are called harmonic Fibonacci numbers. Then they gave a combinatoric identity related to harmonic Fibonacci numbers as follows:

$$\sum_{k=0}^{n-1} F_{k-1}\mathbb{F}_k = F_n\mathbb{F}_n - n. \tag{2}$$

Moreover, in [8], they defined hyperharmonic Fibonacci numbers for $n, r \geq 1$

$$\mathbb{F}_n^{(r)} = \sum_{k=1}^{n} \mathbb{F}_k^{(r-1)},$$

where $\mathbb{F}_n^{(0)} = \frac{1}{F_n}$ and $\mathbb{F}_0 = 0$. At this point, we give some definitions and lemmas related to our study.

Definition 1 Let $A = (a_{ij})$ be any $m \times n$ matrix. The Euclidean norm of matrix A is

$$\|A\|_E = \sqrt{\left(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2\right)}.$$

Definition 2 Let $A = (a_{ij})$ be any $m \times n$ matrix. The spectral norm of matrix A is

$$\|A\|_2 = \sqrt{\max_{1\le i\le n} \lambda_i\bigl(A^H A\bigr)},$$

where $\lambda_i(A^H A)$ is an eigenvalue of $A^H A$ and A^H is the conjugate transpose of matrix A.

Then the following inequalities hold for the Euclidean norm and the spectral norm:

$$\frac{1}{\sqrt{n}}\|A\|_E \le \|A\|_2 \le \|A\|_E, \tag{3}$$

$$\|A\|_2 \le \|A\|_E \le \sqrt{n}\|A\|_2. \tag{4}$$

Lemma 1 [9] *Let A and B be two $m \times n$ matrices. Then we have*

$$\|A \circ B\|_2 \le \|A\|_2\|B\|_2,$$

where $A \circ B$ is the Hadamard product of A and B.

Lemma 2 [9] *Let A and B be two $n \times m$ matrices. We have*

$$\|A \circ B\|_2 \le r_1(A)c_1(B),$$

where

$$r_1(A) = \max_{1\le i\le m}\sqrt{\sum_{j=1}^{n}|a_{ij}|^2},$$

$$c_1(B) = \max_{1\le j\le n}\sqrt{\sum_{i=1}^{m}|b_{ij}|^2}.$$

Definition 3 [10] The difference operator of $f(x)$ is defined as

$$\Delta f(x) = f(x+1) - f(x).$$

Definition 4 [10] A function $f(x)$ with the property that $\Delta f(x) = g(x)$ is called the anti-difference operator of $g(x)$.

Lemma 3 [10] *If $\Delta f(x) = g(x)$, then*

$$\sum_{a}^{b} g(x)\delta_x = \sum_{x=a}^{b-1} g(x) = f(b) - f(a).$$

Lemma 4 [10] *We have*

$$\sum_a^b u(x)\Delta v(x)\delta_x = u(x)v(x)|_a^{b+1} - \sum_a^b v(x+1)\Delta u(x)\delta_x. \tag{5}$$

Lemma 5 [10] *For $m \neq -1$ we have*

$$\sum x^{\underline{m}}\delta_x = \frac{x^{\underline{m+1}}}{m+1},$$

where $x^{\underline{m}} = x(x-1)(x-2)\cdots(x-m+1)$.

3 Main results

Theorem 1 [8] *Let $C_1 = \mathrm{Circ}(\mathbb{F}_0, \mathbb{F}_1, \mathbb{F}_2, \dots, \mathbb{F}_{n-1})$ be an $n \times n$ circulant matrix. The spectral norm of C_1 is*

$$\|C_1\|_2 = n\mathbb{F}_n - \sum_{k=0}^{n-1} \frac{k+1}{F_{k+1}}.$$

Theorem 2 [8] *Let $C^{(k)} = \mathrm{Circ}(\mathbb{F}_0^{(k)}, \mathbb{F}_1^{(k)}, \mathbb{F}_2^{(k)}, \dots, \mathbb{F}_{n-1}^{(k)})$ be an $n \times n$ circulant matrix. The spectral norm of $C^{(k)}$ is*

