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Eigenvalues for iterative systems of nonlinear *m*-point boundary value problems on time scales

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Abstract
In this paper, we determine the eigenvalue intervals of the parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$ for which there exist positive solutions of the iterative systems of *m*-point boundary value problems on time scales. The method involves an application of Guo-Krasnosel'skii fixed point theorem. We give an example to demonstrate our main results.
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1 Introduction

The study of dynamic equations on time scales goes back to Stefan Hilger [1]. Theoretically, this new theory has not only unify continuous and discrete equations, but it has also exhibited much more complicated dynamics on time scales. Moreover, the study of dynamic equations on time scales has led to several important applications, for example, insect population models, biology, neural networks, heat transfer, and epidemic models; see [2–7].

There has been much interest shown in obtaining optimal eigenvalue intervals for the existence of positive solutions of the boundary value problems on time scales, often using Guo-Krasnosel'skii fixed point theorem. To mention a few papers along these lines, see [8–12]. On the other hand, there is not much work concerning the eigenvalues for iterative system of nonlinear boundary value problems on time scales; see [13, 14].

In [15], Ma and Thompson are concerned with determining values λ, by using the Guo-Krasnosel'skii fixed point theorem for which there exist positive solutions of the *m*-point boundary value problem

$$\begin{cases} (p(t)u')' - q(t)u + \lambda f(t,u) = 0, \quad 0 < t < 1, \\ au(0) - bp(0)u'(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \\ cu(1) + dp(1)u'(1) = \sum_{i=1}^{m-2} \beta_i u(\xi_i). \end{cases}$$

In [13], Benchohra *et al.* studied the eigenvalues for iterative system of nonlinear boundary value problems on time scales,

$$u_i^{\triangle\triangle}(t) + \lambda_i a_i(t) f_i\big(u_{i+1}\big(\sigma(t)\big)\big) = 0, \quad 1 \le i \le n, t \in [0,1]_{\mathbb{T}},$$

$$u_{n+1}(t) = u_1(t), \quad t \in [0,1]_{\mathbb{T}},$$

satisfying the boundary conditions,

$$u_i(0) = 0 = u_i\big(\sigma^2(1)\big), \quad 1 \le i \le n.$$

The method involves application of Guo-Krasnosel'skii fixed point theorem for operators on a cone in a Banach space.

In [14], Prasad *et al.* studied the eigenvalues for iterative system of nonlinear boundary value problems on time scales,

$$y_i^{\triangle\triangle}(t) + \lambda_i p_i(t) f_i\big(y_{i+1}(t)\big) = 0, \quad 1 \le i \le n, t \in [t_1, t_m]_{\mathbb{T}},$$
$$y_{n+1}(t) = y_1(t), \quad t \in [t_1, t_m]_{\mathbb{T}},$$

satisfying the m-point boundary conditions,

$$y_i(t_1) = 0,$$
$$\alpha y_i\big(\sigma(t_m)\big) + \beta y_i^{\triangle}\big(\sigma(t_m)\big) = \sum_{k=2}^{m-1} y_i^{\triangle}(t_k), \quad 1 \le i \le n.$$

They used the Guo-Krasnosel'skii fixed point theorem.

Motivated by the above results, in this study, we are concerned with determining the eigenvalue intervals of λ_i, $1 \le i \le n$, for which there exist positive solutions for the iterative system of nonlinear m-point boundary value problems on time scales,

$$\begin{cases} u_i^{\triangle\triangle}(t) + \lambda_i q_i(t) f_i(u_{i+1}(t)) = 0, & t \in [0,1]_{\mathbb{T}}, 1 \le i \le n, \\ u_{n+1}(t) = u_1(t), & t \in [0,1]_{\mathbb{T}}, \end{cases} \tag{1.1}$$

satisfying the m-point boundary conditions,

$$\begin{cases} a u_i(0) - b u_i^{\triangle}(0) = \sum_{j=1}^{m-2} \alpha_j u_i(\xi_j), \\ c u_i(1) + d u_i^{\triangle}(1) = \sum_{j=1}^{m-2} \beta_j u_i(\xi_j), & 1 \le i \le n, \end{cases} \tag{1.2}$$

where $\mathbb{T}$ is a time scale, $0, 1 \in \mathbb{T}$, $[0,1]_{\mathbb{T}} = [0,1] \cap \mathbb{T}$.

Throughout this paper we assume that following conditions hold:

(C1) $a, b, c, d \in [0,\infty)$ with $ac + ad + bc > 0$; $\alpha_j, \beta_j \in [0,\infty)$, $\xi_j \in (0,1)_{\mathbb{T}}$ for $1 \le j \le m-2$,

(C2) $f_i : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, for $1 \le i \le n$,

(C3) $q_i \in \mathcal{C}([0,1]_{\mathbb{T}}, \mathbb{R}^+)$ and q_i does not vanish identically on any closed subinterval of $[0,1]_{\mathbb{T}}$, for $1 \le i \le n$,

(C4) each of $f_{i0} := \lim_{x \to 0^+} \frac{f_i(x)}{x}$ and $f_{i\infty} := \lim_{x \to \infty} \frac{f_i(x)}{x}$, $1 \le i \le n$, exists as positive real number.

In fact, our results are also new when $\mathbb{T} = \mathbb{R}$ (the differential case) and $\mathbb{T} = \mathbb{Z}$ (the discrete case). Therefore, the results can be considered as a contribution to this field.

This paper is organized as follows. In Section 2, we construct the Green's function for the homogeneous problem corresponding to (1.1)-(1.2) and estimate bounds for the Green's function. In Section 3, we determine the eigenvalue intervals for which there exist positive solutions of the boundary value problem (1.1)-(1.2) by using the Guo-Krasnosel'skii fixed point theorem for operators on a cone in a Banach space. Finally, in Section 4, we give an example to demonstrate our main results.

2 Preliminaries

We need the auxiliary lemmas that will be used to prove our main results.

