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High order of accuracy difference schemes for the inverse elliptic problem with Dirichlet condition

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Abstract
The overdetermination problem for elliptic differential equation with Dirichlet boundary condition is considered. The third and fourth orders of accuracy stable difference schemes for the solution of this inverse problem are presented. Stability, almost coercive stability, and coercive inequalities for the solutions of difference problems are established. As a result of the application of established abstract theorems, we get well-posedness of high order difference schemes of the inverse problem for a multidimensional elliptic equation. The theoretical statements are supported by a numerical example.
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Keywords: difference scheme; inverse elliptic problem; high order accuracy; well-posedness; stability; almost coercive stability; coercive stability

1 Introduction

Many problems in various branches of science lead to inverse problems for partial differential equations [1–3]. Inverse problems for partial differential equations have been investigated extensively by many researchers (see [3–18] and the references therein).

Consider the inverse problem of finding a function u and an element p for the elliptic equation

$$\begin{cases} -u_{tt}(t) + Au(t) = f(t) + p, & 0 < t < T, \\ u(0) = \varphi, \quad u(T) = \psi, \quad u(\lambda) = \xi, & 0 < \lambda < T \end{cases} \tag{1.1}$$

in an arbitrary Hilbert space H with a self-adjoint positive definite operator A. Here, λ is a known number, φ, ξ, and ψ are given elements of H.

Existence and uniqueness theorems for problem (1.1) in a Banach space are presented in [5]. The first and second accuracy stable difference schemes for this problem have been constructed in [15]. High order of accuracy stable difference schemes for nonlocal boundary value elliptic problems are presented in [19–21].

Our aim in this work is the construction of the third and fourth order stable accuracy difference schemes for the inverse problem (1.1).

In the present paper, the third and fourth orders of accuracy difference schemes for the approximate solution of problem (1.1) are presented. Stability, almost coercive stability,

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and coercive stability inequalities for the solution of these difference schemes are established.

In the application, we consider the inverse problem for the multidimensional elliptic equation with Dirichlet condition

$$\begin{cases} -u_{tt}(t,x) - \sum_{r=1}^{n}(a_r(x)u_{x_r})_{x_r} + \sigma u = f(t,x) + p(x), \\ \quad x = (x_1,\dots,x_n) \in \Omega, 0 < t < T, \\ u(0,x) = \varphi(x), \qquad u(T,x) = \psi(x), \qquad u(\lambda,x) = \xi(x), \quad x \in \overline{\Omega}, \\ u(t,x) = 0, \quad x \in S, 0 \le t \le T. \end{cases} \tag{1.2}$$

Here, $\Omega = (0,L) \times \cdots \times (0,L)$ is the open cube in the n-dimensional Euclidean space with boundary S, $\overline{\Omega} = \Omega \cup S$, $a_r(x)$ $(x \in \Omega)$, $\varphi(x)$, $\xi(x)$, $\psi(x)$ $(x \in \overline{\Omega})$, $f(t,x)$ $(t \in (0,1), x \in \Omega)$ are given smooth functions, $a_r(x) \ge a > 0$ $(x \in \overline{\Omega})$, and $0 < \lambda < T$, $\sigma > 0$ are given numbers.

The first and second orders of accuracy stable difference schemes for equation (1.2) are presented in [15]. We construct the third and fourth orders of accuracy stable difference schemes for problem (1.2).

The remainder of this paper is organized as follows. In Section 2, we present the third and fourth order difference schemes for problem (1.1) and obtain stability estimates for them. In Section 3, we construct the third and fourth order difference schemes for problem (1.2) and establish their well-posedness. In Section 4, the numerical results are given. Section 5 is our conclusion.

2 High order of accuracy difference schemes for (1.1) and stability inequalities

We use, respectively, the third and fourth order accuracy approximate formulas

$$\begin{aligned} u(\lambda) = {} & \left(\frac{1}{2}\left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau}\right]\right) + \frac{1}{2}\left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau}\right]\right)^2\right) u\left(\left(\left[\frac{\lambda}{\tau}\right] - 1\right)\tau\right) \\ & + \left(1 - \left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau}\right]\right)^2\right) u\left(\left[\frac{\lambda}{\tau}\right]\tau\right) \\ & + \left(-\frac{1}{2}\left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau}\right]\right) + \frac{1}{2}\left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau}\right]\right)^2\right) u\left(\left(\left[\frac{\lambda}{\tau}\right] + 1\right)\tau\right) + o(\tau^3), \end{aligned} \tag{2.1}$$

$$\begin{aligned} u(\lambda) = {} & \left(-\frac{1}{12}\left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau}\right]\right) + \frac{1}{12}\left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau}\right]\right)^3\right) u\left(\left(\left[\frac{\lambda}{\tau}\right] - 2\right)\tau\right) \\ & + \left(\frac{8}{12}\left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau}\right]\right) + \frac{1}{2}\left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau}\right]\right)^2 \right. \\ & \left. - \frac{1}{6}\left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau}\right]\right)^3\right) u\left(\left(\left[\frac{\lambda}{\tau}\right] - 1\right)\tau\right) \\ & + \left(1 - \left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau}\right]\right)^2\right) u\left(\left[\frac{\lambda}{\tau}\right]\tau\right) + \left(-\frac{8}{12}\left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau}\right]\right) + \frac{1}{2}\left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau}\right]\right)^2 \right. \\ & \left. + \frac{1}{6}\left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau}\right]\right)^3\right) u\left(\left(\left[\frac{\lambda}{\tau}\right] - 1\right)\tau\right) \\ & + \left(\frac{1}{12}\left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau}\right]\right) - \frac{1}{12}\left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau}\right]\right)^3\right) u\left(\left(\left[\frac{\lambda}{\tau}\right] + 2\right)\tau\right) + o(\tau^4) \end{aligned} \tag{2.2}$$

for $u(\lambda)$. Here, $l = [\frac{\lambda}{\tau}]$, $[\cdot]$ is a notation for the greatest integer function. Applying formulas (2.1) and (2.2) to $u(\lambda) = \xi$, we get, respectively,

$$\begin{cases} -\tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + Au_k + \frac{\tau^2}{12}A^2 u_k = \theta_k + p, \\ \quad \theta_k = f(t_k) + \frac{\tau^2}{12}(\frac{f(t_{k+1})-2f(t_k)+f(t_{k-1})}{\tau^2} + Af(t_k)), \\ \quad t_k = k\tau, 1 \le k \le N-1, N\tau = T, u_0 = \varphi, u_N = \psi, \\ (\frac{1}{2}(\frac{\lambda}{\tau} - l) + \frac{1}{2}(\frac{\lambda}{\tau} - l)^2)u_{l-1} + (1 - \frac{1}{2}(\frac{\lambda}{\tau} - l)^2)u_l \\ \quad + (-\frac{1}{2}(\frac{\lambda}{\tau} - l) + \frac{1}{2}(\frac{\lambda}{\tau} - l)^2)u_{l+1} = \xi, \end{cases} \tag{2.3}$$

the third order of accuracy difference problem and

$$\begin{cases} -\tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + Au_k + \frac{\tau^2}{12}A^2 u_k = \theta_k + p, \\ \quad \theta_k = f(t_k) + \frac{\tau^2}{12}(\frac{f(t_{k+1})-2f(t_k)+f(t_{k-1})}{\tau^2} + Af(t_k)), \\ \quad t_k = k\tau, 1 \le k \le N-1, N\tau = T, u_0 = \varphi, u_N = \psi, \\ (\frac{1}{12}(\frac{\lambda}{\tau} - l) - \frac{1}{12}(\frac{\lambda}{\tau} - l)^3)u_{l-2} \\ \quad + (-\frac{8}{12}(\frac{\lambda}{\tau} - l) + \frac{1}{2}(\frac{\lambda}{\tau} - l)^2 + \frac{1}{6}(\frac{\lambda}{\tau} - l)^3)u_{l-1} \\ \quad + (1 - (\frac{\lambda}{\tau} - l)^2)u_l \\ \quad + (\frac{8}{12}(\frac{\lambda}{\tau} - l) + \frac{1}{2}(\frac{\lambda}{\tau} - l)^2 - \frac{1}{6}(\frac{\lambda}{\tau} - l)^3)u_{l+1} \\ \quad + (-\frac{1}{12}(\frac{\lambda}{\tau} - l) + \frac{1}{12}(\frac{\lambda}{\tau} - l)^3)u_{l+2} = \xi, \end{cases} \tag{2.4}$$

the fourth order of accuracy difference problem for inverse problem (1.1).

For solving of problems (2.3) and (2.4), we use the algorithm [14], which includes three stages. For finding a solution $\{u_k\}_{k=1}^{N-1}$ of difference problems (2.3) and (2.4) we apply the substitution

$$u_k = v_k + A^{-1}p. \tag{2.5}$$

In the first stage, applying approximation (2.5), we get a nonlocal boundary value difference problem for obtaining $\{v_k\}_{k=0}^{N}$. In the second stage, we put $k = 0$ and find v_0. Then, using the formula

$$p = A\varphi - Av_0, \tag{2.6}$$

we define an element p. In the third stage, by using approximation (2.5), we can obtain the solution $\{u_k\}_{k=1}^{N-1}$ of difference problems (2.3) and (2.4). In the framework of the above mentioned algorithm for $\{v_k\}_{k=0}^{N}$, we get the following auxiliary nonlocal boundary value difference scheme:

$$\begin{cases} -\tau^{-2}(v_{k+1} - 2v_k + v_{k-1}) + Av_k + \frac{\tau^2}{12}A^2 v_k = \theta_k, \\ \quad \theta_k = f(t_k) + \frac{\tau^2}{12}(\frac{f(t_{k+1})-2f(t_k)+f(t_{k-1})}{\tau^2} + Af(t_k)), \\ \quad t_k = k\tau, 1 \le k \le N-1, N\tau = T, \\ v_0 - (\frac{1}{2}(\frac{\lambda}{\tau} - l) + \frac{1}{2}(\frac{\lambda}{\tau} - l)^2)v_{l-1} - (1 - \frac{1}{2}(\frac{\lambda}{\tau} - l)^2)v_l \\ \quad - (-\frac{1}{2}(\frac{\lambda}{\tau} - l) + \frac{1}{2}(\frac{\lambda}{\tau} - l)^2)v_{l+1} = \varphi - \xi, \\ v_N - (\frac{1}{2}(\frac{\lambda}{\tau} - l) + \frac{1}{2}(\frac{\lambda}{\tau} - l)^2)v_{l-1} - (1 - \frac{1}{2}(\frac{\lambda}{\tau} - l)^2)v_l \\ \quad - (-\frac{1}{2}(\frac{\lambda}{\tau} - l) + \frac{1}{2}(\frac{\lambda}{\tau} - l)^2)v_{l+1} = \psi - \xi \end{cases} \tag{2.7}$$