$$\|C^{(k)}\|_2 = \mathbb{F}_{n-1}^{(k+1)}.$$

Theorem 3 *Let*

$$C = \mathrm{Circ}(F_{-1}\mathbb{F}_0, F_0\mathbb{F}_1, \dots, F_{n-2}\mathbb{F}_{n-1}) \tag{6}$$

be an $n \times n$ circulant matrix. Then the spectral norm of the matrix C is

$$\|C\|_2 = F_n\mathbb{F}_n - n.$$

Proof Since C is a circulant matrix, from (1), for all $t = 0, 1, \dots, s-1$,

$$\lambda_t(C) = \sum_{i=0}^{s-1} F_{i-1}\mathbb{F}_i \left(w^t\right)^i.$$

Then, for $t = 0$,

$$\lambda_0(C) = \sum_{i=0}^{s-1} F_{i-1}\mathbb{F}_i \tag{7}$$

and from (2), $\lambda_0(C) = F_n\mathbb{F}_n - n$. Hence, for $1 \leq m \leq n-1$, we have

$$|\lambda_m| = \left|\sum_{i=0}^{s-1} F_{i-1}\mathbb{F}_i \left(w^t\right)^i\right| \leq \left|\sum_{i=0}^{s-1} F_{i-1}\mathbb{F}_i\right| \left|\left(w^t\right)^i\right| \leq \sum_{i=0}^{s-1} F_{i-1}\mathbb{F}_i. \tag{8}$$

Since C is a normal matrix, we have

$$\|C\|_2 = \max_{0\le m\le n-1} |\lambda_m|. \tag{9}$$

From (7), (8), (9), and (2), we have

$$\|C\|_2 = F_n\mathbb{F}_n - n.$$ □

Corollary 1 *We have*

$$\sqrt{\sum_{k=0}^{n-1} F_{k-1}^2\mathbb{F}_n^2} \le F_n\mathbb{F}_n - n \le \sqrt{n\sum_{k=0}^{n-1} F_{k-1}^2\mathbb{F}_n^2}.$$

Proof The proof is trivial from Definition 1 and the relation between the Euclidean norm and the spectral norm in (3). □

Theorem 4 *Let*

$$C_r^{(k)} = \operatorname{Circ}\bigl(\mathbb{F}_0^{(k)}, \mathbb{F}_1^{(k)}, \ldots, \mathbb{F}_{n-1}^{(k)}\bigr) \tag{10}$$

be an $n \times n$ r-circulant matrix. The Euclidean norm of $C_r^{(k)}$ is

$$\bigl\|C_r^{(k)}\bigr\|_E = \Biggl[\frac{n}{2}\bigl(n+1+(n-1)|r|^2\bigr)\bigl(\mathbb{F}_n^{(k)}\bigr)^2 - \frac{1}{2}\sum_{s=0}^{n-1}(s+1)\bigl(2n+s\bigl(|r|^2-1\bigr)\bigr)\bigl(\mathbb{F}_{s+1}^{(k-1)} + 2\mathbb{F}_s^{(k)}\bigr)\mathbb{F}_{s+1}^{(k-1)}\Biggr]^{\frac{1}{2}}.$$

Proof From the definition of the Euclidean norm we have

$$\begin{aligned}\bigl\|C_r^{(k)}\bigr\|_E &= \Biggl[\sum_{s=0}^{n-1}(n-s)\bigl(\mathbb{F}_s^{(k)}\bigr)^2 + \sum_{s=0}^{n-1} s|r|^2\bigl(\mathbb{F}_s^{(k)}\bigr)^2\Biggr]^{\frac{1}{2}} \\ &= \Biggl[\sum_{s=0}^{n-1}\bigl(n+s\bigl(|r|^2-1\bigr)\bigr)\bigl(\mathbb{F}_s^{(k)}\bigr)^2\Biggr]^{\frac{1}{2}}.\end{aligned}$$