We define $\mathbb{B} = \mathcal{C}[0,1]$, which is a Banach space with the norm

$$\|u\| = \sup_{t\in[0,1]_{\mathbb{T}}} \left|u(t)\right|.$$

Let $h \in \mathcal{C}[0,1]$, then we consider the following boundary value problem:

$$\begin{cases} -u_1^{\triangle\triangle}(t) = h(t), & t \in [0,1]_{\mathbb{T}}, \\ au_1(0) - bu_1^{\triangle}(0) = \sum_{j=1}^{m-2} \alpha_j u_1(\xi_j), \\ cu_1(1) + du_1^{\triangle}(1) = \sum_{j=1}^{m-2} \beta_j u_1(\xi_j). \end{cases} \tag{2.1}$$

Denote by θ and φ, the solutions of the corresponding homogeneous equation

$$-u_1^{\triangle\triangle}(t) = 0, \quad t \in [0,1]_{\mathbb{T}}, \tag{2.2}$$

under the initial conditions

$$\begin{cases} \theta(0) = b, & \theta^{\triangle}(0) = a, \\ \varphi(1) = d, & \varphi^{\triangle}(1) = -c. \end{cases} \tag{2.3}$$

Using the initial conditions (2.3), we can deduce from equation (2.2) for θ and φ the following equations:

$$\theta(t) = b + at, \qquad \varphi(t) = d + c(1-t). \tag{2.4}$$

Set

$$\Delta := \begin{vmatrix} -\sum_{j=1}^{m-2} \alpha_j(b + a\xi_j) & \rho - \sum_{j=1}^{m-2} \alpha_j(d + c(1-\xi_j)) \\ \rho - \sum_{j=1}^{m-2} \beta_j(b + a\xi_j) & -\sum_{j=1}^{m-2} \beta_j(d + c(1-\xi_j)) \end{vmatrix} \tag{2.5}$$

and

$$\rho := ad + ac + bc. \tag{2.6}$$

Lemma 2.1 *Let* (C1) *hold. Assume that*

(C5) $\Delta \neq 0$.

If $u_1 \in \mathcal{C}[0,1]$ *is a solution of the equation*

$$u_1(t) = \int_0^1 G(t,s)h(s)\triangle s + A(h)(b + at) + B(h)\bigl(d + c(1-t)\bigr), \tag{2.7}$$

where

$$G(t,s)=\frac{1}{\rho}\begin{cases}(b+a\sigma(s))(d+c(1-t)), & \sigma(s)\le t,\\ (b+at)(d+c(1-\sigma(s))), & t\le s,\end{cases}\tag{2.8}$$

$$A(h):=\frac{1}{\Delta}\begin{vmatrix}\sum_{j=1}^{m-2}\alpha_j\int_0^1 G(\xi_j,s)h(s)\triangle s & \rho-\sum_{j=1}^{m-2}\alpha_j(d+c(1-\xi_j))\\ \sum_{j=1}^{m-2}\beta_j\int_0^1 G(\xi_j,s)h(s)\triangle s & -\sum_{j=1}^{m-2}\beta_j(d+c(1-\xi_j))\end{vmatrix}\tag{2.9}$$

and

$$B(h):=\frac{1}{\Delta}\begin{vmatrix}-\sum_{j=1}^{m-2}\alpha_j(b+a\xi_j) & \sum_{j=1}^{m-2}\alpha_j\int_0^1 G(\xi_j,s)h(s)\triangle s\\ \rho-\sum_{j=1}^{m-2}\beta_j(b+a\xi_j) & \sum_{j=1}^{m-2}\beta_j\int_0^1 G(\xi_j,s)h(s)\triangle s\end{vmatrix},\tag{2.10}$$

then u_1 is a solution of the boundary value problem (2.1).

Proof Let u_1 satisfy the integral equation (2.7), then we have

$$u_1(t)=\int_0^1 G(t,s)h(s)\triangle s+A(h)(b+at)+B(h)\bigl(d+c(1-t)\bigr),$$

i.e.,

$$\begin{aligned}u_1(t)={}&\int_0^t\frac{1}{\rho}\bigl(b+a\bigl(\sigma(s)\bigr)\bigr)\bigl(d+c(1-t)\bigr)h(s)\triangle s\\&+\int_t^1\frac{1}{\rho}(b+at)\bigl(d+c\bigl(1-\sigma(s)\bigr)\bigr)h(s)\triangle s\\&+A(h)(b+at)+B(h)\bigl(d+c(1-t)\bigr),\\ u_1^{\triangle}(t)={}&-\int_0^t\frac{c}{\rho}\bigl(b+a\bigl(\sigma(s)\bigr)\bigr)h(s)\triangle s\\&+\int_t^1\frac{a}{\rho}\bigl(d+c\bigl(1-\sigma(s)\bigr)\bigr)h(s)\triangle s\\&+A(h)a-B(h)c.\end{aligned}$$

Hence

$$\begin{aligned}u_1^{\triangle\triangle}(t)&=\frac{1}{\rho}\bigl(-c\bigl(b+a\bigl(\sigma(t)\bigr)\bigr)-a\bigl(d+c\bigl(1-\sigma(t)\bigr)\bigr)\bigr)h(t)\\&=\frac{1}{\rho}\bigl(-(ad+ac+bc)\bigr)h(t)=-h(t),\\ -u_1^{\triangle\triangle}(t)&=h(t).\end{aligned}$$

Since

$$\begin{aligned}u_1(0)&=\int_0^1\frac{b}{\rho}\bigl(d+c\bigl(1-\sigma(s)\bigr)\bigr)h(s)\triangle s+A(h)b+B(h)(d+c),\\ u_1^{\triangle}(0)&=\int_0^1\frac{a}{\rho}\bigl(d+c\bigl(1-\sigma(s)\bigr)\bigr)h(s)\triangle s+A(h)a-B(h)c,\end{aligned}$$

we have

$$\begin{aligned} au_1(0) - bu_1^{\triangle}(0) &= B(h)\rho \\ &= \sum_{j=1}^{m-2} \alpha_j \biggl[\int_0^1 G(\xi_j, s)h(s)\triangle s + A(h)(b + a\xi_j) \\ &\quad + B(h)\bigl(d + c(1 - \xi_j)\bigr) \biggr]. \end{aligned} \tag{2.11}$$

Since

$$u_1(1) = \int_0^1 \frac{d}{\rho}\bigl(b + a\bigl(\sigma(s)\bigr)\bigr)h(s)\triangle s + A(h)(b + a) + B(h)d,$$
$$u_1^{\triangle}(1) = -\int_0^1 \frac{c}{\rho}\bigl(b + a\bigl(\sigma(s)\bigr)\bigr)h(s)\triangle s + A(h)a - B(h)c,$$

we have

$$\begin{aligned} cu_1(1) + du_1^{\triangle}(1) &= A(h)\rho \\ &= \sum_{j=1}^{m-2} \beta_j \biggl[\int_0^1 G(\xi_j, s)h(s)\triangle s + A(h)(b + a\xi_j) \\ &\quad + B(h)\bigl(d + c(1 - \xi_j)\bigr) \biggr]. \end{aligned} \tag{2.12}$$