for the third order of accuracy difference problem (2.3) and

$$\begin{cases} -\tau^{-2}(v_{k+1} - 2v_k + v_{k-1}) + Av_k + \frac{\tau^2}{12}A^2 v_k = \theta_k, \\ \quad \theta_k = f(t_k) + \frac{\tau^2}{12}(\frac{f(t_{k+1})-2f(t_k)+f(t_{k-1})}{\tau^2} + Af(t_k)), \\ \quad t_k = k\tau, 1 \le k \le N-1, N\tau = T, \\ v_0 - (\frac{1}{12}(\frac{\lambda}{\tau} - l) - \frac{1}{12}(\frac{\lambda}{\tau} - l)^3)v_{l-2} - (-\frac{8}{12}(\frac{\lambda}{\tau} - l) + \frac{1}{2}(\frac{\lambda}{\tau} - l)^2 \\ \quad + \frac{1}{6}(\frac{\lambda}{\tau} - l)^3)v_{l-1} - (1 - (\frac{\lambda}{\tau} - l)^2)v_l - (\frac{8}{12}(\frac{\lambda}{\tau} - l) + \frac{1}{2}(\frac{\lambda}{\tau} - l)^2 \\ \quad - \frac{1}{6}(\frac{\lambda}{\tau} - l)^3)v_{l+1} - (-\frac{1}{12}(\frac{\lambda}{\tau} - l) + \frac{1}{12}(\frac{\lambda}{\tau} - l)^3)v_{l+2} = \varphi - \xi, \\ v_N - (\frac{1}{12}(\frac{\lambda}{\tau} - l) - \frac{1}{12}(\frac{\lambda}{\tau} - l)^3)v_{l-2} - (-\frac{8}{12}(\frac{\lambda}{\tau} - l) + \frac{1}{2}(\frac{\lambda}{\tau} - l)^2 \\ \quad + \frac{1}{6}(\frac{\lambda}{\tau} - l)^3)v_{l-1} - (1 - (\frac{\lambda}{\tau} - l)^2)v_l - (\frac{8}{12}(\frac{\lambda}{\tau} - l) + \frac{1}{2}(\frac{\lambda}{\tau} - l)^2 \\ \quad - \frac{1}{6}(\frac{\lambda}{\tau} - l)^3)v_{l+1} - (-\frac{1}{12}(\frac{\lambda}{\tau} - l) + \frac{1}{12}(\frac{\lambda}{\tau} - l)^3)v_{l+2} = \psi - \xi \end{cases} \tag{2.8}$$

for the fourth order of accuracy difference problem (2.4).

For a self-adjoint positive definite operator A, it follows that [22] $D = \frac{1}{2}(\tau C + \sqrt{4C + \tau^2 C^2})$ is a self-adjoint positive definite operator, where $C = A + \frac{\tau^2}{12}A^2$, $R = (I + \tau D)^{-1}$, I is the identity operator. Moreover, the bounded operator D is defined on the whole space H.

Now we give some lemmas that will be needed below.

Lemma 2.1 *The following estimates hold* [23]:

$$\begin{aligned} &\left\| \exp\left(-k\tau A^{\frac{1}{2}}\right) - R^k \right\|_{H \to H} \le \frac{M(\delta)\tau}{\tau^k}, \quad k \ge 1, \qquad \left\| \left(I - R^{2N}\right)^{-1} \right\|_{H \to H} \le M(\delta), \\ &k\tau \left\| DR^k \right\|_{H \to H} \le M(\delta), \qquad \left\| R^k \right\|_{H \to H} \le M(\delta)(1 + \delta\tau)^{-k}, \quad k \ge 1, \delta > 0, \\ &\left\| D^\beta \left(R^{k+r} - R^k\right) \right\|_{H \to H} \le M(\delta) \frac{(r\tau)^\alpha}{(k\tau)^{\alpha+\beta}}, \quad 1 \le k < k + r \le N, 0 \le \alpha, \beta \le 1. \end{aligned}$$

Lemma 2.2 *The following estimate holds* [23]:

$$\sum_{j=1}^{N-1} \tau \left\| DR^j \right\|_{H \to H} \le M(\delta) Y(\tau, \delta),$$

where

$$Y(\tau, \delta) = \min\left\{ \ln \frac{1}{\tau}, 1 + \left| \ln \|B\|_{H \to H} \right| \right\}.$$

Lemma 2.3 *For $1 \le l \le N-1$, the operator*

$$\begin{aligned} S_1 = I - R^{2N} &- \left(\frac{1}{2}\left(\frac{\lambda}{\tau} - l\right) + \frac{1}{2}\left(\frac{\lambda}{\tau} - l\right)^2 \right)\left(R^{l-1} + R^{2N-l+1} - R^{N-l+1} + R^{N+l-1}\right) \\ &- \left(1 - \frac{1}{2}\left(\frac{\lambda}{\tau} - l\right)^2\right)\left(R^l + R^{2N-l} - R^{N-l} + R^{N+l}\right) \\ &- \left(-\frac{1}{2}\left(\frac{\lambda}{\tau} - l\right) + \frac{1}{2}\left(\frac{\lambda}{\tau} - l\right)^2\right)\left(R^{l+1} + R^{2N-l-1} - R^{N-l-1} + R^{N+l+1}\right) \end{aligned}$$

has an inverse such that

$$G_1 = S_1^{-1}$$

and the estimate

$$\|G_1\|_{H\to H} \le M(\delta) \tag{2.9}$$

is valid.

Proof We have

$$G_1 - G = G_1 G K_1, \tag{2.10}$$

where

$$\begin{aligned}
G &= I - R^{2N} - R^{l} + R^{2N-l} - R^{N-l} + R^{N+l}, \qquad (2.11)\\
K_1 &= -\left(\frac{1}{2}\left(\frac{\lambda}{\tau}-l\right) + \frac{1}{2}\left(\frac{\lambda}{\tau}-l\right)^2\right)\left(R^{l-1} + R^{2N-l+1} - R^{N-l+1} + R^{N+l-1}\right)\\
&\quad - \left(1 - \frac{1}{2}\left(\frac{\lambda}{\tau}-l\right)^2\right)\left(R^{l} + R^{2N-l} - R^{N-l} + R^{N+l}\right) - \left(-\frac{1}{2}\left(\frac{\lambda}{\tau}-l\right) + \frac{1}{2}\left(\frac{\lambda}{\tau}-l\right)^2\right)\\
&\quad \times \left(R^{l+1} + R^{2N-l-1} - R^{N-l-1} + R^{N+l+1}\right).
\end{aligned}$$

Applying estimates of Lemma 2.1, we have

$$\begin{aligned}
\|K_1\|_{H\to H} &= \bigg\| -\left(\frac{1}{2}\left(\frac{\lambda}{\tau}-l\right) + \frac{1}{2}\left(\frac{\lambda}{\tau}-l\right)^2\right)\left(R^{l-1} + R^{2N-l+1} - R^{N-l+1} + R^{N+l-1}\right)\\
&\quad - \left(1 - \frac{1}{2}\left(\frac{\lambda}{\tau}-l\right)^2\right)\left(R^{l} + R^{2N-l} - R^{N-l} + R^{N+l}\right)\\
&\quad - \left(-\frac{1}{2}\left(\frac{\lambda}{\tau}-l\right) + \frac{1}{2}\left(\frac{\lambda}{\tau}-l\right)^2\right)\left(R^{l+1} + R^{2N-l-1} - R^{N-l-1} + R^{N+l+1}\right)\bigg\|_{H\to H}\\
&\le M_1(\delta)\tau. \qquad (2.12)
\end{aligned}$$

By using the triangle inequality, formula (2.10), estimates (2.9), (2.12), and Lemma 2.2 of paper [15], we obtain

$$\|G_1\|_{H\to H} = \|G\|_{H\to H} + \|G_1\|_{H\to H}\|G\|_{H\to H} \le M(\delta) + \|G_1\|_{H\to H} M(\delta) M_1(\delta)\tau$$

for any small positive parameter τ. From that follows estimate (2.9). Lemma 2.3 is proved. □

Lemma 2.4 *For* $1 \le l \le N-1$, *the operator*

$$\begin{aligned}
S_2 &= I - R^{2N} - \left(\frac{1}{12}\left(\frac{\lambda}{\tau}-l\right) - \frac{1}{12}\left(\frac{\lambda}{\tau}-l\right)^3\right)\\
&\quad \times \left(R^{l-2} + R^{2N-l+2} - R^{N-l+2} + R^{N+l-2}\right) - \left(-\frac{8}{12}\left(\frac{\lambda}{\tau}-l\right) + \frac{1}{2}\left(\frac{\lambda}{\tau}-l\right)^2\right.\\
&\quad \left. + \frac{1}{6}\left(\frac{\lambda}{\tau}-l\right)^3\right)\left(R^{l-1} + R^{2N-l+1} - R^{N-l+1} + R^{N+l-1}\right) + \left(\frac{\lambda}{\tau}-l\right)^2
\end{aligned}$$

$$\times \left(R^{l}+R^{2N-l}-R^{N-l}+R^{N+l}\right)-\left(\frac{8}{12}\left(\frac{\lambda}{\tau}-l\right)+\frac{1}{2}\left(\frac{\lambda}{\tau}-l\right)^{2}-\frac{1}{6}\left(\frac{\lambda}{\tau}-l\right)^{3}\right)$$
$$\times \left(R^{l+1}+R^{2N-l-1}-R^{N-l-1}+R^{N+l+1}\right)-\left(-\frac{1}{12}\left(\frac{\lambda}{\tau}-l\right)+\frac{1}{12}\left(\frac{\lambda}{\tau}-l\right)^{3}\right)$$
$$\times \left(R^{l+2}+R^{2N-l-2}-R^{N-l-2}+R^{N+l+2}\right)$$

has an inverse

$$G_2 = S_2^{-1}$$

and the estimate

$$\|G_2\|_{H\to H} \leq M(\delta) \tag{2.13}$$

is satisfied.