Now we will use the property of the difference operator in Lemma 4. Let $u(s) = (\mathbb{F}_s^{(k)})^2$ and $\Delta v(s) = n + s(|r|^2 - 1)$. Then using the definition of the hyperharmonic Fibonacci numbers we obtain $\Delta u(s) = \mathbb{F}_{s+1}^{(k-1)}(\mathbb{F}_{s+1}^{(k-1)} + 2\mathbb{F}_s^{(k)})$ and $v(s) = ns + \frac{s^2}{2}(|r|^2 - 1)$. By using (5), we have

$$\bigl\|C_r^{(k)}\bigr\|_E = \Biggl[\frac{n}{2}\bigl(n+1+(n-1)|r|^2\bigr)\bigl(\mathbb{F}_n^{(k)}\bigr)^2 - \frac{1}{2}\sum_{s=0}^{n-1}(s+1)\bigl(2n+s\bigl(|r|^2-1\bigr)\bigr)\bigl(\mathbb{F}_{s+1}^{(k-1)} + 2\mathbb{F}_s^{(k)}\bigr)\mathbb{F}_{s+1}^{(k-1)}\Biggr]^{\frac{1}{2}}.$$ □

Corollary 2 *Let $C_r = \operatorname{Circ}(\mathbb{F}_0, \mathbb{F}_1, \ldots, \mathbb{F}_{n-1})$ be an $n \times n$ r-circulant matrix. The Euclidean norm of C_r is*

$$\|C_r\|_E = \Bigg[\bigg(n^2 + \frac{n^{\underline{2}}}{2}\big(|r|^2 - 1\big)\bigg)\mathbb{F}_n^2 - \sum_{s=0}^{n-1}\bigg(n(s+1) + \frac{(s+1)^{\underline{2}}}{2}\big(|r|^2 - 1\big)\bigg)\bigg(2\mathbb{F}_s + \frac{1}{F_{s+1}}\bigg)\frac{1}{F_{s+1}}\Bigg]^{\frac{1}{2}}.$$

Proof It is clear that the proof can be completed if we take $k = 1$ in Theorem 4. □

Corollary 3 [8] *Let $C_1 = \operatorname{Circ}(\mathbb{F}_0, \mathbb{F}_1, \ldots, \mathbb{F}_{n-1})$ be an $n \times n$ matrix. The Euclidean norm is*

$$\|C_1\|_E = \Bigg[n^2\mathbb{F}_n^2 - n\sum_{k=0}^{n-1}\frac{k+1}{F_{k+1}}\bigg(2\mathbb{F}_k + \frac{1}{F_{k+1}}\bigg)\Bigg]^{\frac{1}{2}}.$$

Proof It is easily seen that the proof can be completed if we take $k = r = 1$ in Theorem 4. □

Now we give upper and lower bounds for the spectral norms of r-circulant matrices.

Theorem 5 *Let $C_r^{(k)} = \operatorname{Circ}(\mathbb{F}_0^{(k)}, \mathbb{F}_1^{(k)}, \ldots, \mathbb{F}_{n-1}^{(k)})$ be an $n \times n$ r-circulant matrix.*

(i) *If $|r| \geq 1$, then*

$$\frac{1}{\sqrt{n}}\mathbb{F}_{n-1}^{(k+1)} \leq \|C_r^{(k)}\|_2 \leq |r|\sqrt{n-1}\,\mathbb{F}_{n-1}^{(k+1)}.$$

(ii) *If $|r| < 1$, then*

$$\frac{|r|}{\sqrt{n}}\mathbb{F}_{n-1}^{(k+1)} \leq \|C_r^{(k)}\|_2 \leq \sqrt{n-1}\,\mathbb{F}_{n-1}^{(k+1)}.$$