From (2.11) and (2.12), we get

$$\begin{cases} [-\sum_{j=1}^{m-2} \alpha_j(b + a\xi_j)]A(h) + [\rho - \sum_{i=1}^{m-2} \alpha_j(d + c(1 - \xi_j))]B(h) \\ \quad = \sum_{i=1}^{m-2} \alpha_j \int_0^1 G(\xi_j, s)h(s)\triangle s, \\ [\rho - \sum_{j=1}^{m-2} \beta_j(b + a\xi_j)]A(h) + [-\sum_{i=1}^{m-2} \beta_j(d + c(1 - \xi_j))]B(h) \\ \quad = \sum_{j=1}^{m-2} \beta_j \int_0^1 G(\xi_j, s)h(s)\triangle s, \end{cases}$$

which implies that $A(h)$ and $B(h)$ satisfy (2.9) and (2.10), respectively. □

Lemma 2.2 *Let* (C1) *hold. Assume*
(C6) $\Delta < 0$, $\rho - \sum_{j=1}^{m-2} \beta_j(b + a\xi_j) > 0$, $a - \sum_{j=1}^{m-2} \alpha_j > 0$.
Then for $u_1 \in \mathcal{C}[0,1]$ *with* $h \geq 0$, *the solution* u_1 *of the problem* (2.1) *satisfies*

$$u_1(t) \geq 0 \quad \textit{for } t \in [0,1]_{\mathbb{T}}.$$

Proof It is an immediate subsequence of the facts that $G \geq 0$ on $[0,1]_{\mathbb{T}} \times [0,1]_{\mathbb{T}}$ and $A(h) \geq 0$, $B(h) \geq 0$. □

Lemma 2.3 *Let* (C1) *and* (C6) *hold. Assume*
(C7) $c - \sum_{j=1}^{m-2} \beta_j < 0$.
Then the solution $u_1 \in \mathcal{C}[0,1]$ *of the problem* (2.1) *satisfies* $u_1^{\triangle}(t) \geq 0$ *for* $t \in [0,1]_{\mathbb{T}}$.

Proof Assume that the inequality $u_1^{\triangle}(t) < 0$ holds. Since $u_1^{\triangle}(t)$ is nonincreasing on $[0,1]_{\mathbb{T}}$, one can verify that

$$u_1^{\triangle}(1) \leq u_1^{\triangle}(t), \quad t \in [0,1]_{\mathbb{T}}.$$

From the boundary conditions of the problem (2.1), we have

$$-\frac{c}{d}u_1(1) + \frac{1}{d}\sum_{i=1}^{m-2}\beta_i u_1(\xi_i) \leq u_1^{\triangle}(t) < 0.$$

The last inequality yields

$$-cu_1(1) + \sum_{i=1}^{m-2}\beta_i u_1(\xi_i) < 0.$$

Therefore, we obtain

$$\sum_{i=1}^{m-2}\beta_i u_1(1) < \sum_{i=1}^{m-2}\beta_i u_1(\xi_i) < cu_1(1),$$

i.e.,

$$\left(c - \sum_{i=1}^{m-2}\beta_i\right)u_1(1) > 0.$$

According to Lemma 2.2, we have $u_1(1) \geq 0$. So, $c - \sum_{i=1}^{m-2}\beta_i > 0$. However, this contradicts to condition (C7). Consequently, $u_1^{\triangle}(t) \geq 0$ for $t \in [0,1]_{\mathbb{T}}$. □

Lemma 2.4 *Let* (C1) *and* $h \geq 0$ *hold. Let* $\mu \in (0,1/2)_{\mathbb{T}}$ *be a constant. Then the unique solution* u_1 *of the problem* (2.1) *satisfies*

$$\min_{t\in[\mu,1-\mu]_{\mathbb{T}}} u_1(t) \geq \gamma \|u_1\|,$$

where $\|u_1\| = \sup_{t\in[0,1]_{\mathbb{T}}} u_1(t)$ *and*

$$\gamma := \min\left\{\frac{b + a\mu}{b + a}, \frac{d + c\mu}{d + c}\right\}. \tag{2.13}$$

Proof We have from (2.8) that

$$0 \leq G(t,s) \leq G\big(\sigma(s),s\big), \quad t \in [0,1]_{\mathbb{T}}, \tag{2.14}$$

which implies

$$u_1(t) \leq \int_0^1 G\big(\sigma(s),s\big)h(s)\triangle s + A(h)(b + a) + B(h)(d + c) \tag{2.15}$$

for all $t \in [0,1]_{\mathbb{T}}$. Applying (2.8), we have for $t \in [\mu, 1-\mu]_{\mathbb{T}}$,

$$\begin{aligned} \frac{G(t,s)}{G(\sigma(s),s)} &= \begin{cases} \frac{d+c(1-t)}{d+c(1-\sigma(s))}, & 0 \leq \sigma(s) \leq t \leq 1, \\ \frac{b+at}{b+a\sigma(s)}, & 0 \leq t \leq s \leq 1 \end{cases} \\ &\geq \begin{cases} \frac{d+c\mu}{d+c}, & 0 \leq \sigma(s) \leq t \leq 1-\mu, \\ \frac{b+a\mu}{b+a}, & \mu \leq t \leq s \leq 1 \end{cases} \geq \gamma, \end{aligned} \tag{2.16}$$

where

$$\gamma := \min\left\{\frac{b+a\mu}{b+a}, \frac{d+c\mu}{d+c}\right\}.$$

Thus for $t \in [\mu, 1-\mu]_{\mathbb{T}}$,

$$\begin{aligned} u_1(t) &= \int_0^1 G(t,s)h(s)\triangle s + A(h)(b+at) + B(h)\big(d+c(1-t)\big) \\ &\geq \gamma\left(\int_0^1 G\big(\sigma(s),s\big)h(s)\triangle s + A(h)(b+a) + B(h)(d+c)\right) \geq \gamma\|u_1\|. \end{aligned}$$

So, the proof is completed. □

We note that an n-tuple $(u_1(t), u_2(t), \ldots, u_n(t))$ is a solution of the boundary value problem (1.1)-(1.2) if and only if

$$\begin{aligned} u_1(t) &= \lambda_1 \int_0^1 G(t,s_1)q_1(s_1)f_1\left(\lambda_2 \int_0^1 G(s_1,s_2)q_2(s_2)\cdots\right. \\ &\quad f_{n-1}\left(\lambda_n \int_0^1 G(s_{n-1},s_n)q_n(s_n)f_n\big(u_1(s_n)\big)\triangle s_n\right)\cdots\triangle s_2\Bigg)\triangle s_1 \\ &\quad + A\big(\lambda_1 q_1(\cdot)f_1\big(u_2(\cdot)\big)\big)(b+at) + B\big(\lambda_1 q_1(\cdot)f_1\big(u_2(\cdot)\big)\big)\big(d+c(1-t)\big), \quad t \in [0,1]_{\mathbb{T}}, \\ u_i(t) &= \lambda_i \int_0^1 G(t,s)q_i(s)f_i\big(u_{i+1}(s)\big)\triangle s + A\big(\lambda_i q_i(\cdot)f_i\big(u_{i+1}(\cdot)\big)\big)(b+at) \\ &\quad + B\big(\lambda_i q_i(\cdot)f_i\big(u_{i+1}(\cdot)\big)\big)\big(d+c(1-t)\big), \quad 2 \leq i \leq n, t \in [0,1]_{\mathbb{T}} \end{aligned}$$

