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L_p Harmonic radial combinations of star bodies

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Abstract

For star bodies, the L_p harmonic radial combinations were defined and studied in several papers. In this paper, we study the mean chord of L_p harmonic radial combinations of star bodies and get an upper bound for dual mixed volumes of L_p harmonic radial combination of star bodies and their polar bodies. Furthermore, we obtain a dual Urysohn type inequality and a dual Bieberbach type inequality.

MSC: 52A20; 52A40

Keywords: star body; L_p harmonic radial combination; Firey linear combination; mean chord; mean width

1 Introduction

The classical Brunn-Minkowski theory originated with Minkowski when he combined his concept of mixed volume with the Brunn-Minkowski inequality, which is the core of convex geometric analysis. This theory was developed from a few basic concepts such as support function, vector addition, and volume. Since Firey introduced his new L_p addition in 1960s (see [1]), the new L_p Brunn-Minkowski theory was born in Lutwak's papers [2, 3] and it has witnessed a rapid growth (see, *e.g.*, [4–13]).

In the 1970s, Lutwak introduced the dual mixed volume and hence developed the dual Brunn-Minkowski theory, which helped achieving a major breakthrough in the solution of the Busemann-Petty problem in the 1990s. The L_p harmonic radial combination of convex bodies was first investigated by Firey (see [14, 15]). Then, the L_p harmonic radial combination was extended to star bodies by Lutwak [3], and it plays a key role in the dual L_p Brunn-Minkowski theory.

For star bodies, the L_p harmonic radial combination was introduced and studied in several papers (see, *e.g.*, [1, 3, 16–20]). The aim of this paper is to study them further, that is, we mainly investigate the mean chord of L_p harmonic radial combination of star bodies.

Let $\mathscr{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in $\mathbb{R}^n$ and $\mathscr{K}_o^n$ denote the subset of $\mathscr{K}^n$ consisting of all convex bodies that contain the origin in their interiors. Let $\mathscr{S}_o^n$ denote the set of star bodies (star-shaped, continuous radial functions) in $\mathbb{R}^n$ containing the origin in their interiors. The unit ball in $\mathbb{R}^n$ and its surface will be denoted by B and S^{n-1}, respectively. The volume of B will be denoted by ω_n, the $(n-1)$-dimensional volume α_{n-1} of S^{n-1} is $\alpha_{n-1} = n\omega_n$.

The *Minkowski addition* of two convex bodies K and L is defined as

$$K + L = \{x + y : x \in K, y \in L\}.$$

The *scalar multiplication* αK of K, where $\alpha \geq 0$, is defined as

$$\alpha K = \{\alpha x : x \in K\}.$$

For each direction $u \in S^{n-1}$, the support function $h(K,u)$ of the convex body K can be defined by $h(K,u) = \max\{u \cdot x : x \in K\}$, where $u \cdot x$ denotes the usual inner product of x and u in $\mathbb{R}^n$. The radial function $\rho(K,u)$ of the star body K is $\rho(K,u) = \sup\{\lambda > 0 : \lambda u \in K\}$ for $u \in S^{n-1}$. Usually, we note $\rho_K(u) = \rho(K,u)$. The polar body of a convex body K, denoted by K^*, is another convex body defined by $K^* = \{y : x \cdot y \leq 1 \text{ for all } x \in K\}$. For $K \in \mathscr{K}_o^n$, the polar body has the well-known property that

$$h\bigl(K^*, u\bigr) = \frac{1}{\rho(K,u)}, \qquad \rho\bigl(K^*, u\bigr) = \frac{1}{h(K,u)}.$$

For real $p \geq 1$, $K, L \in \mathscr{K}_o^n$, and $\alpha, \beta \geq 0$ (not both zero), the *Firey linear combination*, $\alpha \cdot K +_p \beta \cdot L \in \mathscr{K}_o^n$, was defined by (see [2, 21])

$$h(\alpha \cdot K +_p \beta \cdot L, u)^p = \alpha h(K,u)^p + \beta h(L,u)^p, \quad u \in S^{n-1}. \tag{1.1}$$

In [22], the mean width of the Firey linear combinations of convex bodies was studied, and the lower bound of the mean width of the Firey linear combinations of convex body and its polar body was given.