Proof Since we have the matrix

$$C_r^{(k)} = \begin{pmatrix} \mathbb{F}_0^{(k)} & \mathbb{F}_1^{(k)} & \mathbb{F}_2^{(k)} & \cdots & \mathbb{F}_{n-2}^{(k)} & \mathbb{F}_{n-1}^{(k)} \\ r\mathbb{F}_{n-1}^{(k)} & \mathbb{F}_0^{(k)} & \mathbb{F}_1^{(k)} & \cdots & \mathbb{F}_{n-3}^{(k)} & \mathbb{F}_{n-2}^{(k)} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ r\mathbb{F}_2^{(k)} & r\mathbb{F}_3^{(k)} & r\mathbb{F}_4^{(k)} & \cdots & \mathbb{F}_0^{(k)} & \mathbb{F}_1^{(k)} \\ r\mathbb{F}_1^{(k)} & r\mathbb{F}_2^{(k)} & r\mathbb{F}_3^{(k)} & \cdots & r\mathbb{F}_{n-1}^{(k)} & \mathbb{F}_0^{(k)} \end{pmatrix},$$

we have

$$\|C_r^{(k)}\|_E = \sqrt{\sum_{s=0}^{n-1}(n-s)\big(\mathbb{F}_s^{(k)}\big)^2 + \sum_{s=0}^{n-1}s|r|^2\big(\mathbb{F}_s^{(k)}\big)^2}.$$

(i) In [8], for the sum of the squares of hyperharmonic Fibonacci numbers, we have

$$\frac{1}{\sqrt{n}}\mathbb{F}_{n-1}^{(r+1)} \leq \sqrt{\sum_{k=0}^{n-1}\big(\mathbb{F}_k^{(r)}\big)^2} \leq \mathbb{F}_{n-1}^{(r+1)}. \tag{11}$$

Since $|r| \geq 1$ and by (11), we have

$$\|C_r^{(k)}\|_E \geq \sqrt{\sum_{s=0}^{n-1}(n-s)\big(\mathbb{F}_s^{(k)}\big)^2 + \sum_{s=0}^{n-1} s\big(\mathbb{F}_s^{(k)}\big)^2} \geq \sqrt{n\sum_{s=0}^{n-1}\big(\mathbb{F}_s^{(k)}\big)^2} \geq \mathbb{F}_{n-1}^{(k+1)}.$$

From (3)

$$\frac{1}{\sqrt{n}}\mathbb{F}_{n-1}^{(k+1)} \leq \|C_r^{(k)}\|_2.$$

On the other hand, let the matrices A and B be defined by

$$A = \begin{pmatrix} \mathbb{F}_0^{(k)} & 1 & 1 & \cdots & 1 & 1 \\ r & \mathbb{F}_0^{(k)} & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ r & r & r & \cdots & \mathbb{F}_0^{(k)} & 1 \\ r & r & r & \cdots & r & \mathbb{F}_0^{(k)} \end{pmatrix}$$

and

$$B = \begin{pmatrix} \mathbb{F}_0^{(k)} & \mathbb{F}_1^{(k)} & \mathbb{F}_2^{(k)} & \cdots & \mathbb{F}_{n-2}^{(k)} & \mathbb{F}_{n-1}^{(k)} \\ \mathbb{F}_{n-1}^{(k)} & \mathbb{F}_0^{(k)} & \mathbb{F}_1^{(k)} & \cdots & \mathbb{F}_{n-3}^{(k)} & \mathbb{F}_{n-2}^{(k)} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \mathbb{F}_2^{(k)} & \mathbb{F}_3^{(k)} & \mathbb{F}_4^{(k)} & \cdots & \mathbb{F}_0^{(k)} & \mathbb{F}_1^{(k)} \\ \mathbb{F}_1^{(k)} & \mathbb{F}_2^{(k)} & \mathbb{F}_3^{(k)} & \cdots & \mathbb{F}_{n-1}^{(k)} & \mathbb{F}_0^{(k)} \end{pmatrix}.$$