Proof We can get

$$G_2 - G = G_2 G K_2, \tag{2.14}$$

where G is defined by formula (2.11) and

$$\begin{aligned}
K_2 = {}&-\left(\frac{1}{12}\left(\frac{\lambda}{\tau}-l\right)-\frac{1}{12}\left(\frac{\lambda}{\tau}-l\right)^{3}\right)\left(R^{l-2}+R^{2N-l+2}-R^{N-l+2}+R^{N+l-2}\right)\\
&-\left(-\frac{8}{12}\left(\frac{\lambda}{\tau}-l\right)+\frac{1}{2}\left(\frac{\lambda}{\tau}-l\right)^{2}+\frac{1}{6}\left(\frac{\lambda}{\tau}-l\right)^{3}\right)\\
&\times \left(R^{l-1}+R^{2N-l+1}-R^{N-l+1}+R^{N+l-1}\right)+\left(\frac{\lambda}{\tau}-l\right)^{2}\left(R^{l}+R^{2N-l}-R^{N-l}+R^{N+l}\right)\\
&-\left(\frac{8}{12}\left(\frac{\lambda}{\tau}-l\right)+\frac{1}{2}\left(\frac{\lambda}{\tau}-l\right)^{2}-\frac{1}{6}\left(\frac{\lambda}{\tau}-l\right)^{3}\right)\\
&\times \left(R^{l+1}+R^{2N-l-1}-R^{N-l-1}+R^{N+l+1}\right)\\
&-\left(-\frac{1}{12}\left(\frac{\lambda}{\tau}-l\right)+\frac{1}{12}\left(\frac{\lambda}{\tau}-l\right)^{3}\right)\left(R^{l+2}+R^{2N-l-2}-R^{N-l-2}+R^{N+l+2}\right).
\end{aligned}$$

Applying estimates of Lemma 2.1, we have

$$\|K_2\|_{H\to H} \leq M_2(\delta)\tau. \tag{2.15}$$

Using the triangle inequality, formula (2.14), estimates (2.13), (2.15), and Lemma 2.3 of paper [15], we get

$$\|G_2\|_{H\to H} = \|G\|_{H\to H} + \|G_2\|_{H\to H}\|G\|_{H\to H} \leq M(\delta) + \|G_2\|_{H\to H} M(\delta) M_2(\delta)\tau$$

for any small positive parameter τ. From that follows estimate (2.13). Lemma 2.4 is proved. □

Let $C_\tau(H)$ and $\mathcal{C}_\tau^{\alpha,\alpha}(H)$ be the spaces of all H-valued grid functions $\{\theta_k\}_{k=1}^{N-1}$ in the corresponding norms,

$$\|\{\theta_k\}_{k=1}^{N-1}\|_{C_\tau(H)} = \max_{1\le k\le N-1} \|\theta_k\|_H,$$

$$\|\{\theta_k\}_{k=1}^{N-1}\|_{\mathcal{C}_\tau^{\alpha,\alpha}(H)} = \|\{\theta_k\}_{k=1}^{N-1}\|_{C_\tau(H)} + \sup_{1\le k<k+n\le N-1} \frac{(k\tau+n\tau)^\alpha (T-k\tau)^\alpha \|\theta_{k+n}-\theta_k\|_H}{(n\tau)^\alpha}.$$

Theorem 2.1 *Assume that A is a self-adjoint positive definite operator, $\varphi,\psi,\xi\in D(A)$ and $\{\theta_k\}_{k=1}^{N-1}\in\mathcal{C}_\tau^{\alpha,\alpha}(H)$ $(0<\alpha<1)$. Then, the solution $(\{u_k\}_{k=1}^{N-1},p)$ of difference problem* (2.3) *obeys the following stability estimates*:

$$\|\{u_k\}_{k=1}^{N-1}\|_{C_\tau(H)} \le M(\delta)\big[\|\varphi\|_H+\|\psi\|_H+\|\xi\|_H+\|\{\theta_k\}_{k=1}^{N-1}\|_{C_\tau(H)}\big], \tag{2.16}$$

$$\|A^{-1}p\|_H \le M(\delta)\big[\|\varphi\|_H+\|\psi\|_H+\|\xi\|_H+\|\{\theta_k\}_{k=1}^{N-1}\|_{C_\tau(H)}\big], \tag{2.17}$$

$$\|p\|_H \le M(\delta)\left[\|A\varphi\|_H+\|A\psi\|_H+\|A\xi\|_H+\frac{1}{\alpha(1-\alpha)}\|\{\theta_k\}_{k=1}^{N-1}\|_{\mathcal{C}_\tau^{\alpha,\alpha}(H)}\right]. \tag{2.18}$$

Proof We will obtain the representation formula for the solution of problem (2.7). Applying the formula [23], we get

$$\begin{aligned} v_k &= \left(I-R^{2N}\right)^{-1}\Bigg[\left(\left(R^k-R^{2N-k}\right)v_0+\left(R^{N-k}-R^{N+k}\right)v_N\right) \\ &\quad -\left(R^{N-k}-R^{N+k}\right)(I+\tau D)(2I+\tau D)^{-1}D^{-1}\sum_{i=1}^{N-1}\left(R^{N-i}-R^{N+i}\right)\theta_i\tau\Bigg] \\ &\quad +(I+\tau D)(2I+\tau D)^{-1}D^{-1}\sum_{i=1}^{N-1}\left(R^{|k-i|}-R^{k+i}\right)\theta_i\tau. \end{aligned} \tag{2.19}$$

By using formula (2.19) and nonlocal boundary conditions

$$\begin{aligned} v_0 &= \left(\frac{1}{2}\left(\frac{\lambda}{\tau}-l\right)+\frac{1}{2}\left(\frac{\lambda}{\tau}-l\right)^2\right)v_{l-1}+\left(1-\frac{1}{2}\left(\frac{\lambda}{\tau}-l\right)^2\right)v_l \\ &\quad +\left(-\frac{1}{2}\left(\frac{\lambda}{\tau}-l\right)+\frac{1}{2}\left(\frac{\lambda}{\tau}-l\right)^2\right)v_{l+1}+\varphi-\xi, \\ v_N &= v_0+\psi-\varphi, \end{aligned}$$

we get the system of equations

$$\begin{aligned} v_0 &= \left(\frac{1}{2}\left(\frac{\lambda}{\tau}-l\right)+\frac{1}{2}\left(\frac{\lambda}{\tau}-l\right)^2\right)\Bigg\{\left(I-R^{2N}\right)^{-1} \\ &\quad \times\Bigg[\left(\left(R^{l-1}-R^{2N-l+1}\right)v_0+\left(R^{N-l-1}-R^{N+l-1}\right)v_N\right)-\left(R^{N-l+1}-R^{N+l-1}\right) \\ &\quad \times(I+\tau D)(2I+\tau D)^{-1}D^{-1}\sum_{i=1}^{N-1}\left(R^{N-i}-R^{N+i}\right)\theta_i\tau\Bigg]+(I+\tau D)(2I+\tau D)^{-1} \end{aligned}$$

$$\begin{aligned}
&\times D^{-1}\sum_{i=1}^{N-1}\left(R^{|l-1-i|}-R^{l-1+i}\right)\theta_i\tau\Bigg\} + \left(1-\frac{1}{2}\left(\frac{\lambda}{\tau}-l\right)^2\right)\Bigg\{\left(I-R^{2N}\right)^{-1} \\
&\times\Bigg[\left(\left(R^{l}-R^{2N-l}\right)\nu_0+\left(R^{N-l}-R^{N+l}\right)\nu_N\right)-\left(R^{N-l}-R^{N+l}\right)(I+\tau D) \\
&\times(2I+\tau D)^{-1}D^{-1}\sum_{i=1}^{N-1}\left(R^{N-i}-R^{N+i}\right)\theta_i\tau\Bigg] + (I+\tau D)(2I+\tau D)^{-1}D^{-1} \\
&\times\sum_{i=1}^{N-1}\left(R^{|l-i|}-R^{l+i}\right)\theta_i\tau\Bigg\} + \left(-\frac{1}{2}\left(\frac{\lambda}{\tau}-l\right)+\frac{1}{2}\left(\frac{\lambda}{\tau}-l\right)^2\right)\Bigg\{\left(I-R^{2N}\right)^{-1} \\
&-\left(R^{N-l-1}-R^{N+l+1}\right)(I+\tau D)(2I+\tau D)^{-1}D^{-1}\Bigg[\left(\left(R^{l+1}-R^{2N-l-1}\right)\nu_0\right. \\
&\left.+\left(R^{N-l-1}-R^{N+l+1}\right)\nu_N\right)\sum_{i=1}^{N-1}\left(R^{N-i}-R^{N+i}\right)\theta_i\tau\Bigg] \\
&+(I+\tau D)(2I+\tau D)^{-1}D^{-1}\sum_{i=1}^{N-1}\left(R^{|l+1-i|}-R^{l+1+i}\right)\theta_i\tau\Bigg\} + \varphi-\xi, \qquad (2.20)
\end{aligned}$$