and

$$u_{n+1}(t) = u_1(t), \quad t \in [0,1]_{\mathbb{T}},$$

where

$$\begin{aligned} &A\big(\lambda_i q_i(\cdot)f_i\big(u_{i+1}(\cdot)\big)\big) \\ &\quad := \frac{1}{\Delta}\begin{vmatrix} \sum_{j=1}^{m-2}\alpha_j\lambda_i\int_0^1 G(\xi_j,s)q_i(s)f_i(u_{i+1}(s))\triangle s & \rho - \sum_{j=1}^{m-2}\alpha_j(d+c(1-\xi_j)) \\ \sum_{j=1}^{m-2}\beta_j\lambda_i\int_0^1 G(\xi_j,s)q_i(s)f_i(u_{i+1}(s))\triangle s & -\sum_{j=1}^{m-2}\beta_j(d+c(1-\xi_j)) \end{vmatrix}, \\ &B\big(\lambda_i q_i(\cdot)f_i\big(u_{i+1}(\cdot)\big)\big) \\ &\quad := \frac{1}{\Delta}\begin{vmatrix} -\sum_{j=1}^{m-2}\alpha_j(b+a\xi_j) & \sum_{j=1}^{m-2}\alpha_j\lambda_i\int_0^1 G(\xi_j,s)q_i(s)f_i(u_{i+1}(s))\triangle s \\ \rho - \sum_{j=1}^{m-2}\beta_j(b+a\xi_j) & \sum_{j=1}^{m-2}\beta_j\lambda_i\int_0^1 G(\xi_j,s)q_i(s)f_i(u_{i+1}(s))\triangle s \end{vmatrix}. \end{aligned}$$

To determine the eigenvalue intervals of the boundary value problem (1.1)-(1.2), we will use the following Guo-Krasnosel'skii fixed point theorem [16].

Theorem 2.1 [16] *Let $\mathbb{B}$ be a Banach space, and let $\mathcal{P} \subset \mathbb{B}$ be a cone in $\mathbb{B}$. Assume Ω_1 and Ω_2 are open subsets of $\mathbb{B}$ with $0 \in \Omega_1$ and $\bar{\Omega}_1 \subset \Omega_2$, and let*

$$T : \mathcal{P} \cap (\bar{\Omega}_2 \setminus \Omega_1) \to \mathcal{P}$$

be a completely continuous operator such that either

(i) $\|Tu\| \leq \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_1$, *and* $\|Tu\| \geq \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_2$, *or*

(ii) $\|Tu\| \geq \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_1$, *and* $\|Tu\| \leq \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_2$.

Then T has a fixed point in $\mathcal{P} \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

3 Positive solutions in a cone

In this section, we establish criteria to determine the eigenvalue intervals for which the boundary value problem (1.1)-(1.2) has at least one positive solution in a cone. We construct a cone $\mathcal{P} \subset \mathbb{B}$ by

$$\mathcal{P} = \left\{ u \in \mathbb{B} : u(t) \geq 0 \text{ on } [0,1]_{\mathbb{T}} \text{ and } \min_{t \in [\mu, 1-\mu]_{\mathbb{T}}} u(t) \geq \gamma \|u\| \right\},$$

where γ is given in (2.13).

Now, we define an integral operator $T : \mathcal{P} \to \mathbb{B}$, for $u_1 \in \mathcal{P}$, by

$$\begin{aligned} Tu_1(t) = {} & \lambda_1 \int_0^1 G(t, s_1) q_1(s_1) f_1 \biggl(\lambda_2 \int_0^1 G(s_1, s_2) q_2(s_2) \cdots \\ & f_{n-1} \biggl(\lambda_n \int_0^1 G(s_{n-1}, s_n) q_n(s_n) f_n\bigl(u_1(s_n)\bigr) \triangle s_n \biggr) \cdots \triangle s_2 \biggr) \triangle s_1 \\ & + A\bigl(\lambda_1 q_1(\cdot) f_1\bigl(u_2(\cdot)\bigr)\bigr)(b + at) + B\bigl(\lambda_1 q_1(\cdot) f_1\bigl(u_2(\cdot)\bigr)\bigr)\bigl(d + c(1-t)\bigr). \end{aligned} \tag{3.1}$$

Notice from (C1)-(C6) and Lemma 2.2 that, for $u_1 \in \mathcal{P}$, $Tu_1(t) \geq 0$ on $t \in [0,1]_{\mathbb{T}}$. Also, we have from (2.8), that

$$\begin{aligned} Tu_1(t) \leq {} & \lambda_1 \int_0^1 G\bigl(\sigma(s_1), s_1\bigr) q_1(s_1) f_1 \biggl(\lambda_2 \int_0^1 G(s_1, s_2) q_2(s_2) \cdots \\ & f_{n-1} \biggl(\lambda_n \int_0^1 G(s_{n-1}, s_n) q_n(s_n) f_n\bigl(u_1(s_n)\bigr) \triangle s_n \biggr) \cdots \triangle s_2 \biggr) \triangle s_1 \\ & + A\bigl(\lambda_1 q_1(\cdot) f_1\bigl(u_2(\cdot)\bigr)\bigr)(b + a) + B\bigl(\lambda_1 q_1(\cdot) f_1\bigl(u_2(\cdot)\bigr)\bigr)(d + c), \end{aligned}$$

so that

$$\begin{aligned} \|Tu_1\| \leq {} & \lambda_1 \int_0^1 G\bigl(\sigma(s_1), s_1\bigr) q_1(s_1) f_1 \biggl(\lambda_2 \int_0^1 G(s_1, s_2) q_2(s_2) \cdots \\ & f_{n-1} \biggl(\lambda_n \int_0^1 G(s_{n-1}, s_n) q_n(s_n) f_n\bigl(u_1(s_n)\bigr) \triangle s_n \biggr) \cdots \triangle s_2 \biggr) \triangle s_1 \\ & + A\bigl(\lambda_1 q_1(\cdot) f_1\bigl(u_2(\cdot)\bigr)\bigr)(b + a) + B\bigl(\lambda_1 q_1(\cdot) f_1\bigl(u_2(\cdot)\bigr)\bigr)(d + c). \end{aligned} \tag{3.2}$$