That is, $C_r^{(k)} = A \circ B$. Then we obtain

$$r_1(A) = \max_{1\leq i\leq n}\sqrt{\sum_{j=1}^{n}|a_{ij}|^2} = \sqrt{\sum_{j=1}^{n}|a_{nj}|^2} = \sqrt{(n-1)|r|^2}$$

and

$$c_1(B) = \max_{1\leq j\leq n}\sqrt{\sum_{i=1}^{n}|b_{ij}|^2} = \sqrt{\sum_{i=1}^{n}|b_{in}|^2} = \sqrt{\sum_{s=0}^{n-1}\big(\mathbb{F}_s^{(k)}\big)^2}.$$

Hence, from (11) and Lemma 1, we have

$$\|C_r^{(k)}\|_2 \leq |r|\sqrt{n-1}\mathbb{F}_{n-1}^{(k+1)}.$$

Thus, we have

$$\frac{1}{\sqrt{n}}\mathbb{F}_{n-1}^{(k+1)} \leq \|C_r^{(k)}\|_2 \leq |r|\sqrt{n-1}\mathbb{F}_{n-1}^{(k+1)}.$$

(ii) From $|r| < 1$ and from (11), we have

$$\begin{aligned}\|C_r^{(k)}\|_E &= \sqrt{\sum_{s=0}^{n-1}(n-s)\left(\mathbb{F}_s^{(k)}\right)^2 + \sum_{s=0}^{n-1} s|r|^2\left(\mathbb{F}_s^{(k)}\right)^2} \\ &\geq \sqrt{\sum_{s=0}^{n-1}(n-s)|r|^2\left(\mathbb{F}_s^{(k)}\right)^2 + \sum_{s=0}^{n-1} s|r|^2\left(\mathbb{F}_s^{(k)}\right)^2} \\ &= |r|\sqrt{n\sum_{s=0}^{n-1}\left(\mathbb{F}_s^{(k)}\right)^2} \\ &\geq |r|\mathbb{F}_{n-1}^{(k+1)}.\end{aligned}$$

From (3),

$$\frac{|r|}{\sqrt{n}}\mathbb{F}_{n-1}^{(k+1)} \leq \|C_r^{(k)}\|_2.$$

On the other hand, let the matrices A and B be defined by

$$A = \begin{pmatrix} \mathbb{F}_0^{(k)} & 1 & 1 & \cdots & 1 & 1 \\ r & \mathbb{F}_0^{(k)} & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ r & r & r & \cdots & \mathbb{F}_0^{(k)} & 1 \\ r & r & r & \cdots & r & \mathbb{F}_0^{(k)} \end{pmatrix}$$

and

$$B = \begin{pmatrix} \mathbb{F}_0^{(k)} & \mathbb{F}_1^{(k)} & \mathbb{F}_2^{(k)} & \cdots & \mathbb{F}_{n-2}^{(k)} & \mathbb{F}_{n-1}^{(k)} \\ \mathbb{F}_{n-1}^{(k)} & \mathbb{F}_0^{(k)} & \mathbb{F}_1^{(k)} & \cdots & \mathbb{F}_{n-3}^{(k)} & \mathbb{F}_{n-2}^{(k)} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \mathbb{F}_2^{(k)} & \mathbb{F}_3^{(k)} & \mathbb{F}_4^{(k)} & \cdots & \mathbb{F}_0^{(k)} & \mathbb{F}_1^{(k)} \\ \mathbb{F}_1^{(k)} & \mathbb{F}_2^{(k)} & \mathbb{F}_3^{(k)} & \cdots & \mathbb{F}_{n-1}^{(k)} & \mathbb{F}_0^{(k)} \end{pmatrix}.$$

Thus $C_r^{(k)} = A \circ B$. Then we obtain

$$r_1(A) = \max_{1\leq i\leq n}\sqrt{\sum_{j=1}^{n}|a_{ij}|^2} = \sqrt{\left(\mathbb{F}_0^{(k)}\right)^2 + n - 1} = \sqrt{n-1}$$

and

$$c_1(B) = \max_{1\leq j\leq n}\sqrt{\sum_{i=1}^{n}|b_{ij}|^2} = \sqrt{\sum_{s=0}^{n-1}\left(\mathbb{F}_s^{(k)}\right)^2}.$$