$$\nu_N = \nu_0 + \psi - \varphi.$$

Solving system (2.20), we obtain

$$\begin{aligned}
\nu_0 = {}&\left(\frac{1}{2}\left(\frac{\lambda}{\tau}-l\right)+\frac{1}{2}\left(\frac{\lambda}{\tau}-l\right)^2\right)G_1\left(R^{N-l+1}-R^{N+l-1}\right)(I+\tau D) \\
&\times(2I+\tau D)^{-1}D^{-1}\sum_{i=1}^{N-1}\left(R^{N-i}-R^{N+i}\right)\theta_i\tau + G_1\left(I-R^{2N}\right)(I+\tau D) \\
&\times(2I+\tau D)^{-1}D^{-1}\sum_{i=1}^{N-1}\left(R^{|l-1-i|}-R^{l-1+i}\right)\theta_i\tau + \left(1-\frac{1}{2}\left(\frac{\lambda}{\tau}-l\right)^2\right)G_1 \\
&\times\left(R^{N-l-1}-R^{N+l+1}\right)(I+\tau D)(2I+\tau D)^{-1}D^{-1}\sum_{i=1}^{N-1}\left(R^{N-i}-R^{N+i}\right)\theta_i\tau \\
&+G_1\left(I-R^{2N}\right)(I+\tau D)(2I+\tau D)^{-1}D^{-1}\sum_{i=1}^{N-1}\left(R^{|l+1-i|}-R^{l+1+i}\right)\theta_i\tau \\
&+\left(-\frac{1}{2}\left(\frac{\lambda}{\tau}-l\right)+\frac{1}{2}\left(\frac{\lambda}{\tau}-l\right)^2\right)G_1\left(R^{N-l-1}-R^{N+l+1}\right)(I+\tau D) \\
&\times(2I+\tau D)^{-1}D^{-1}\sum_{i=1}^{N-1}\left(R^{N-i}-R^{N+i}\right)\theta_i\tau + G_1\left(I-R^{2N}\right)(I+\tau D) \\
&\times(2I+\tau D)^{-1}D^{-1}\sum_{i=1}^{N-1}\left(R^{|l+1-i|}-R^{l+1+i}\right)\theta_i\tau + G_1\left(I-R^{2N}\right)(\varphi-\xi) \\
&+G_1\left(\left(\frac{1}{2}\left(\frac{\lambda}{\tau}-l\right)+\frac{1}{2}\left(\frac{\lambda}{\tau}-l\right)^2\right)\left(R^{N-l+1}-R^{N+l-1}\right)\right. \\
&+\left(1-\frac{1}{2}\left(\frac{\lambda}{\tau}-l\right)^2\right)\left(R^{N-l-1}-R^{N+l+1}\right)
\end{aligned}$$

$$+ \left(-\frac{1}{2}\left(\frac{\lambda}{\tau}-l\right)+\frac{1}{2}\left(\frac{\lambda}{\tau}-l\right)^2\right)\left(R^{N-l-1}-R^{N+l+1}\right)\Big)(\psi-\varphi), \tag{2.21}$$

$$v_N = v_0 + \psi - \varphi. \tag{2.22}$$

Therefore, difference problem (2.7) has a unique solution $\{v_k\}_{k=0}^{N}$ which is defined by formulas (2.19), (2.21), and (2.22). Applying formulas (2.19), (2.21), (2.22), and the method of the monograph [23], we get

$$\left\|\{v_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)} \le M(\delta)\left[\|\varphi\|_H + \|\psi\|_H + \|\xi\|_H + \left\|\{\theta_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)}\right]. \tag{2.23}$$

The proofs of estimates (2.17), (2.18) are based on formula (2.5) and estimate (2.23). Using formula (2.5) and estimates (2.23), (2.17), we obtain inequality (2.16). Theorem 2.1 is proved. □

Theorem 2.2 *Suppose that A is a self-adjoint positive definite operator, $\varphi, \psi, \xi \in D(A)$ and $\{\theta_k\}_{k=1}^{N-1} \in \mathcal{C}_\tau^{\alpha,\alpha}(H)$ $(0 < \alpha < 1)$. Then, the solution $(\{u_k\}_{k=1}^{N-1}, p)$ of difference problem* (2.4) *obeys the stability estimates* (2.16), (2.17), *and* (2.18).

Proof By using the representation formula (2.19) for the solution of (2.8), formula (2.19), and the nonlocal boundary conditions

$$\begin{aligned}
v_0 &= \left(\frac{1}{12}\left(\frac{\lambda}{\tau}-l\right)-\frac{1}{12}\left(\frac{\lambda}{\tau}-l\right)^3\right)v_{l-2} \\
&\quad + \left(-\frac{8}{12}\left(\frac{\lambda}{\tau}-l\right)+\frac{1}{2}\left(\frac{\lambda}{\tau}-l\right)^2+\frac{1}{6}\left(\frac{\lambda}{\tau}-l\right)^3\right)v_{l-1} \\
&\quad + \left(1-\left(\frac{\lambda}{\tau}-l\right)^2\right)v_l + \left(\frac{8}{12}\left(\frac{\lambda}{\tau}-l\right)+\frac{1}{2}\left(\frac{\lambda}{\tau}-l\right)^2-\frac{1}{6}\left(\frac{\lambda}{\tau}-l\right)^3\right)v_{l+1} \\
&\quad + \left(-\frac{1}{12}\left(\frac{\lambda}{\tau}-l\right)+\frac{1}{12}\left(\frac{\lambda}{\tau}-l\right)^3\right)v_{l+2} + \varphi - \xi, \\
v_N &= v_0 + \psi - \varphi,
\end{aligned}$$

we obtain the system of equations

$$\begin{aligned}
v_0 &= \left(\frac{1}{12}\left(\frac{\lambda}{\tau}-l\right)-\frac{1}{12}\left(\frac{\lambda}{\tau}-l\right)^3\right)\Bigg\{\left(I-R^{2N}\right)^{-1} \\
&\quad \times \Bigg[\left(\left(R^{l-2}-R^{2N-l+2}\right)v_0 + \left(R^{N-l-2}-R^{N+l-2}\right)v_N\right) - \left(R^{N-l+2}-R^{N+l-2}\right)(I+\tau D) \\
&\quad \times (2I+\tau D)^{-1}D^{-1}\sum_{i=1}^{N-1}\left(R^{N-i}-R^{N+i}\right)\theta_i\tau\Bigg] + (I+\tau D)(2I+\tau D)^{-1}D^{-1} \\
&\quad \times \sum_{i=1}^{N-1}\left(R^{|l-2-i|}-R^{l-2+i}\right)\theta_i\tau\Bigg\} - \left(R^{N-l-1}-R^{N+l+1}\right)(I+\tau D) \\
&\quad \times (2I+\tau D)^{-1}D^{-1}\sum_{i=1}^{N-1}\left(R^{N-i}-R^{N+i}\right)\theta_i\tau + (I+\tau D)(2I+\tau D)^{-1}D^{-1}
\end{aligned}$$

$$\times \sum_{i=1}^{N-1}\left(R^{|l+1-i|}-R^{l+1+i}\right)\theta_i\tau + \left(-\frac{8}{12}\left(\frac{\lambda}{\tau}-l\right)+\frac{1}{2}\left(\frac{\lambda}{\tau}-l\right)^2+\frac{1}{6}\left(\frac{\lambda}{\tau}-l\right)^3\right)$$

$$\times \left\{\left(I-R^{2N}\right)^{-1}\left[\left(\left(R^{l-1}-R^{2N-l+1}\right)\nu_0+\left(R^{N-l-1}-R^{N+l-1}\right)\nu_N\right)\right.\right.$$

$$\left. - \left(R^{N-l+1}-R^{N+l-1}\right)(I+\tau D)(2I+\tau D)^{-1}D^{-1}\sum_{i=1}^{N-1}\left(R^{N-i}-R^{N+i}\right)\theta_i\tau\right]$$

$$\left. + (I+\tau D)(2I+\tau D)^{-1}D^{-1}\sum_{i=1}^{N-1}\left(R^{|l-2-i|}-R^{l-2+i}\right)\theta_i\tau\right\}$$

$$+ \left(1-\left(\frac{\lambda}{\tau}-l\right)^2\right)\left\{\left(I-R^{2N}\right)^{-1}\left[\left(\left(R^{l}-R^{2N-l}\right)\nu_0+\left(R^{N-l}-R^{N+l}\right)\nu_N\right)\right.\right.$$

$$- \left(R^{N-l}-R^{N+l}\right)(I+\tau D)(2I+\tau D)^{-1}D^{-1}$$

$$\left.\left. \times \sum_{i=1}^{N-1}\left(R^{N-i}-R^{N+i}\right)\theta_i\tau\right] + (I+\tau D)(2I+\tau D)^{-1}D^{-1}\sum_{i=1}^{N-1}\left(R^{|l-i|}-R^{l+i}\right)\theta_i\tau\right\}$$

$$+ \left(\frac{8}{12}\left(\frac{\lambda}{\tau}-l\right)+\frac{1}{2}\left(\frac{\lambda}{\tau}-l\right)^2-\frac{1}{6}\left(\frac{\lambda}{\tau}-l\right)^3\right)\left\{\left(I-R^{2N}\right)^{-1}\right.$$

$$\times \left[\left(\left(R^{l+1}-R^{2N-l-1}\right)\nu_0+\left(R^{N-l-1}-R^{N+l+1}\right)\nu_N\right)-\left(R^{N-l-1}-R^{N+l+1}\right)(I+\tau D)\right.$$

$$\left. \times (2I+\tau D)^{-1}D^{-1}\sum_{i=1}^{N-1}\left(R^{N-i}-R^{N+i}\right)\theta_i\tau\right] + (I+\tau D)(2I+\tau D)^{-1}D^{-1}$$

$$\left. \times \sum_{i=1}^{N-1}\left(R^{|l+1-i|}-R^{l+1+i}\right)\theta_i\tau\right\} + \left(-\frac{1}{12}\left(\frac{\lambda}{\tau}-l\right)+\frac{1}{12}\left(\frac{\lambda}{\tau}-l\right)^3\right)\left\{\left(I-R^{2N}\right)^{-1}\right.$$

$$\times \left[\left(\left(R^{l+2}-R^{2N-l-2}\right)\nu_0+\left(R^{N-l-2}-R^{N+l+2}\right)\nu_N\right)-\left(R^{N-l-2}-R^{N+l+2}\right)\right.$$

$$\left. \times (I+\tau D)(2I+\tau D)^{-1}D^{-1}\sum_{i=1}^{N-1}\left(R^{N-i}-R^{N+i}\right)\theta_i\tau\right]$$