Next, if $u_1 \in \mathcal{P}$, we have from Lemma 2.4 and (3.2) that

$$\begin{aligned}
&\min_{t\in[\mu,1-\mu]_{\mathbb{T}}} Tu_1(t) \\
&\quad= \min_{t\in[\mu,1-\mu]_{\mathbb{T}}} \Bigg\{\lambda_1 \int_0^1 G(t,s_1)q_1(s_1)f_1\Bigg(\lambda_2 \int_0^1 G(s_1,s_2)q_2(s_2)\cdots \\
&\qquad f_{n-1}\Bigg(\lambda_n \int_0^1 G(s_{n-1},s_n)q_n(s_n)f_n\big(u_1(s_n)\big)\triangle s_n\Bigg)\cdots\triangle s_2\Bigg)\triangle s_1 \\
&\qquad + A\big(\lambda_1 q_1(\cdot)f_1\big(u_2(\cdot)\big)\big)(b+at) + B\big(\lambda_1 q_1(\cdot)f_1\big(u_2(\cdot)\big)\big)\big(d+c(1-t)\big)\Bigg\} \\
&\quad\geq \gamma\Bigg(\lambda_1 \int_0^1 G\big(\sigma(s_1),s_1\big)q_1(s_1)f_1\Bigg(\lambda_2 \int_0^1 G(s_1,s_2)q_2(s_2)\cdots \\
&\qquad f_{n-1}\Bigg(\lambda_n \int_0^1 G(s_{n-1},s_n)q_n(s_n)f_n\big(u_1(s_n)\big)\triangle s_n\Bigg)\cdots\triangle s_2\Bigg)\triangle s_1 \\
&\qquad + A\big(\lambda_1 q_1(\cdot)f_1\big(u_2(\cdot)\big)\big)(b+a) + B\big(\lambda_1 q_1(\cdot)f_1\big(u_2(\cdot)\big)\big)(d+c)\Bigg) \\
&\quad\geq \gamma\|Tu_1\|.
\end{aligned}$$

Hence, $Tu_1 \in \mathcal{P}$ and $T : \mathcal{P} \to \mathcal{P}$. In addition, the operator T is completely continuous by an application of the Arzela-Ascoli theorem.

Now, we investigate suitable fixed points of T belonging to the cone $\mathcal{P}$. For convenience we introduce the following notations.

Let

$$M_1 = \max_{1\leq i\leq n}\left\{\left[\gamma^2 \int_\mu^{1-\mu} G\big(\sigma(s),s\big)q_i(s)\triangle s f_{i\infty}\right]^{-1}\right\}$$

and

$$M_2 = \min_{1\leq i\leq n}\left\{\left[\left(\int_0^1 G\big(\sigma(s),s\big)q_i(s)\triangle s + A\big(q_i(\cdot)\big)(b+a) + B\big(q_i(\cdot)\big)(d+c)\right)f_{i0}\right]^{-1}\right\}.$$

Theorem 3.1 *Suppose conditions* (C1)-(C7) *are satisfied. Then, for each* $\lambda_1, \lambda_2, \dots, \lambda_n$ *satisfying*

$$M_1 < \lambda_i < M_2, \quad 1 \leq i \leq n, \tag{3.3}$$

there exists an n-tuple $(u_1, u_2, \dots, u_n)$ *satisfying* (1.1)-(1.2) *such that* $u_i(t) > 0$, $1 \leq i \leq n$, *on* $[0,1]_{\mathbb{T}}$.

Proof Let λ_k, $1 \leq k \leq n$, be as in (3.3). Now, let $\epsilon > 0$ be chosen such that

$$\max_{1\leq i\leq n}\left\{\left[\gamma^2 \int_\mu^{1-\mu} G\big(\sigma(s),s\big)q_i(s)\triangle s(f_{i\infty} - \epsilon)\right]^{-1}\right\} \leq \min_{1\leq k\leq n} \lambda_k$$

and

$$\max_{1\leq k\leq n} \lambda_k \leq \min_{1\leq i\leq n}\left\{\left[\left(\int_0^1 G\big(\sigma(s),s\big)q_i(s)\triangle s + A\big(q_i(\cdot)\big)(b+a) + B\big(q_i(\cdot)\big)(d+c)\right)(f_{i0} + \epsilon)\right]^{-1}\right\}.$$

We investigate fixed points of the completely continuous operator $T:\mathcal{P}\to\mathcal{P}$ defined by (3.1). Now, from the definitions of f_{i0}, $1\leq i\leq n$, there exists an $H_1>0$ such that, for each $1\leq i\leq n$,

$$f_i(x)\leq (f_{i0}+\epsilon)x,\quad 0<x\leq H_1.$$

Let $u_1\in\mathcal{P}$ with $\|u_1\|=H_1$. We have from (2.14) and the choice of ϵ, for $0\leq s_{n-1}\leq 1$,

$$\begin{aligned}
&\lambda_n\int_0^1 G(s_{n-1},s_n)q_n(s_n)f_n\big(u_1(s_n)\big)\triangle s_n\\
&\quad\leq \lambda_n\int_0^1 G\big(\sigma(s_n),s_n\big)q_n(s_n)f_n\big(u_1(s_n)\big)\triangle s_n\\
&\quad\leq \lambda_n\int_0^1 G\big(\sigma(s_n),s_n\big)q_n(s_n)(f_{n0}+\epsilon)u_1(s_n)\triangle s_n\\
&\quad\leq \lambda_n\int_0^1 G\big(\sigma(s_n),s_n\big)q_n(s_n)\triangle s_n(f_{n0}+\epsilon)\|u_1\|\\
&\quad\leq \|u_1\|\\
&\quad\leq H_1.
\end{aligned}$$

It follows in a similar manner from (2.14), for $0\leq s_{n-2}\leq 1$, that

$$\begin{aligned}
&\lambda_{n-1}\int_0^1 G(s_{n-2},s_{n-1})q_{n-1}(s_{n-1})f_{n-1}\left(\lambda_n\int_0^1 G(s_{n-1},s_n)q_n(s_n)f_n\big(u_1(s_n)\big)\triangle s_n\right)\triangle s_{n-1}\\
&\quad\leq \lambda_{n-1}\int_0^1 G\big(\sigma(s_{n-1}),s_{n-1}\big)q_{n-1}(s_{n-1})\triangle s_{n-1}(f_{n-1,0}+\epsilon)\|u_1\|\\
&\quad\leq \|u_1\|=H_1.
\end{aligned}$$