Therefore, from (11) and Lemma 1, we have

$$\|C_r^{(k)}\|_2 \leq \sqrt{n-1}\mathbb{F}_{n-1}^{(k+1)}.$$

Thus, we have

$$\frac{|r|}{\sqrt{n}}\mathbb{F}_{n-1}^{(k+1)} \leq \left\| C_r^{(k)} \right\|_2 \leq \sqrt{n-1}\mathbb{F}_{n-1}^{(k+1)}. \qquad \square$$

Corollary 4 *Let* $C_r = \operatorname{Circ}(\mathbb{F}_0, \mathbb{F}_1, \ldots, \mathbb{F}_{n-1})$ *be an* $n \times n$ *r-circulant matrix.*

(i) *If* $|r| \geq 1$, *then*

$$\frac{1}{\sqrt{n}}\mathbb{F}_{n-1}^{(2)} \leq \|C_r\|_2 \leq |r|\sqrt{n-1}\mathbb{F}_{n-1}^{(2)}.$$

(ii) *If* $|r| < 1$, *then*

$$\frac{|r|}{\sqrt{n}}\mathbb{F}_{n-1}^{(2)} \leq \|C_r\|_2 \leq \sqrt{n-1}\mathbb{F}_{n-1}^{(2)}.$$

Proof It is easily seen that the proof can be completed if we take $k = 1$ in Theorem 5. $\square$

4 Numerical examples

In this section, we present some numerical examples by using Maple 11.

Example 1 Let $C = \operatorname{Circ}(F_{-1}\mathbb{F}_0, F_0\mathbb{F}_1, \ldots, F_{n-2}\mathbb{F}_{n-1})$ be as in (6). We obtain the spectral norms of some $n \times n$ C matrices, with the aid of Theorem 3 (see Table 1).

Example 2 Let $C_r^{(k)} = \operatorname{Circ}(\mathbb{F}_0^{(k)}, \mathbb{F}_1^{(k)}, \ldots, \mathbb{F}_4^{(k)})$ be 5×5 r-circulant matrix as in (10). We obtain Euclidean norms of $C_r^{(k)}$ for some values of r and k, with the aid of Theorem 4 (see Table 2).

Example 3 Let $C_r^{(k)} = \operatorname{Circ}(\mathbb{F}_0^{(k)}, \mathbb{F}_1^{(k)}, \ldots, \mathbb{F}_4^{(k)})$ be 5×5 r-circulant matrix as in (10). We obtain some lower and upper bounds for the spectral norms of $C_r^{(k)}$ for some values of r and k, with the aid of Theorem 5 (see Tables 3 and 4).

Table 1 Spectral norms of *C*

n	$\|C\|_2$
$n = 5$	$0.1016666667 \times 10^{2}$
$n = 10$	$0.1731757972 \times 10^{3}$
$n = 50$	$0.4228842484 \times 10^{11}$
$n = 100$	$0.1190154990 \times 10^{22}$
$n = 500$	$0.4684460937 \times 10^{105}$
$n = 1{,}000$	$0.1460426641 \times 10^{210}$