$$\left. + (I+\tau D)(2I+\tau D)^{-1}D^{-1}\sum_{i=1}^{N-1}\left(R^{|l+2-i|}-R^{l+2+i}\right)\theta_i\tau\right\} + \varphi - \xi, \tag{2.24}$$

$$\nu_N = \nu_0 + \psi - \varphi.$$

Solving system (2.24), we have

$$\nu_0 = \left(\frac{1}{12}\left(\frac{\lambda}{\tau}-l\right)-\frac{1}{12}\left(\frac{\lambda}{\tau}-l\right)^3\right)G_2\left(R^{N-l+2}-R^{N+l-2}\right)(I+\tau D)$$

$$\times (2I+\tau D)^{-1}D^{-1}\sum_{i=1}^{N-1}\left(R^{N-i}-R^{N+i}\right)\theta_i\tau + G_2\left(I-R^{2N}\right)(I+\tau D)$$

$$\times (2I+\tau D)^{-1}D^{-1}\sum_{i=1}^{N-1}\left(R^{|l-2-i|}-R^{l-2+i}\right)\theta_i\tau$$

$$
\begin{aligned}
&+\left(-\frac{8}{12}\left(\frac{\lambda}{\tau}-l\right)+\frac{1}{2}\left(\frac{\lambda}{\tau}-l\right)^2+\frac{1}{6}\left(\frac{\lambda}{\tau}-l\right)^3\right)G_2\left(R^{N-l+1}-R^{N+l-1}\right)\\
&\times (I+\tau D)(2I+\tau D)^{-1}D^{-1}\sum_{i=1}^{N-1}\left(R^{N-i}-R^{N+i}\right)\theta_i\tau+G_2\left(I-R^{2N}\right)(I+\tau D)\\
&\times (2I+\tau D)^{-1}D^{-1}\sum_{i=1}^{N-1}\left(R^{|l-1-i|}-R^{l-1+i}\right)\theta_i\tau+\left(1-\left(\frac{\lambda}{\tau}-l\right)^2\right)G_2\\
&\times \left(R^{N-l}-R^{N+l}\right)(I+\tau D)(2I+\tau D)^{-1}D^{-1}\sum_{i=1}^{N-1}\left(R^{N-i}-R^{N+i}\right)\theta_i\tau\\
&+G_2\left(I-R^{2N}\right)(I+\tau D)(2I+\tau D)^{-1}D^{-1}\sum_{i=1}^{N-1}\left(R^{|l-i|}-R^{l+i}\right)\theta_i\tau\\
&+\left(\frac{8}{12}\left(\frac{\lambda}{\tau}-l\right)+\frac{1}{2}\left(\frac{\lambda}{\tau}-l\right)^2-\frac{1}{6}\left(\frac{\lambda}{\tau}-l\right)^3\right)G_2\left(R^{N-l-1}-R^{N+l+1}\right)\\
&\times (I+\tau D)(2I+\tau D)^{-1}D^{-1}\sum_{i=1}^{N-1}\left(R^{N-i}-R^{N+i}\right)\theta_i\tau+G_2\left(I-R^{2N}\right)(I+\tau D)\\
&\times (2I+\tau D)^{-1}D^{-1}\sum_{i=1}^{N-1}\left(R^{|l+1-i|}-R^{l+1+i}\right)\theta_i\tau+\left(-\frac{1}{12}\left(\frac{\lambda}{\tau}-l\right)+\frac{1}{12}\left(\frac{\lambda}{\tau}-l\right)^3\right)\\
&\times G_2\left(R^{N-l-2}-R^{N+l+2}\right)(I+\tau D)(2I+\tau D)^{-1}D^{-1}\sum_{i=1}^{N-1}\left(R^{N-i}-R^{N+i}\right)\theta_i\tau\\
&+G_2\left(I-R^{2N}\right)(I+\tau D)(2I+\tau D)^{-1}D^{-1}\sum_{i=1}^{N-1}\left(R^{|l+2-i|}-R^{l+2+i}\right)\theta_i\tau\\
&+G_2\left(I-R^{2N}\right)(\varphi-\xi)+G_2\left(\left(\frac{1}{12}\left(\frac{\lambda}{\tau}-l\right)-\frac{1}{12}\left(\frac{\lambda}{\tau}-l\right)^3\right)\left(R^{N-l-2}-R^{N+l+2}\right)\right.\\
&+\left(-\frac{8}{12}\left(\frac{\lambda}{\tau}-l\right)+\frac{1}{2}\left(\frac{\lambda}{\tau}-l\right)^2+\frac{1}{6}\left(\frac{\lambda}{\tau}-l\right)^3\right)\left(R^{N-l+1}-R^{N+l-1}\right)\\
&+\left(1-\left(\frac{\lambda}{\tau}-l\right)^2\right)\left(R^{N-l}-R^{N+l}\right)\\
&+\left(\frac{8}{12}\left(\frac{\lambda}{\tau}-l\right)+\frac{1}{2}\left(\frac{\lambda}{\tau}-l\right)^2-\frac{1}{6}\left(\frac{\lambda}{\tau}-l\right)^3\right)\left(R^{N-l-1}-R^{N+l+1}\right)\\
&\left.+\left(-\frac{1}{12}\left(\frac{\lambda}{\tau}-l\right)+\frac{1}{12}\left(\frac{\lambda}{\tau}-l\right)^3\right)\left(R^{N-l-2}-R^{N+l+2}\right)\right)(\psi-\varphi),
\end{aligned}
\tag{2.25}
$$

$$
v_N=v_0+\psi-\varphi. \tag{2.26}
$$

So, the difference problem (2.8) has a unique solution $\{v_k\}_{k=0}^N$, which is defined by formulas (2.19), (2.25), and (2.26). By using formulas (2.19), (2.25), (2.26), and the method of the monograph [23], we can get the stability estimate (2.23) for the solution of difference problem (2.8). The proofs of estimates (2.17), (2.18) are based on (2.5) and (2.23). Applying formula (2.5) and estimates (2.23), (2.17), we get estimate (2.16). Theorem 2.2 is proved. □

Theorem 2.3 *Assume that A is a self-adjoint positive definite operator, $\varphi,\psi,\xi\in D(A)$ and $\{\theta_k\}_{k=1}^{N-1}\in C_\tau(H)$. Then, the solutions $(\{u_k\}_{k=1}^{N-1},p)$ of difference problems* (2.3) *and* (2.4) *obey* *the following almost coercive inequality*:

$$\begin{aligned}&\bigl\|\{\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\}_{k=1}^{N-1}\bigr\|_{C_\tau(H)}+\left\|\left\{\left(A+\frac{\tau^2}{12}A^2\right)u_k\right\}_{k=1}^{N-1}\right\|_{C_\tau(H)}+\|p\|_H\\&\quad\le M(\delta)\left[\min\left\{\ln\frac{1}{\tau},1+\bigl|\ln\|B\|_{H\to H}\bigr|\right\}\bigl\|\{\theta_k\}_{k=1}^{N-1}\bigr\|_{C_\tau(H)}\right.\\&\qquad\left.+\left\|\left(A+\frac{\tau^2}{12}A^2\right)\varphi\right\|_H+\left\|\left(A+\frac{\tau^2}{12}A^2\right)\psi\right\|_H+\left\|\left(A+\frac{\tau^2}{12}A^2\right)\xi\right\|_H\right].\end{aligned}\tag{2.27}$$

Theorem 2.4 *Assume that A is a self-adjoint positive definite operator, $\varphi,\psi,\xi\in D(A)$ and $\{\theta_k\}_{k=1}^{N-1}\in C_\tau^{\alpha,\alpha}(H)$ $(0<\alpha<1)$. Then, the solutions $(\{u_k\}_{k=1}^{N-1},p)$ of difference problems* (2.3) *and* (2.4) *obey the following coercive inequality*:

$$\begin{aligned}&\bigl\|\{\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\}_{k=1}^{N-1}\bigr\|_{C_\tau^{\alpha,\alpha}(H)}+\left\|\left\{\left(A+\frac{\tau^2}{12}A^2\right)u_k\right\}_{k=1}^{N-1}\right\|_{C_\tau^{\alpha,\alpha}(H)}+\|p\|_H\\&\quad\le M(\delta)\left[\frac{1}{\alpha(1-\alpha)}\bigl\|\{\theta_k\}_{k=1}^{N-1}\bigr\|_{C_\tau^{\alpha,\alpha}(H)}+\left\|\left(A+\frac{\tau^2}{12}A^2\right)\varphi\right\|_H\right.\\&\qquad\left.+\left\|\left(A+\frac{\tau^2}{12}A^2\right)\psi\right\|_H+\left\|\left(A+\frac{\tau^2}{12}A^2\right)\xi\right\|_H\right].\end{aligned}\tag{2.28}$$

The proofs of Theorems 2.3 and 2.4 are based on formulas (2.5), (2.19), (2.21), (2.22), (2.25), (2.26), Lemmas 2.1 and 2.2.

3 High order of accuracy difference schemes for the problem (1.2) and their well-posedness

Now, we consider problem (1.2). The differential expression [22, 23]

$$A^x u(x)=-\sum_{r=1}^{n}(a_r(x)u_{x_r})_{x_r}+\sigma u$$

defines a self-adjoint strongly positive definite operator A^x acting on $L_2(\overline{\Omega})$ with the domain

$$D(A^x)=\{u(x)\in W_2(\overline{\Omega}),u(x)=0,x\in S\}.$$

The discretization of problem (1.2) is carried out in two steps. In the first step, we define the grid spaces

$$\begin{aligned}\widetilde{\Omega}_h&=\{x=x_m=(h_1m_1,\dots,h_nm_n);m=(m_1,\dots,m_n),\\&\qquad m_r=0,\dots,N_r,h_rN_r=1,r=1,\dots,n\},\\\Omega_h&=\widetilde{\Omega}_h\cap\Omega,\qquad S_h=\widetilde{\Omega}_h\cap S.\end{aligned}$$

To the differential operator A^x generated by problem (1.2) we assign the difference operator A_h^x defined by the formula

$$A_h^x u^h(x)=-\sum_{r=1}^{n}\bigl(a_r(x)u_{\bar{x}_r}^h\bigr)_{x_r,j_r}+\sigma u^h(x)\tag{3.1}$$

acting in the space of grid functions $u^h(x)$, satisfying the condition $u^h(x) = 0$ for all $x \in S_h$.