For real $p \geq 1$, $K, L \in \mathscr{S}_o^n$, and $\alpha, \beta \geq 0$ (not both zero), the L_p *harmonic radial combination*, $\alpha \cdot K \widehat{+}_p \beta \cdot L \in \mathscr{S}_o^n$, was defined by (see [3])

$$\rho(\alpha \cdot K \widehat{+}_p \beta \cdot L, u)^{-p} = \alpha \rho(K,u)^{-p} + \beta \rho(L,u)^{-p}, \quad u \in S^{n-1}. \tag{1.2}$$

In this paper, we give some good properties of L_p harmonic radial combination of star bodies from the definition directly. Besides these properties, we also establish an upper bound for dual mixed volumes $\widetilde{V}_i(\cdot,\cdot)$ of L_p harmonic radial combination of star bodies and their polar bodies as follows.

Theorem 1.1 *Let $K \in \mathscr{K}_o^n$, real $p \geq 1$, and $\alpha, \beta \geq 0$, then for n is even*

$$\widetilde{V}_{\frac{n}{2}}\bigl(\alpha \cdot K \widehat{+}_p \beta \cdot K^*, \alpha \cdot K^* \widehat{+}_p \beta \cdot K\bigr) \leq \omega_n,$$

with equality if and only if K is a unit ball centered at the origin.

In [23], Hadwiger defined the mean width $\overline{b}(K)$ of $K \in \mathscr{K}_o^n$. Here we prove the following.

Theorem 1.2 (Dual Urysohn type inequality) *Let $K, L \in \mathscr{K}_o^n$, real $p \geq 1$, and $\alpha, \beta \geq 0$ (not both zero), then*

$$2^n \omega_n \overline{b}(\alpha \cdot K \widehat{+}_p \beta \cdot L)^{-n} \leq V\bigl(\alpha \cdot K^* +_p \beta \cdot L^*\bigr),$$

the equality holds if and only if $\alpha \cdot K \widehat{+}_p \beta \cdot L$ is an ellipsoid.

This immediately yields the following inequality.

Theorem 1.3 (Dual Bieberbach type inequality) *Let $K, L \in \mathscr{K}_o^n$, real $p \geq 1$, and $\alpha, \beta \geq 0$ (not both zero), then*

$$2^n \omega_n D(\alpha \cdot K \widehat{+}_p \beta \cdot L)^{-n} \leq V\bigl(\alpha \cdot K^* +_p \beta \cdot L^*\bigr),$$

where $D(K)$ denotes the diameter of K and the equality holds if and only if $\alpha \cdot K \widehat{+}_p \beta \cdot L$ is an ellipsoid.

2 Preliminaries

2.1 Mixed volumes and mean width

Let $K_1, \ldots, K_m$ be compact convex sets in $\mathbb{R}^n$ and $\lambda_1, \ldots, \lambda_m \geq 0$. The volume of $\lambda_1 K_1 + \cdots + \lambda_m K_m$ is a homogeneous nth degree polynomial in $\lambda_1, \ldots, \lambda_m$,

$$V(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{i_1,\ldots,i_n=1}^{m} V(K_{i_1}, \ldots, K_{i_n}) \lambda_{i_1} \cdots \lambda_{i_n}.$$

The coefficients $V(K_{i_1}, \ldots, K_{i_n})$ are nonnegative, symmetric in the indices, and are called mixed volumes of $K_{i_1}, \ldots, K_{i_n}$.

If $K_1, \ldots, K_n \in \mathscr{K}^n$, the mixed surface area measure $S(K_1, \ldots, K_{n-1}; \cdot)$ is the unique finite Borel measure on S^{n-1} such that for all $K \in \mathscr{K}^n$,

$$V(K, K_1, \ldots, K_{n-1}) = \frac{1}{n} \int_{S^{n-1}} h(K, u)\, dS(K_1, \ldots, K_{n-1}; u).$$

Let

$$V_i(K, L) = V(\underbrace{K, \ldots, K}_{n-i}, \underbrace{L, \ldots, L}_{i}).$$

If L is the unit ball B, then the mixed volumes $V(\underbrace{K, \ldots, K}_{n-i}, \underbrace{B, \ldots, B}_{i}) = V_i(K, B)$ are called the quermassintegrals of K and denoted by $W_i(K)$. The quermassintegrals are generalizations of the surface area and the volume. Indeed, it can be shown that

$$W_0(K) = V(K), \qquad nW_1(K) = S(K), \qquad W_n(K) = \omega_n, \qquad \frac{2}{\omega_n} W_{n-1}(K) = \overline{b}(K). \quad (2.1)$$