Continuing with this bootstrapping argument, we have, for $0\leq t\leq 1$,

$$\begin{aligned}
&\lambda_1\int_0^1 G(t,s_1)q_1(s_1)f_1\left(\lambda_2\int_0^1 G(s_1,s_2)q_2(s_2)\cdots f_n\big(u_1(s_n)\big)\triangle s_n\cdots\triangle s_2\right)\triangle s_1\\
&\quad\leq \lambda_1\int_0^1 G\big(\sigma(s_1),s_1\big)q_1(s_1)\triangle s_1(f_{10}+\epsilon)H_1,\\
&A\big(\lambda_1 q_1(\cdot)f_1\big(u_2(\cdot)\big)\big)\\
&\quad\leq \frac{\lambda_1}{\Delta}\begin{vmatrix}\sum_{j=1}^{m-2}\alpha_j\int_0^1 G(\xi_j,s)q_1(s)\triangle s & \rho-\sum_{j=1}^{m-2}\alpha_j(d+c(1-\xi_j))\\ \sum_{j=1}^{m-2}\beta_j\int_0^1 G(\xi_j,s)q_1(s)\triangle s & -\sum_{j=1}^{m-2}\beta_j(d+c(1-\xi_j))\end{vmatrix}\big\|f_1(u_2)\big\|\\
&\quad\leq \lambda_1 A\big(q_1(\cdot)\big)\big\|f_1(u_2)\big\|,\\
&B\big(\lambda_1 q_1(\cdot)f_1\big(u_2(\cdot)\big)\big)\\
&\quad\leq \frac{\lambda_1}{\Delta}\begin{vmatrix}-\sum_{j=1}^{m-2}\alpha_j(b+a(\xi_j)) & \sum_{j=1}^{m-2}\alpha_j\int_0^1 G(\xi_j,s)q_1(s)\triangle s\\ \rho-\sum_{j=1}^{m-2}\beta_j(b+a(\xi_j)) & \sum_{j=1}^{m-2}\beta_j\int_0^1 G(\xi_j,s)q_1(s)\triangle s\end{vmatrix}\big\|f_1(u_2)\big\|\\
&\quad\leq \lambda_1 B\big(q_1(\cdot)\big)\big\|f_1(u_2)\big\|,
\end{aligned}$$

so that, for $0 \le t \le 1$,

$$\begin{aligned} Tu_1(t) &\le \lambda_1 \biggl(\int_0^1 G\bigl(\sigma(s_1), s_1\bigr) q_1(s_1) \triangle s_1 (f_{10} + \epsilon) H_1 \\ &\quad + A\bigl(q_1(\cdot)\bigr) \bigl\| f_1(u_2) \bigr\| (b + a) + B\bigl(q_1(\cdot)\bigr) \bigl\| f_1(u_2) \bigr\| (d + c) \biggr) \\ &\le \lambda_1 \biggl(\int_0^1 G\bigl(\sigma(s_1), s_1\bigr) q_1(s_1) \triangle s_1 \\ &\quad + A\bigl(q_1(\cdot)\bigr)(b + a) + B\bigl(q_1(\cdot)\bigr)(d + c) \biggr) (f_{10} + \epsilon) H_1 \\ &\le H_1 = \|u_1\|. \end{aligned}$$

Hence, $\|Tu_1\| \le H_1 = \|u_1\|$. If we set

$$\Omega_1 = \bigl\{ u \in \mathbb{B} \mid \|u\| < H_1 \bigr\},$$

then

$$\|Tu_1\| \le \|u_1\| \quad \text{for } u_1 \in \mathcal{P} \cap \partial\Omega_1. \tag{3.4}$$

Next, from the definitions of $f_{i\infty}$, $1 \le i \le n$, there exists $\bar{H}_2 > 0$ such that, for each $1 \le i \le n$,

$$f_i(x) \ge (f_{i\infty} - \epsilon)x, \quad x \ge \bar{H}_2.$$

Let

$$H_2 = \max\biggl\{ 2H_1, \frac{\bar{H}_2}{\gamma} \biggr\}.$$

Let $u_1 \in \mathcal{P}$ and $\|u_1\| = H_2$. Then, we have from Lemma 2.4

$$\min_{t \in [\mu, 1-\mu]_{\mathbb{T}}} u_1(t) \ge \gamma \|u_1\| \ge \bar{H}_2.$$

Consequently, from Lemma 2.4 and the choice of ϵ, for $0 \le s_{n-1} \le 1$, we have

$$\begin{aligned} &\lambda_n \int_0^1 G(s_{n-1}, s_n) q_n(s_n) f_n\bigl(u_1(s_n)\bigr) \triangle s_n \\ &\quad \ge \gamma \lambda_n \int_\mu^{1-\mu} G\bigl(\sigma(s_n), s_n\bigr) q_n(s_n) f_n\bigl(u_1(s_n)\bigr) \triangle s_n \\ &\quad \ge \gamma \lambda_n \int_\mu^{1-\mu} G\bigl(\sigma(s_n), s_n\bigr) q_n(s_n) (f_{n\infty} - \epsilon) u_1(s_n) \triangle s_n \\ &\quad \ge \gamma^2 \lambda_n \int_\mu^{1-\mu} G\bigl(\sigma(s_n), s_n\bigr) q_n(s_n) \triangle s_n (f_{n\infty} - \epsilon) \|u_1\| \\ &\quad \ge \|u_1\| = H_2. \end{aligned}$$

It follows in a similar manner from Lemma 2.4 and the choice of ϵ, for $0 \leq s_{n-2} \leq 1$,

$$\begin{aligned}
&\lambda_{n-1} \int_0^1 G(s_{n-2}, s_{n-1}) q_{n-1}(s_{n-1}) f_{n-1}\left(\lambda_n \int_0^1 G(s_{n-1}, s_n) q_n(s_n) f_n\big(u_1(s_n)\big) \triangle s_n\right) \triangle s_{n-1} \\
&\quad \geq \gamma \lambda_{n-1} \int_\mu^{1-\mu} G\big(\sigma(s_{n-1}), s_{n-1}\big) q_{n-1}(s_{n-1}) \triangle s_{n-1} (f_{n-1,\infty} - \epsilon) H_2 \\
&\quad \geq \gamma^2 \lambda_{n-1} \int_\mu^{1-\mu} G\big(\sigma(s_{n-1}), s_{n-1}\big) q_{n-1}(s_{n-1}) \triangle s_{n-1} (f_{n-1,\infty} - \epsilon) H_2 \\
&\quad \geq H_2.
\end{aligned}$$

Again, using a bootstrapping argument, we have

$$\begin{aligned}
&\lambda_1 \int_0^1 G(t, s_1) q_1(s_1) f_1\left(\lambda_2 \int_0^1 G(s_1, s_2) q_2(s_2) \cdots f_n\big(u_1(s_n)\big) \triangle s_n \cdots \triangle s_2\right) \triangle s_1 \\
&\quad \geq H_2,
\end{aligned}$$

so that

$$Tu_1(t) \geq H_2 = \|u_1\|.$$

Hence, $\|Tu_1\| \geq \|u_1\|$. So if we set

$$\Omega_2 = \big\{u \in \mathbb{B} \,|\, \|u\| < H_2\big\},$$

then

$$\|Tu_1\| \geq \|u_1\| \quad \text{for } u_1 \in \mathcal{P} \cap \partial\Omega_2. \tag{3.5}$$