To formulate our results, let $L_{2h} = L_2(\widetilde{\Omega}_h)$ and $W_{2h}^2 = W_2^2(\widetilde{\Omega}_h)$ be spaces of the grid functions $\zeta^h(x) = \{\zeta(h_1 m_1, \dots, h_n m_n)\}$ defined on $\widetilde{\Omega}_h$, equipped with the norms

$$\|\zeta\|_{L_{2h}} = \left(\sum_{x\in\widetilde{\Omega}_h} \left|\zeta^h(x)\right|^2 h_1 \cdots h_n\right)^{1/2},$$

$$\begin{aligned}\left\|\zeta^h\right\|_{W_{2h}^2} = \left\|\zeta^h\right\|_{L_{2h}} &+ \left(\sum_{x\in\widetilde{\Omega}_h} \sum_{r=1}^{n} \left|\left(\zeta^h\right)_{x_r}\right|^2 h_1 \cdots h_n\right)^{1/2} \\ &+ \left(\sum_{x\in\widetilde{\Omega}_h} \sum_{r=1}^{n} \left|\left(\zeta^h(x)\right)_{x_r \bar{x}_r, m_r}\right|^2 h_1 \cdots h_n\right)^{1/2}.\end{aligned}$$

Applying formula (2.5) to A_h^x, we arrive for $v^h(t,x)$ functions, at auxiliary nonlocal boundary value problem for a system of ordinary differential equations

$$\begin{cases} -\frac{d^2 v^h(t,x)}{dt^2} + A_h^x v^h(t,x) = f^h(t,x), & 0 < t < T, x \in \widetilde{\Omega}_h, \\ v^h(0,x) - v^h(\lambda,x) = \varphi(x) - \xi(x), & x \in \widetilde{\Omega}_h \\ v^h(T,x) - v^h(\lambda,x) = \psi(x) - \xi(x), & x \in \widetilde{\Omega}_h. \end{cases} \tag{3.2}$$

We define function $p^h(x)$ by formula

$$p^h(x) = A_h^x \varphi^h(x) - A_h^x v^h(0,x), \quad x \in \widetilde{\Omega}_h. \tag{3.3}$$

In the second step, auxiliary nonlocal problem (3.2) is replaced by the third order of accuracy difference scheme

$$\begin{aligned}
&-\frac{v_{k+1}^h(x) - 2v_k^h(x) + v_{k-1}^h(x)}{\tau^2} + A_h^x v_k^h(x) + \frac{\tau^2}{12}\left(A_h^x\right)^2 v_k^h(x) = \theta_k^h(x), \\
&\quad \theta_k^h(x) = f^h(t_k,x) + \frac{\tau^2}{12}\left(\frac{f^h(t_{k+1},x) - 2f^h(t_k,x) + f^h(t_{k-1},x)}{\tau^2} + A_h^x f^h(t_k,x)\right), \\
&v_0^h(x) - \left(\frac{1}{2}\left(\frac{\lambda}{\tau} - l\right) + \frac{1}{2}\left(\frac{\lambda}{\tau} - l\right)^2\right) v_{l-1}^h(x) - \left(1 - \frac{1}{2}\left(\frac{\lambda}{\tau} - l\right)^2\right) v_l^h(x) \\
&\quad - \left(-\frac{1}{2}\left(\frac{\lambda}{\tau} - l\right) + \frac{1}{2}\left(\frac{\lambda}{\tau} - l\right)^2\right) v_{l+1}^h(x) = \varphi^h(x) - \xi^h(x), \quad x \in \widetilde{\Omega}_h, \\
&v_N^h(x) - \left(\frac{1}{2}\left(\frac{\lambda}{\tau} - l\right) + \frac{1}{2}\left(\frac{\lambda}{\tau} - l\right)^2\right) v_{l-1}^h(x) - \left(1 - \frac{1}{2}\left(\frac{\lambda}{\tau} - l\right)^2\right) v_l^h(x) \\
&\quad - \left(-\frac{1}{2}\left(\frac{\lambda}{\tau} - l\right) + \frac{1}{2}\left(\frac{\lambda}{\tau} - l\right)^2\right) v_{l+1}^h(x) = \varphi^h(x) - \xi^h(x), \quad x \in \widetilde{\Omega}_h, \\
&\quad l = \left[\frac{\lambda}{\tau}\right],
\end{aligned} \tag{3.4}$$

and by the fourth order of accuracy difference scheme

$$
\begin{aligned}
&-\frac{v_{k+1}^h(x)-2v_k^h(x)+v_{k-1}^h(x)}{\tau^2}+A_h^x v_k^h(x)+\frac{\tau^2}{12}\left(A_h^x\right)^2 v_k^h(x)=\theta_k^h(x),\\
&\theta_k^h(x)=f^h(t_k,x)+\frac{\tau^2}{12}\left(\frac{f^h(t_{k+1},x)-2f^h(t_k,x)+f^h(t_{k-1},x)}{\tau^2}+A_h^x f^h(t_k,x)\right),\\
&t_k=k\tau, 1\le k\le N-1, x\in\widetilde{\Omega}_h,\\
&v_0^h(x)-\left(\frac{1}{12}\left(\frac{\lambda}{\tau}-l\right)-\frac{1}{12}\left(\frac{\lambda}{\tau}-l\right)^3\right)v_{l-2}^h(x)\\
&\quad-\left(-\frac{8}{12}\left(\frac{\lambda}{\tau}-l\right)+\frac{1}{2}\left(\frac{\lambda}{\tau}-l\right)^2+\frac{1}{6}\left(\frac{\lambda}{\tau}-l\right)^3\right)v_{l-1}^h(x)-\left(1-\left(\frac{\lambda}{\tau}-l\right)^2\right)v_l^h(x)\\
&\quad-\left(\frac{8}{12}\left(\frac{\lambda}{\tau}-l\right)+\frac{1}{2}\left(\frac{\lambda}{\tau}-l\right)^2-\frac{1}{6}\left(\frac{\lambda}{\tau}-l\right)^3\right)v_{l+1}^h(x)\\
&\quad-\left(-\frac{1}{12}\left(\frac{\lambda}{\tau}-l\right)+\frac{1}{12}\left(\frac{\lambda}{\tau}-l\right)^3\right)v_{l+2}^h(x)=\varphi^h(x)-\xi^h(x),\quad x\in\widetilde{\Omega}_h,\\
&v_N^h(x)-\left(\frac{1}{12}\left(\frac{\lambda}{\tau}-l\right)-\frac{1}{12}\left(\frac{\lambda}{\tau}-l\right)^3\right)v_{l-2}^h(x)\\
&\quad-\left(-\frac{8}{12}\left(\frac{\lambda}{\tau}-l\right)+\frac{1}{2}\left(\frac{\lambda}{\tau}-l\right)^2+\frac{1}{6}\left(\frac{\lambda}{\tau}-l\right)^3\right)v_{l-1}^h(x)-\left(1-\left(\frac{\lambda}{\tau}-l\right)^2\right)v_l^h(x)\\
&\quad-\left(\frac{8}{12}\left(\frac{\lambda}{\tau}-l\right)+\frac{1}{2}\left(\frac{\lambda}{\tau}-l\right)^2-\frac{1}{6}\left(\frac{\lambda}{\tau}-l\right)^3\right)v_{l+1}^h(x)\\
&\quad-\left(-\frac{1}{12}\left(\frac{\lambda}{\tau}-l\right)+\frac{1}{12}\left(\frac{\lambda}{\tau}-l\right)^3\right)v_{l+2}^h(x)=\varphi^h(x)-\xi^h(x),\quad x\in\widetilde{\Omega}_h.
\end{aligned}
\tag{3.5}
$$

Let τ and $|h|=\sqrt{h_1^2+\cdots+h_n^2}$ be sufficiently small positive numbers.

Theorem 3.1 *The solutions of difference schemes* (3.4) *and* (3.5) *obey the following stability estimates*:

$$
\begin{aligned}
&\left\|\{u_k^h\}_1^{N-1}\right\|_{C_\tau(L_{2h})}\le M(\delta)\left[\|\varphi^h\|_{L_{2h}}+\|\psi^h\|_{L_{2h}}+\|\xi^h\|_{L_{2h}}+\left\|\{f_k^h\}_1^{N-1}\right\|_{C_\tau(L_{2h})}\right],\\
&\|p^h\|_{L_{2h}}\le M(\delta)\left[\|A\varphi^h\|_{W_{2h}^2}+\|A\psi^h\|_{W_{2h}^2}+\|A\xi^h\|_{W_{2h}^2}+\frac{1}{\alpha(1-\alpha)}\left\|\{f_k^h\}_1^{N-1}\right\|_{C_\tau^{\alpha,\alpha}(L_{2h})}\right].
\end{aligned}
$$

Theorem 3.2 *The solutions of difference schemes* (3.4) *and* (3.5) *obey the following almost coercive stability estimate*:

$$
\begin{aligned}
&\left\|\left\{\frac{u_{k+1}^h-2u_k^h+u_{k-1}^k}{\tau^2}\right\}_1^{N-1}\right\|_{C_\tau(L_{2h})}+\left\|\left\{\left(A+\frac{\tau^2}{12}A^2\right)u_k\right\}_{k=1}^{N-1}\right\|_{C_\tau(W_{2h}^2)}+\|p^h\|_{L_{2h}}\\
&\quad\le M(\delta)\left[\ln\left(\frac{1}{\tau+h}\right)\left\|\{f_k^h\}_1^N\right\|_{C_\tau(L_{2h})}+\left\|\left(A+\frac{\tau^2}{12}A^2\right)\varphi^h\right\|_{W_{2h}^2}\right.\\
&\qquad\left.+\left\|\left(A+\frac{\tau^2}{12}A^2\right)\psi^h\right\|_{W_{2h}^2}+\left\|\left(A+\frac{\tau^2}{12}A^2\right)\xi^h\right\|_{W_{2h}^2}\right].
\end{aligned}
$$

Theorem 3.3 *The solutions of difference schemes* (3.4) *and* (3.5) *obey the following coercive stability estimate*:

$$\left\| \left\{ \frac{u_{k+1}^h - 2u_k^h + u_{k-1}^k}{\tau^2} \right\}_1^{N-1} \right\|_{C_\tau^{\alpha,\alpha}(L_{2h})} + \left\| \{u_k\}_{k=1}^{N-1} \right\|_{C_\tau^{\alpha,\alpha}(W_{2h}^2)} + \left\| p^h \right\|_{L_{2h}}$$
$$\leq M(\delta)\left[\frac{1}{\alpha(1-\alpha)} \left\| \{f_k^h\}_1^N \right\|_{C_\tau^{\alpha,\alpha}(L_{2h})} + \left\| \varphi^h \right\|_{W_{2h}^2} + \left\| \psi^h \right\|_{W_{2h}^2} + \left\| \xi^h \right\|_{W_{2h}^2} \right].$$

The proofs of Theorems 3.1-3.3 are based on the abstract Theorems 2.1-2.4, symmetry properties of the operator A_h^x in L_{2h} and the following theorem on the coercivity inequality for the solution of the elliptic difference problem in L_{2h}.