Table 2 Euclidean norms of $C_r^{(k)}$ for $n = 5$

k/r	$k = 1$	$k = 2$	$k = 3$	$k = 4$
$r = -2$	$\frac{\sqrt{9{,}935}}{6}$	$\frac{\sqrt{61{,}598}}{6}$	$\frac{\sqrt{246{,}743}}{6}$	$\frac{\sqrt{755{,}966}}{6}$
$r = -0.5$	$\frac{\sqrt{7{,}415}}{12}$	$\frac{\sqrt{37{,}127}}{12}$	$\frac{\sqrt{136{,}007}}{12}$	$\frac{\sqrt{397{,}559}}{12}$
$r = 0.1$	$\frac{\sqrt{133{,}655}}{60}$	$\frac{\sqrt{593{,}351}}{60}$	$\frac{\sqrt{2{,}038{,}631}}{60}$	$\frac{\sqrt{5{,}736{,}887}}{60}$
$r = 0.9$	$\frac{\sqrt{306{,}055}}{60}$	$\frac{\sqrt{1{,}709{,}431}}{60}$	$\frac{\sqrt{6{,}577{,}111}}{60}$	$\frac{\sqrt{19{,}743{,}847}}{60}$
$r = 1$	$\frac{\sqrt{3{,}470}}{6}$	$\frac{\sqrt{19{,}745}}{6}$	$\frac{5\sqrt{3{,}062}}{6}$	$\frac{\sqrt{230{,}705}}{6}$
$r = 1.1$	$\frac{\sqrt{392{,}255}}{60}$	$\frac{\sqrt{2{,}267{,}471}}{60}$	$\frac{\sqrt{8{,}846{,}351}}{60}$	$\frac{\sqrt{26{,}747{,}327}}{60}$
$r = 10$	$\frac{\sqrt{216{,}815}}{6}$	$\frac{\sqrt{1{,}400{,}894}}{6}$	$\frac{\sqrt{5{,}692{,}919}}{6}$	$\frac{\sqrt{17{,}564{,}318}}{6}$

Table 3 Some lower and upper bounds for the spectral norms of $C_r^{(k)}$ for $n = 5$ and $|r| \geq 1$

	$\lvert r\rvert \geq 1$	$\frac{1}{\sqrt{n}}\mathbb{F}_{n-1}^{(k+1)}$	$\lVert C_r^{(k)}\rVert_2$	$\lvert r\rvert\sqrt{n-1}\mathbb{F}_{n-1}^{(k+1)}$
$k = 1$	$r = -2$	3.726779962	13.07393997	33.33333333
	$r = 1$	3.726779962	8.333333332	16.66666667
	$r = 1.1$	3.726779962	6.187213310	18.33333333
$k = 2$	$r = -2$	7.975309119	29.94984421	71.33333333
	$r = 1$	7.975309119	17.83333333	35.66666667
	$r = 1.1$	7.975309119	19.01179206	39.23333333
$k = 3$	$r = -2$	14.45990625	56.50736302	129.3333333
	$r = 1$	14.45990625	32.33333334	64.66666667
	$r = 1.1$	14.45990625	34.60097132	71.13333333
$k = 4$	$r = -2$	23.62778496	94.70523788	211.3333333
	$r = 1$	23.62778496	52.83333332	105.6666667
	$r = 1.1$	23.62778496	56.68238740	116.2333333

Table 4 Some lower and upper bounds for the spectral norms of $C_r^{(k)}$ for $n = 5$ and $|r| < 1$

	$\lvert r\rvert < 1$	$\frac{\lvert r\rvert}{\sqrt{n}}\mathbb{F}_{n-1}^{(k+1)}$	$\lVert C_r^{(k)}\rVert_2$	$\sqrt{n-1}\mathbb{F}_{n-1}^{(k+1)}$
$k = 1$	$r = -0.5$	1.863389981	6.051069352	16.66666667
	$r = 0.1$	0.372677996	5.912862964	16.66666667
	$r = 0.9$	3.354101966	7.874554252	16.66666667
$k = 2$	$r = -0.5$	3.987654558	13.14607243	35.66666667
	$r = 0.1$	0.797530912	12.64953828	35.66666667
	$r = 0.9$	7.177778206	16.74610810	35.66666667
$k = 3$	$r = -0.5$	7.229953125	24.39979647	64.66666667
	$r = 0.1$	1.445990625	23.46328510	64.66666667
	$r = 0.9$	13.01391563	30.26866878	64.66666667
$k = 4$	$r = -0.5$	11.81389248	40.74250279	105.6666667
	$r = 0.1$	2.362778496	39.32798158	105.6666667
	$r = 0.9$	21.26500646	49.37728355	105.6666667

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Each of the authors, NT, and CK, contributed to each part of this work equally and read and approved the final version of the manuscript.

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