Applying Theorem 2.1 to (3.4) and (3.5), we see that T has a fixed point $u_1 \in \mathcal{P} \cap (\bar{\Omega}_2 \setminus \Omega_1)$. Therefore, setting $u_{n+1} = u_1$, we obtain a positive solution $(u_1, u_2, \ldots, u_n)$ of (1.1)-(1.2) given iteratively by

$$\begin{aligned}
u_k(t) &= \lambda_k \int_0^1 G(t, s) q_k(s) f_k\big(u_{k+1}(s)\big) \triangle s + A\big(\lambda_k q_k(\cdot) f_k\big(u_{k+1}(\cdot)\big)\big)(b + at) \\
&\quad + B\big(\lambda_k q_k(\cdot) f_k\big(u_{k+1}(\cdot)\big)\big)\big(d + c(1-t)\big), \quad k = n, n-1, \ldots, 1.
\end{aligned}$$

The proof is completed. □

For our next result, we define the positive numbers M_3 and M_4 by

$$M_3 = \max_{1 \leq i \leq n} \left\{ \left[\gamma^2 \int_\mu^{1-\mu} G\big(\sigma(s), s\big) q_i(s) \triangle s f_{i0} \right]^{-1} \right\}$$

and

$$M_4 = \min_{1 \leq i \leq n} \left\{ \left[\left(\int_0^1 G\big(\sigma(s), s\big) q_i(s) \triangle s + A\big(q_i(\cdot)\big)(b + a) + B\big(q_i(\cdot)\big)(d + c) \right) f_{i\infty} \right]^{-1} \right\}.$$

Theorem 3.2 *Suppose conditions* (C1)-(C7) *are satisfied. Then, for each* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *satisfying*

$$M_3 < \lambda_i < M_4, \quad 1 \leq i \leq n, \tag{3.6}$$

there exists an n-tuple $(u_1, u_2, \ldots, u_n)$ *satisfying* (1.1)-(1.2) *such that* $u_i(t) > 0$, $1 \leq i \leq n$, *on* $[0,1]_{\mathbb{T}}$.

Proof Let λ_k, $1 \leq k \leq n$, be as in (3.6). Now, let $\epsilon > 0$ be chosen such that

$$\max_{1\leq i\leq n}\left\{\left[\gamma^2\int_\mu^{1-\mu} G\big(\sigma(s),s\big)q_i(s)\triangle s(f_{i0}-\epsilon)\right]^{-1}\right\} \leq \min_{1\leq k\leq n}\lambda_k$$

and

$$\max_{1\leq k\leq n}\lambda_k \leq \min_{1\leq i\leq n}\left\{\left[\left(\int_0^1 G\big(\sigma(s),s\big)q_i(s)\triangle s + A\big(q_i(\cdot)\big)(b+a)B\big(q_i(\cdot)\big)(d+c)\right)(f_{i\infty}+\epsilon)\right]^{-1}\right\}.$$

Let T be the cone preserving, completely continuous operator that was defined by (3.1). From the definition of f_{i0}, $1 \leq i \leq n$, there exists $\bar{H}_3 > 0$ such that, for each $1 \leq i \leq n$,

$$f_i(x) \geq (f_{i0}-\epsilon)x, \quad 0 < x \leq \bar{H}_3.$$

Also, from the definition of f_{i0}, it follows that $f_{i0}(0) = 0$, $1 \leq i \leq n$, and so there exist $0 < K_n < K_{n-1} < \cdots < K_2 < \bar{H}_3$ such that

$$\lambda_i f_i(t) \leq \frac{K_{i-1}}{\int_0^1 G(\sigma(s),s)q_i(s)\triangle s}, \quad t \in [0,K_i]_{\mathbb{T}}, 3 \leq i \leq n$$

and

$$\lambda_2 f_2(t) \leq \frac{\bar{H}_3}{\int_0^1 G(\sigma(s),s)q_2(s)\triangle s}, \quad t \in [0,K_2]_{\mathbb{T}}.$$

Choose $u_1 \in \mathcal{P}$ with $\|u_1\| = K_n$. Then we have

$$\begin{aligned}
&\lambda_n \int_0^1 G(s_{n-1},s_n)q_n(s_n)f_n\big(u_1(s_n)\big)\triangle s_n \\
&\quad \leq \lambda_n \int_0^1 G\big(\sigma(s_n),s_n\big)q_n(s_n)f_n\big(u_1(s_n)\big)\triangle s_n \\
&\quad \leq \frac{\int_0^1 G(\sigma(s_n),s_n)q_n(s_n)K_{n-1}\triangle s_n}{\int_0^1 G(\sigma(s_n),s_n)q_n(s_n)\triangle s_n} \\
&\quad = K_{n-1}.
\end{aligned}$$

Continuing with this bootstrapping argument, we get

$$\begin{aligned}
&\lambda_2 \int_0^1 G(s_1,s_2)q_2(s_2)f_2\left(\lambda_3 \int_0^1 G(s_2,s_3)q_3(s_3)\cdots f_n\big(u_1(s_n)\big)\triangle s_n \cdots \triangle s_3\right)\triangle s_2 \\
&\quad \leq \bar{H}_3.
\end{aligned}$$

Then

$$\begin{aligned} Tu_1(t) &\geq \lambda_1 \int_0^1 G(t,s_1)q_1(s_1)f_1\left(\lambda_2 \int_0^1 G(s_1,s_2)q_2(s_2)\cdots f_n\big(u_1(s_n)\big)\triangle s_n \cdots \triangle s_2\right)\triangle s_1 \\ &\geq \gamma^2\lambda_1 \int_\mu^{1-\mu} G\big(\sigma(s_1),s_1\big)q_1(s_1)(f_{10}-\epsilon)\|u_1\|\triangle s_1 \\ &\geq \|u_1\|. \end{aligned}$$

So, $\|Tu_1\| \geq \|u_1\|$. If we put

$$\Omega_3 = \big\{u \in \mathbb{B} \mid \|u\| < K_n\big\},$$

then

$$\|Tu_1\| \geq \|u_1\| \quad \text{for } u_1 \in \mathcal{P} \cap \partial\Omega_3. \tag{3.7}$$

Since each $f_{i\infty}$ is assumed to be a positive real number, it follows that f_i, $1 \leq i \leq n$, is unbounded at ∞.