Here $\overline{b}(K)$ is the mean width of $K \in \mathscr{K}_o^n$, defined by Hadwiger (see [23]),

$$\overline{b}(K) = \frac{2}{n\omega_n} \int_{S^{n-1}} h(K, u)\, d\sigma(u),$$

where $d\sigma$ is the $(n-1)$-dimensional volume element on S^{n-1}, *i.e.*, the area element on S^{n-1}. Furthermore, the mixed width-integrals, $A(K_1, \ldots, K_n)$, of $K_1, \ldots, K_n \in \mathscr{K}^n$ was also defined by Lutwak (see [24])

$$A(K_1, K_2, \ldots, K_n) = \frac{1}{n} \int_{S^{n-1}} b(K_1, u) b(K_2, u) \cdots b(K_n, u)\, d\sigma(u), \quad (2.2)$$

where $b(K, u) = \frac{h(K,u)+h(K,-u)}{2}$ is half the width of K in the direction u.

2.2 Dual mixed volumes and mean chord

Let $L_i \in \mathscr{S}_o^n$, for $1 \le i \le n$, the dual mixed volumes $\widetilde{V}(L_1, L_2, \ldots, L_n)$ were defined by (see [25, 26])

$$\widetilde{V}(L_1, L_2, \ldots, L_n) = \frac{1}{n} \int_{S^{n-1}} \rho_{L_1}(u) \rho_{L_2}(u) \cdots \rho_{L_n}(u)\, d\sigma.$$

Let

$$\widetilde{V}_i(K, L) = \widetilde{V}(\underbrace{K, \ldots, K}_{n-i}, \underbrace{L, \ldots, L}_{i}),$$

then

$$\widetilde{V}_i(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho_K^{n-i}(u) \rho_L^i(u)\, d\sigma. \tag{2.3}$$

We shall also introduce the dual concept of the mean width of a convex body: for $L \in \mathscr{S}_o^n$, the mean chord of L, $\widetilde{d}(L)$, can be defined by

$$\widetilde{d}(L) = \frac{2}{n\omega_n} \int_{S^{n-1}} \rho(L, u)\, d\sigma(u). \tag{2.4}$$

Furthermore, the mixed chord-integrals $B(L_1, \ldots, L_n)$ of $L_1, \ldots, L_n \in \mathscr{S}_o^n$ were defined by Lu (see [27])

$$B(L_1, \ldots, L_n) = \frac{1}{n} \int_{S^{n-1}} d(L_1, u) \cdots d(L_n, u)\, d\sigma(u), \tag{2.5}$$

where $d(L, u) = \frac{\rho(L,u)+\rho(L,-u)}{2}$ is half the chord of L in the direction u.

3 Main results and proofs

In the following, we obtain some good properties and inequalities for the L_p harmonic radial combinations of star bodies from the definitions directly.

Theorem 3.1 *Let $K, L \in \mathscr{S}_o^n$, real $p \ge 1$, $\alpha, \beta \ge 0$, and $\alpha + \beta = 1$. Then*

$$\widetilde{d}(\alpha \cdot K \widehat{+}_p \beta \cdot L) \le \alpha \widetilde{d}(K) + \beta \widetilde{d}(L).$$

Proof According to the definition of L_p harmonic radial combination of star bodies (1.2) and the fact that $f(x) = x^{-\frac{1}{p}}$ is convex, we have

$$\rho(\alpha \cdot K \widehat{+}_p \beta \cdot L, u) \le \alpha \rho(K, u) + \beta \rho(L, u), \quad \alpha + \beta = 1. \tag{3.1}$$

So, using the definition of mean chord (2.4), we have

$$\begin{aligned} \widetilde{d}(\alpha \cdot K \widehat{+}_p \beta \cdot L) &= \frac{2}{n\omega_n} \int_{S^{n-1}} \rho(\alpha \cdot K \widehat{+}_p \beta \cdot L, u)\, d\sigma(u) \le \frac{2}{n\omega_n} \int_{S^{n-1}} \alpha \rho(K, u)\, d\sigma(u) \\ &\quad + \frac{2}{n\omega_n} \int_{S^{n-1}} \beta \rho(L, u)\, d\sigma(u) = \alpha \widetilde{d}(K) + \beta \widetilde{d}(L). \end{aligned}$$

This completes the proof. □

Theorem 3.2 (Positive multisublinear) *Let $K, L \in \mathscr{K}_o^n$, real $p \geq 1$, $\alpha, \beta \geq 0$, and $\alpha + \beta = 1$. Then for any $K_2, \ldots, K_n \in \mathscr{K}^n$,*

$$A(\alpha \cdot K +_p \beta \cdot L, K_2, \ldots, K_n) \geq \alpha A(K, K_2, \ldots, K_n) + \beta A(L, K_2, \ldots, K_n). \tag{3.2}$$

Proof According to the definition of Firey linear combination of convex bodies (1.1) and the fact that $f(x) = x^{\frac{1}{p}}$ is concave, we have

$$h(\alpha \cdot K +_p \beta \cdot L, u) \geq \alpha h(K, u) + \beta h(L, u), \quad \alpha + \beta = 1.$$

Then

$$\begin{aligned} b(\alpha \cdot K +_p \beta \cdot L, u) &= \frac{h(\alpha \cdot K +_p \beta \cdot L, u) + h(\alpha \cdot K +_p \beta \cdot L, -u)}{2} \\ &\geq \frac{1}{2}\bigl(\alpha h(K, u) + \beta h(L, u)\bigr) + \frac{1}{2}\bigl(\alpha h(K, -u) + \beta h(L, -u)\bigr) \\ &= \alpha b(K, u) + \beta b(L, u). \end{aligned}$$

So using definition (2.2), we can get

$$\begin{aligned} A(\alpha \cdot K +_p \beta \cdot L, K_2, \ldots, K_n) &= \frac{1}{n} \int_{S^{n-1}} b(\alpha \cdot K +_p \beta \cdot L, u) b(K_2, u) \cdots b(K_n, u)\, d\sigma(u) \\ &\geq \frac{1}{n} \int_{S^{n-1}} \alpha b(K, u) b(K_2, u) \cdots b(K_n, u)\, d\sigma(u) \\ &\quad + \frac{1}{n} \int_{S^{n-1}} \beta b(L, u) b(K_2, u) \cdots b(K_n, u)\, d\sigma(u) \\ &= \alpha A(K, K_2, \ldots, K_n) + \beta A(L, K_2, \ldots, K_n). \end{aligned}$$

This completes the proof. □

Just like Theorem 3.2, we have one more general property than that of Theorem 3.1 as follows. It is also the dual of inequality (3.2).

Theorem 3.3 (Positive multisublinear) *Let $K, L \in \mathscr{S}_o^n$, $p \geq 1$, $\alpha, \beta \geq 0$, and $\alpha + \beta = 1$. Then for any $K_2, \ldots, K_n \in \mathscr{S}_o^n$,*

$$B(\alpha \cdot K \widehat{+}_p \beta \cdot L, K_2, \ldots, K_n) \leq \alpha B(K, K_2, \ldots, K_n) + \beta B(L, K_2, \ldots, K_n).$$

Proof As in the proof of Theorem 3.2, by (3.1) we have

$$\begin{aligned} d(\alpha \cdot K \widehat{+}_p \beta \cdot L, u) &= \frac{\rho(\alpha \cdot K \widehat{+}_p \beta \cdot L, u) + \rho(\alpha \cdot K \widehat{+}_p \beta \cdot L, -u)}{2} \\ &\leq \frac{1}{2}\bigl(\alpha \rho(K, u) + \beta \rho(L, u)\bigr) \\ &\quad + \frac{1}{2}\bigl(\alpha \rho(K, -u) + \beta \rho(L, -u)\bigr) \\ &= \alpha d(K, u) + \beta d(L, u). \end{aligned}$$