For each $1 \leq i \leq n$, set

$$f_i^*(x) = \sup_{0\leq s\leq x} f_i(s).$$

Then, for each $1 \leq i \leq n$, f_i^* is a nondecreasing real-valued function, $f_i \leq f_i^*$, and

$$\lim_{x\to\infty} \frac{f_i^*(x)}{x} = f_{i\infty}.$$

Next, by definition of $f_{i\infty}$, $1 \leq i \leq n$, there exists $\bar{H}_4$ such that, for each $1 \leq i \leq n$,

$$f_i^*(x) \leq (f_{i\infty} + \epsilon)x, \quad x \geq \bar{H}_4.$$

It follows that there exists $H_4 > \max\{2\bar{H}_3, \bar{H}_4\}$ such that, for each $1 \leq i \leq n$,

$$f_i^*(x) \leq f_i^*(H_4), \quad 0 < x \leq H_4.$$

Choose $u_1 \in \mathcal{P}$ with $\|u_1\| = H_4$. Then, using the bootstrapping argument, we have

$$\begin{aligned} &\lambda_1 \int_0^1 G(t,s_1)q_1(s_1)f_1(\lambda_2\cdots)\triangle s_1 \\ &\quad\leq \lambda_1 \int_0^1 G(t,s_1)q_1(s_1)f_1^*(\lambda_2\cdots)\triangle s_1 \\ &\quad\leq \lambda_1 \int_0^1 G\big(\sigma(s_1),s_1\big)q_1(s_1)f_1^*(H_4)\triangle s_1 \\ &\quad\leq \lambda_1 \int_0^1 G\big(\sigma(s_1),s_1\big)q_1(s_1)\triangle s_1(f_{1\infty}+\epsilon)H_4. \end{aligned}$$

So we have

$$\begin{aligned} Tu_1(t) &\le \lambda_1\biggl(\int_0^1 G\bigl(\sigma(s_1),s_1\bigr)q_1(s_1)\triangle s_1(f_{1\infty}+\epsilon)H_4 \\ &\quad + A\bigl(q_1(\cdot)\bigr)\bigl\|f_1(u_2)\bigr\|(b+a) + B\bigl(q_1(\cdot)\bigr)\bigl\|f_1(u_2)\bigr\|(d+c)\biggr) \\ &\le \lambda_1\biggl(\int_0^1 G\bigl(\sigma(s_1),s_1\bigr)q_1(s_1)\triangle s_1(f_{1\infty}+\epsilon)H_4 \\ &\quad + A\bigl(q_1(\cdot)\bigr)\bigl\|f_1^*(u_2)\bigr\|(b+a) + B\bigl(q_1(\cdot)\bigr)\bigl\|f_1^*(u_2)\bigr\|(d+c)\biggr) \\ &\le \lambda_1\biggl(\int_0^1 G\bigl(\sigma(s_1),s_1\bigr)q_1(s_1)\triangle s_1 + A\bigl(q_1(\cdot)\bigr)(b+a) + B\bigl(q_1(\cdot)\bigr)(d+c)\biggr) \\ &\quad \times (f_{1\infty}+\epsilon)H_4 \\ &\le H_4 = \|u_1\|. \end{aligned}$$

Hence, $\|Tu_1\| \le \|u_1\|$. So, if we set

$$\Omega_4 = \bigl\{u \in \mathbb{B} \mid \|u\| < H_4\bigr\},$$

then

$$\|Tu_1\| \le \|u_1\| \quad \text{for } u_1 \in \mathcal{P} \cap \partial\Omega_4. \tag{3.8}$$

Applying Theorem 2.1 to (3.7) and (3.8), we see that T has a fixed point $u_1 \in \mathcal{P} \cap (\bar{\Omega}_4 \setminus \Omega_3)$, which in turn with $u_{n+1} = u_1$, we obtain an n-tuple $(u_1, u_2, \ldots, u_n)$ satisfying (1.1)-(1.2) for the chosen values of λ_i, $1 \le i \le n$. The proof is completed. □

4 An example

Example 4.1 In BVP (1.1)-(1.2), suppose that $\mathbb{T} = [0,1]$, $n = m = 3$, $q_1(t) = q_2(t) = q_3(t) = 1$, $a = c = 2$, $b = d = 1$, $\xi_1 = \frac{1}{2}$, $\mu = \frac{1}{4}$, $\alpha_1 = \frac{1}{2}$ and $\beta_1 = 3$ *i.e.*,

$$\begin{cases} u_i''(t) + \lambda_i f_i(u_{i+1}(t)) = 0, & t \in [0,1], 1 \le i \le 3, \\ u_4(t) = u_1(t), & t \in [0,1]_{\mathbb{T}}, \end{cases} \tag{4.1}$$

satisfying the following boundary conditions:

$$\begin{cases} 2u_i(0) - u_i'(0) = \frac{1}{2}u_i(\frac{1}{2}), \\ 2u_i(1) + u_i'(1) = 3u_i(\frac{1}{2}), & 1 \le i \le 3, \end{cases} \tag{4.2}$$

where

$$\begin{aligned} f_1(u_2) &= u_2\bigl(1{,}000 - 999e^{-u_2}\bigr)\bigl(520 - 512e^{-2u_2}\bigr), \\ f_2(u_3) &= u_3\bigl(700 - 698e^{-2u_3}\bigr)\bigl(1{,}500 - 1{,}498e^{-3u_2}\bigr), \\ f_3(u_1) &= u_1\bigl(800 - 796e^{-u_1}\bigr)\bigl(400 - 396e^{-5u_1}\bigr). \end{aligned}$$

It is easy to see that (C1)-(C7) are satisfied. By simple calculation, we get $\rho = 8$, $\theta(t) = 1 + 2t$, $\varphi(t) = 3 - 2t$, $\Delta = -8$, $\gamma = \frac{1}{2}$, $A(1) = 9$, $B(1) = \frac{3}{2}$ and

$$G(t,s) = \frac{1}{8}\begin{cases} (1+2s)(3-2t), & s \le t, \\ (1+2t)(3-2s), & t \le s. \end{cases}$$

We obtain

$$\begin{aligned} &f_{10} = 8, \qquad f_{20} = 4, \qquad f_{30} = 16, \\ &f_{1\infty} = 520{,}000, \qquad f_{2\infty} = 1{,}050{,}000, \qquad f_{3\infty} = 320{,}000, \\ &M_1 = \max\{0.0000039279869, 0.00000194528875, 0.000006382978723\} \end{aligned}$$

and

$$M_2 = \min\{0.00355450236, 0.007109004739, 0.001777251184\}.$$

Applying Theorem 3.1, we get the optimal eigenvalue interval $0.000006382978723 < \lambda_i < 0.001777251184$, $i = 1, 2, 3$, for which the boundary value problem (4.1)-(4.2) has a positive solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and typed, read and approved the final manuscript.

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