From definition (2.5) we have

$$\begin{aligned}
&B(\alpha\cdot K\widehat{+}_p\beta\cdot L,K_2,\ldots,K_n)\\
&\quad=\frac{1}{n}\int_{S^{n-1}} d(\alpha\cdot K\widehat{+}_p\beta\cdot L,u)\,d(K_2,u)\cdots d(K_n,u)\,d\sigma(u)\\
&\quad\leq\frac{1}{n}\int_{S^{n-1}} \alpha\,d(K,u)\,d(K_2,u)\cdots d(K_n,u)\,d\sigma(u)\\
&\qquad+\frac{1}{n}\int_{S^{n-1}} \beta\,d(L,u)\,d(K_2,u)\cdots d(K_n,u)\,d\sigma(u)\\
&\quad=\alpha B(K,K_2,\ldots,K_n)+\beta B(L,K_2,\ldots,K_n).
\end{aligned}$$

This completes the proof. □

Next, we give the proof of Theorem 1.1 which was illustrated in Section 1. We shall prove a generalized form of an upper bound for the dual mixed volume.

Theorem 3.4 *Let $K\in\mathscr{K}_o^n$, $p\geq 1$, and $\alpha,\beta\geq 0$, then*

$$\widetilde{V}_i\bigl(\alpha\cdot K\widehat{+}_p\beta\cdot K^*,\alpha\cdot K^*\widehat{+}_p\beta\cdot K\bigr)\leq R^{n-2i}\omega_n$$

with equality if and only if K is the unit ball centered at the origin, where $R=\max\{\rho(\alpha\cdot K\widehat{+}_p\beta\cdot K^*,u),u\in S^{n-1}\}$.

Proof From the arithmetic-geometric mean inequality, we have

$$\rho(\alpha\cdot K\widehat{+}_p\beta\cdot L,u)^{-p}=\alpha\rho(K,u)^{-p}+\beta\rho(L,u)^{-p}\geq\rho(K,u)^{-\alpha p}\rho(L,u)^{-\beta p},$$

that is,

$$\rho(\alpha\cdot K\widehat{+}_p\beta\cdot L,u)\leq\rho(K,u)^{\alpha}\rho(L,u)^{\beta},$$

where the equality holds if and only if $\rho(K,u)=\rho(L,u)$.

If we let $L=K^*$ and use $h(K^*,u)=\frac{1}{\rho(K,u)}$, then we have

$$\begin{aligned}
\rho\bigl(\alpha\cdot K\widehat{+}_p\beta\cdot K^*,u\bigr)&\leq\rho(K,u)^{\alpha}\rho\bigl(K^*,u\bigr)^{\beta}\\
&=\frac{\rho(K,u)^{\alpha}}{h(K,u)^{\beta}}\leq\frac{h(K,u)^{\alpha}}{h(K,u)^{\beta}}.
\end{aligned}$$

The second inequality follows since $\rho(K,u)\leq h(K,u)$.

In the same manner, we have

$$\rho\bigl(\alpha\cdot K^*\widehat{+}_p\beta\cdot K,u\bigr)\leq\frac{h(K,u)^{\beta}}{h(K,u)^{\alpha}}.$$

Then

$$\rho\bigl(\alpha\cdot K\widehat{+}_p\beta\cdot K^*,u\bigr)\rho\bigl(\alpha\cdot K^*\widehat{+}_p\beta\cdot K,u\bigr)\leq 1.$$

Using definition (2.3) of dual mixed volume, we have

$$\begin{aligned}
&\widetilde{V}_i\bigl(\alpha\cdot K\widehat{+}_p\beta\cdot K^*,\alpha\cdot K^*\widehat{+}_p\beta\cdot K\bigr)\\
&\quad=\frac{1}{n}\int_{S^{n-1}}\rho\bigl(\alpha\cdot K\widehat{+}_p\beta\cdot K^*,u\bigr)^{n-i}\rho\bigl(\alpha\cdot K^*\widehat{+}_p\beta\cdot K,u\bigr)^{i}\,du\\
&\quad\le\frac{1}{n}\int_{S^{n-1}}\max_{u\in S^{n-1}}\rho\bigl(\alpha\cdot K\widehat{+}_p\beta\cdot K^*,u\bigr)^{n-2i}\,du\\
&\quad=R^{n-2i}\omega_n.
\end{aligned}$$

From the equality conditions of the arithmetic-geometric mean inequality and $h(K,u)=\rho(K,u)$, the equality holds if and only if K is the unit ball centered at the origin. This completes the proof. □

Remark 3.1 Theorem 1.1 is just the case $i=\frac{n}{2}$ of Theorem 3.4, and we complete the proof of Theorem 1.1.

In the following, we will obtain a dual Urysohn type inequality and a dual Bieberbach type inequality.

Lemma 3.1 (see [26]) *Let $K\in\mathscr{K}_o^n$, then*

$$\omega_n^{n-i+1}W_{n-1}^{i-n}(K)\le W_i\bigl(K^*\bigr),$$

where the equality holds if and only if K is an n-ball centered at the origin.

Theorem 3.5 *Let $K,L\in\mathscr{K}_o^n$, $p\ge 1$, and $\alpha,\beta\ge 0$, then*

$$\omega_n^{n-i+1}W_{n-1}^{i-n}(\alpha\cdot K\widehat{+}_p\beta\cdot L)\le W_i\bigl(\alpha\cdot K^*+_p\beta\cdot L^*\bigr),$$

where the equality holds if and only if $\alpha\cdot K\widehat{+}_p\beta\cdot L$ is an ellipsoid.

Proof From the definitions of Firey linear combinations and L_p harmonic radial combinations, adding the relation $h(K^*,u)=\frac{1}{\rho(K,u)}$, we have

$$\begin{aligned}
h^p\bigl((\alpha\cdot K\widehat{+}_p\beta\cdot L)^*,u\bigr)&=\frac{1}{\rho^p_{\alpha\cdot K\widehat{+}_p\beta\cdot L}(u)}=\rho^{-p}_{\alpha\cdot K\widehat{+}_p\beta\cdot L}(u)\\
&=\alpha\rho^{-p}(K,u)+\beta\rho^{-p}(L,u)\\
&=\alpha h^p\bigl(K^*,u\bigr)+\beta h^p\bigl(L^*,u\bigr)\\
&=h^p\bigl(\alpha\cdot K^*+_p\beta\cdot L^*,u\bigr).
\end{aligned}$$

Thus

$$(\alpha\cdot K\widehat{+}_p\beta\cdot L)^*=\alpha\cdot K^*+_p\beta\cdot L^*.$$

By Lemma 3.1, we have

$$\omega_n^{n-i+1} W_{n-1}^{i-n}(\alpha \cdot K \widehat{+}_p \beta \cdot L) \leq W_i\big(\alpha \cdot K^* +_p \beta \cdot L^*\big),$$

the condition for the equality to hold can be obtained from Lemma 3.1 directly. □

If we let $i = 0$ in Theorem 3.5, then we have the following.

Corollary 3.1 *Let $K, L \in \mathscr{K}_o^n$, $p \geq 1$, and $\alpha, \beta \geq 0$. Then*

$$\omega_n^{n+1} W_{n-1}^{-n}(\alpha \cdot K \widehat{+}_p \beta \cdot L) \leq V\big(\alpha \cdot K^* +_p \beta \cdot L^*\big),$$

where the equality holds if and only if $\alpha \cdot K \widehat{+}_p \beta \cdot L$ is an ellipsoid.

At the same time, by the last equation in (2.1) we can obtain the *dual Urysohn type inequality* (see [26] for the ***dual Urysohn inequality***):

Corollary 3.2 (Theorem 1.2) *Let $K, L \in \mathscr{K}_o^n$, $p \geq 1$, and $\alpha, \beta \geq 0$. Then*

$$2^n \omega_n \overline{b}(\alpha \cdot K \widehat{+}_p \beta \cdot L)^{-n} \leq V\big(\alpha \cdot K^* +_p \beta \cdot L^*\big),$$

the equality holds if and only if $\alpha \cdot K \widehat{+}_p \beta \cdot L$ is an ellipsoid.

This immediately yields the *dual Bieberbach type inequality* (see [26] for the ***dual Bieberbach inequality***):

Corollary 3.3 (Theorem 1.3) *Let $K, L \in \mathscr{K}_o^n$, $p \geq 1$, and $\alpha, \beta \geq 0$. Then*

$$2^n \omega_n D(\alpha \cdot K \widehat{+}_p \beta \cdot L)^{-n} \leq V\big(\alpha \cdot K^* +_p \beta \cdot L^*\big),$$

where $D(K)$ denotes the diameter of K and the equality holds if and only if $\alpha \cdot K \widehat{+}_p \beta \cdot L$ is an ellipsoid.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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