Theorem 3.4 [24] *For the solution of the elliptic difference problem*

$$\begin{cases} A_h^x u^h(x) = \omega^h(x), & x \in \widetilde{\Omega}_h, \\ u^h(x) = 0, & x \in S_h, \end{cases}$$

the following coercivity inequality holds:

$$\sum_{r=1}^{n} \left\| (u_k^h)_{\bar{x}_r \bar{x}_r, j_r} \right\|_{L_{2h}} \leq M \left\| \omega^h \right\|_{L_{2h}},$$

where M does not depend on h and ω^h.

4 Numerical results

In this section, by using the third and fourth order of the accuracy approximation, we obtain an approximate solution of the inverse problem

$$\begin{cases} -\frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial}{\partial x}\left(\frac{\partial u(t,x)}{\partial x}\right) + u(t,x) = f(t,x) + p(x), & 0 < x < \pi, 0 < t < T, \\ f(t,x) = (\exp(-t) + 2t)\sin(x), \\ u(0,x) = 2\sin(x), & 0 \leq x \leq \pi, \\ u(T,x) = (\exp(-T) + T + 1)\sin(x), & 0 \leq x \leq \pi, \\ u(\lambda,x) = (\exp(-\lambda) + \lambda + 1)\sin(x), & 0 \leq x \leq \pi, \\ u(t,0) = u(t,\pi) = 0, & 0 \leq t \leq T \ (T = 2, \lambda = \frac{4}{7}T) \end{cases} \tag{4.1}$$

for the elliptic equation. Note that $u(t,x) = (\exp(-t) + t + 1)\sin(x)$ and $p(x) = 2\sin(x)$ are the exact solutions of equation (4.1).

For the approximate solution of the nonlocal boundary value problem (3.2), consider the set of grid points

$$[0,T]_\tau \times [0,\pi]_h$$
$$= \left\{ (t_k, x_n) : t_k = k\tau, k = 1, \ldots, N-1, x_n = nh, n = 1, \ldots, M-1 \right\},$$

which depends on the small parameters $\tau = \frac{T}{N}$ and $h = \frac{\pi}{M}$.

Applying approximations (3.4) and (3.5), we get, respectively, the third order of the accuracy difference scheme

$$\begin{cases} -\frac{v_n^{k+1}-2v_n^k+v_n^{k-1}}{\tau^2} - \frac{v_{n+1}^k-2v_n^k+v_{n-1}^k}{h^2} + v_n^k \\ \qquad - \frac{\tau^2}{12}[-\frac{1}{h^2}(-\frac{v_{n+2}^k-2v_{n+1}^k+v_n^k}{h^2} + v_{n+1}^k) + \frac{2}{h^2}(-\frac{v_{n+1}^k-2v_n^k+v_{n-1}^k}{h^2} + v_n^k) \\ \qquad - \frac{1}{h^2}(\frac{v_n^k-2v_{n-1}^k+v_{n-2}^k}{h^2} + v_{n-1}^k) - \frac{v_{n+1}^k-2v_n^k+v_{n-1}^k}{h^2} + v_n^k] \\ \quad = (\exp(-t_k) + 2t_k)\sin(x_n) + \frac{\tau^2}{12}(\exp(-t_k) + 4t_k)\sin(x_n), \\ \qquad k = 1,\dots,N-1, n = 1,\dots,M-2, \\ v_0^k = v_M^k = 0, \qquad v_1^k = \frac{4}{5}v_2^k - \frac{1}{5}v_3^k, \qquad v_{M-1}^k = \frac{4}{5}v_{M-2}^k - \frac{1}{5}v_{M-3}^k, \\ \quad k = 0,\dots,N, \\ v_n^0 - (\frac{1}{2}(\frac{\lambda}{\tau} - l) + \frac{1}{2}(\frac{\lambda}{\tau} - l)^2)v_n^{l-1} - (1 - \frac{1}{2}(\frac{\lambda}{\tau} - l)^2)v_n^l \\ \quad - (-\frac{1}{2}(\frac{\lambda}{\tau} - l) + \frac{1}{2}(\frac{\lambda}{\tau} - l)^2)v_n^{l+1} = (1 - \exp(-\lambda) - \lambda)\sin(x_n), \\ \quad n = 0,\dots,M, \\ v_n^N - (\frac{1}{2}(\frac{\lambda}{\tau} - l) + \frac{1}{2}(\frac{\lambda}{\tau} - l)^2)v_n^{l-1} - (1 - \frac{1}{2}(\frac{\lambda}{\tau} - l)^2)v_n^l \\ \qquad - (-\frac{1}{2}(\frac{\lambda}{\tau} - l) + \frac{1}{2}(\frac{\lambda}{\tau} - l)^2)v_n^{l+1} \\ \quad = (\exp(-t_N) - \exp(-\lambda) + t_N - \lambda)\sin(x_n), \\ \qquad n = 0,\dots,M, \end{cases} \tag{4.2}$$

and the fourth order of the accuracy difference scheme

$$\begin{cases} -\frac{v_n^{k+1}-2v_n^k+v_n^{k-1}}{\tau^2} - \frac{v_{n+1}^k-2v_n^k+v_{n-1}^k}{h^2} + v_n^k \\ \qquad - \frac{\tau^2}{12}[-\frac{1}{h^2}(-\frac{v_{n+2}^k-2v_{n+1}^k+v_n^k}{h^2} + v_{n+1}^k) + \frac{2}{h^2}(-\frac{v_{n+1}^k-2v_n^k+v_{n-1}^k}{h^2} + v_n^k) \\ \qquad - \frac{1}{h^2}(\frac{v_n^k-2v_{n-1}^k+v_{n-2}^k}{h^2} + v_{n-1}^k) - \frac{v_{n+1}^k-2v_n^k+v_{n-1}^k}{h^2} + v_n^k] \\ \quad = (\exp(-t_k) + 2t_k)\sin(x_n) + \frac{\tau^2}{12}(\exp(-t_k) + 4t_k)\sin(x_n), \\ \qquad k = 1,\dots,N-1, n = 2,\dots,M-2, \\ v_0^k = v_M^k = 0, \qquad v_1^k = \frac{4}{5}v_2^k - \frac{1}{5}v_3^k, \qquad v_{M-1}^k = \frac{4}{5}v_{M-2}^k - \frac{1}{5}v_{M-3}^k, \\ \quad k = 0,\dots,N, \\ v_n^0 - (\frac{1}{12}(\frac{\lambda}{\tau} - l) - \frac{1}{12}(\frac{\lambda}{\tau} - l)^3)v_n^{l-2} \\ \qquad - (-\frac{8}{12}(\frac{\lambda}{\tau} - l) + \frac{1}{2}(\frac{\lambda}{\tau} - l)^2 + \frac{1}{6}(\frac{\lambda}{\tau} - l)^3)v_n^{l-1} - (1 - (\frac{\lambda}{\tau} - l)^2)v_n^l \\ \qquad - (\frac{8}{12}(\frac{\lambda}{\tau} - l) + \frac{1}{2}(\frac{\lambda}{\tau} - l)^2 - \frac{1}{6}(\frac{\lambda}{\tau} - l)^3)v_n^{l+1} \\ \qquad - (-\frac{1}{12}(\frac{\lambda}{\tau} - l) + \frac{1}{12}(\frac{\lambda}{\tau} - l)^3)v_n^{l+2} \\ \quad = (1 - \exp(-\lambda) - \lambda)\sin(x_n), \quad n = 0,\dots,M, \\ v_n^N - (\frac{1}{12}(\frac{\lambda}{\tau} - l) - \frac{1}{12}(\frac{\lambda}{\tau} - l)^3)v_n^{l-2} \\ \qquad - (-\frac{8}{12}(\frac{\lambda}{\tau} - l) + \frac{1}{2}(\frac{\lambda}{\tau} - l)^2 + \frac{1}{6}(\frac{\lambda}{\tau} - l)^3)v_n^{l-1} - (1 - (\frac{\lambda}{\tau} - l)^2)v_n^l \\ \qquad - (\frac{8}{12}(\frac{\lambda}{\tau} - l) + \frac{1}{2}(\frac{\lambda}{\tau} - l)^2 - \frac{1}{6}(\frac{\lambda}{\tau} - l)^3)v_n^{l+1} \\ \qquad - (-\frac{1}{12}(\frac{\lambda}{\tau} - l) + \frac{1}{12}(\frac{\lambda}{\tau} - l)^3)v_n^{l+2} \\ \quad = (\exp(-t_N) - \exp(-\lambda) + t_N - \lambda)\sin(x_n), \quad n = 0,\dots,M \end{cases} \tag{4.3}$$

for the approximate solutions of the auxiliary nonlocal boundary value problem (3.2). Applying approximation (3.3) and the second order of the accuracy in x in the approximation of A, we get the following values of the p function in the grid points:

$$p_n = -\frac{1}{h^2}\big((\varphi_{n+1} - v_{n+1}^0) - 2(\varphi_n - v_n^0) + (\varphi_{n-1} - v_{n-1}^0)\big) + (\varphi_n - v_n^0)$$

$$n = 1,\dots,M-1. \tag{4.4}$$

In this step, applying to the boundary value problem for the function $w(t,x)$ for the third and fourth order approximation in the variable t, we get, respectively, the third order of the accuracy difference scheme

$$\begin{cases} -\frac{w_n^{k+1}-2w_n^k+v_n^{k-1}}{\tau^2}-\frac{w_{n+1}^k-2w_n^k+w_{n-1}^k}{h^2}+w_n^k \\ \quad -\frac{\tau^2}{12}[-\frac{1}{h^2}(-\frac{w_{n+2}^k-2w_{n+1}^k+w_n^k}{h^2}+w_{n+1}^k) \\ \quad +\frac{2}{h^2}(-\frac{w_{n+1}^k-2w_n^k+w_{n-1}^k}{h^2}+w_n^k)-\frac{1}{h^2}(\frac{w_n^k-2w_{n-1}^k+w_{n-2}^k}{h^2}+w_{n-1}^k) \\ \quad -\frac{w_{n+1}^k-2w_n^k+w_{n-1}^k}{h^2}+w_n^k]=p(x_n)+\frac{\tau^2}{6}p(x_n), \\ \quad k=1,\dots,N-1, n=2,\dots,M-2, \\ w_0^k=w_M^k=0, \qquad w_1^k=\frac{4}{5}w_2^k-\frac{1}{5}w_3^k, \qquad w_{M-1}^k=\frac{4}{5}w_{M-2}^k-\frac{1}{5}w_{M-3}^k, \\ \quad k=0,\dots,N, \\ w_n^0=(\exp(-\lambda)+\lambda+1)\sin(x_n)-(\frac{1}{2}(\frac{\lambda}{\tau}-l)+\frac{1}{2}(\frac{\lambda}{\tau}-l)^2)v_n^{l-1} \\ \qquad -(1-\frac{1}{2}(\frac{\lambda}{\tau}-l)^2)v_n^l-(-\frac{1}{2}(\frac{\lambda}{\tau}-l)+\frac{1}{2}(\frac{\lambda}{\tau}-l)^2)v_n^{l+1}, \\ \qquad n=0,\dots,M, \\ w_n^N=(\exp(-\lambda)+\lambda+1)\sin(x_n)-(\frac{1}{2}(\frac{\lambda}{\tau}-l)+\frac{1}{2}(\frac{\lambda}{\tau}-l)^2)v_n^{l-1} \\ \qquad -(1-\frac{1}{2}(\frac{\lambda}{\tau}-l)^2)v_n^l-(-\frac{1}{2}(\frac{\lambda}{\tau}-l)+\frac{1}{2}(\frac{\lambda}{\tau}-l)^2)v_n^{l+1}, \\ \qquad n=0,\dots,M, \end{cases} \tag{4.5}$$

and the fourth order of the accuracy difference scheme

$$\begin{cases} -\frac{w_n^{k+1}-2w_n^k+v_n^{k-1}}{\tau^2}-\frac{w_{n+1}^k-2w_n^k+w_{n-1}^k}{h^2}+w_n^k \\ \quad -\frac{\tau^2}{12}[-\frac{1}{h^2}(-\frac{w_{n+2}^k-2w_{n+1}^k+w_n^k}{h^2}+w_{n+1}^k) \\ \quad +\frac{2}{h^2}(-\frac{w_{n+1}^k-2w_n^k+w_{n-1}^k}{h^2}+w_n^k)-\frac{1}{h^2}(\frac{w_n^k-2w_{n-1}^k+w_{n-2}^k}{h^2}+w_{n-1}^k) \\ \quad -\frac{w_{n+1}^k-2w_n^k+w_{n-1}^k}{h^2}+w_n^k]=p(x_n)+\frac{\tau^2}{6}p(x_n), \\ \quad k=1,\dots,N-1, n=2,\dots,M-2, \\ w_0^k=w_M^k=0, \qquad w_1^k=\frac{4}{5}w_2^k-\frac{1}{5}w_3^k, \qquad w_{M-1}^k=\frac{4}{5}w_{M-2}^k-\frac{1}{5}w_{M-3}^k, \\ \quad k=0,\dots,N, \\ w_n^0=(\exp(-\lambda)+\lambda+1)\sin(x_n)-(\frac{1}{12}(\frac{\lambda}{\tau}-l) \\ \qquad -\frac{1}{12}(\frac{\lambda}{\tau}-l)^3)v_n^{l-2}-(-\frac{8}{12}(\frac{\lambda}{\tau}-l)+\frac{1}{2}(\frac{\lambda}{\tau}-l)^2 \\ \qquad +\frac{1}{6}(\frac{\lambda}{\tau}-l)^3)v_n^{l-1}-(1-(\frac{\lambda}{\tau}-l)^2)v_n^l \\ \qquad -(\frac{8}{12}(\frac{\lambda}{\tau}-l)+\frac{1}{2}(\frac{\lambda}{\tau}-l)^2-\frac{1}{6}(\frac{\lambda}{\tau}-l)^3)v_n^{l+1} \\ \qquad -(-\frac{1}{12}(\frac{\lambda}{\tau}-l)+\frac{1}{12}(\frac{\lambda}{\tau}-l)^3)v_n^{l+2}, \\ w_n^N=(\exp(-\lambda)+\lambda+1)\sin(x_n)-(\frac{1}{12}(\frac{\lambda}{\tau}-l) \\ \qquad -\frac{1}{12}(\frac{\lambda}{\tau}-l)^3)v_n^{l-2}-(-\frac{8}{12}(\frac{\lambda}{\tau}-l)+\frac{1}{2}(\frac{\lambda}{\tau}-l)^2 \\ \qquad +\frac{1}{6}(\frac{\lambda}{\tau}-l)^3)v_n^{l-1}-(1-(\frac{\lambda}{\tau}-l)^2)v_n^l \\ \qquad -(\frac{8}{12}(\frac{\lambda}{\tau}-l)+\frac{1}{2}(\frac{\lambda}{\tau}-l)^2-\frac{1}{6}(\frac{\lambda}{\tau}-l)^3)v_n^{l+1} \\ \qquad -(-\frac{1}{12}(\frac{\lambda}{\tau}-l)+\frac{1}{12}(\frac{\lambda}{\tau}-l)^3)v_n^{l+2}. \end{cases} \tag{4.6}$$

We can rewrite the difference scheme (4.2) in the following matrix form:

$$AV_{n+2}+BV_{n+1}+CV_n+DV_{n-1}+EV_{n-2}=I\theta_n, \quad n=2,\dots,M-2, \tag{4.7}$$

$$V_0=\vec{0}, \qquad V_M=\vec{0},$$

$$V_1=\frac{4}{5}V_2-\frac{1}{5}V_3, \qquad V_{M-1}=\frac{4}{5}V_{M-2}-\frac{1}{5}V_{M-3}.$$

Here, I is the $(N+1)\times(N+1)$ identity matrix, θ_n is $(N+1)\times 1$ column matrix, A, B, C, D, E are $(N+1)\times(N+1)$ square matrices. Moreover,

$$A = E = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)}, \tag{4.8}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & y & z & q & 0 & \cdots & 0 & 0 & 0 & 0 \\ r & c & r & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & r & c & r & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & r & c & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & c & r & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & r & c & r & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & r & c & r \\ 0 & 0 & 0 & 0 & \cdots & 0 & y & z & q & 0 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B = D = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)}, \tag{4.9}$$

$$a = \frac{\tau^2}{12h^4}, \qquad b = -\frac{1}{h^2} - \frac{\tau^2}{3h^4} - \frac{\tau^2}{6h^2}, \qquad c = 1 + \frac{2}{\tau^2} + \frac{2}{h^2} - \frac{\tau^2}{12}\left(\frac{6}{h^4} + \frac{4}{h^2} + 1\right),$$

$$r = -\frac{1}{\tau^2}, \qquad y = -\left(\frac{1}{2}\left(\frac{\lambda}{\tau} - l\right) + \frac{1}{2}\left(\frac{\lambda}{\tau} - l\right)^2\right), \qquad z = -\left(1 - \frac{1}{2}\left(\frac{\lambda}{\tau} - l\right)^2\right),$$

$$q = -\left(-\frac{1}{2}\left(\frac{\lambda}{\tau} - l\right) + \frac{1}{2}\left(\frac{\lambda}{\tau} - l\right)^2\right),$$

$$\theta_n = \begin{bmatrix} \theta_n^0 \\ \vdots \\ \theta_n^N \end{bmatrix}, \qquad \theta_n^k = \bigl(\exp(-t_k) + 2t_k\bigr)\sin(x_n) + \frac{\tau^2}{12}\bigl(\exp(-t_k) + 4t_k\bigr)\sin(x_n), \tag{4.10}$$

$k = 1, \ldots, N-1, n = 1, \ldots, M-1,$

$\theta_n^0 = \bigl(1 - \exp(-\lambda) - \lambda\bigr)\sin(x_n), \theta_n^N = \bigl(\exp(-t_N) - \exp(-\lambda) + t_N - \lambda\bigr)\sin(x_n),$

$$n = 1, \ldots, M-1,$$

$$V_s = \begin{bmatrix} v_s^0 \\ \vdots \\ v_s^N \end{bmatrix}_{(N+1)\times 1}, \quad s = n-1, n, n+1.$$

For the solution of the linear matrix equation (4.7), we use the modified Gauss elimination method [25]. Namely, we seek a solution of equation (4.7) by the formula

$$V_n = \alpha_n V_{n+1} + \beta_n V_{n+2} + \gamma_n, \quad n = M-2, \ldots, 0. \tag{4.11}$$

Here, α_n and β_n ($n = 1, \ldots, M$) are $(N+1) \times (N+1)$ square matrices, γ_n ($n = 1, \ldots, M$) are $(N+1) \times 1$ column matrices which are defined by

$$\begin{aligned} &F_n = (C + D\alpha_{n-1} + E\beta_{n-2} + E\alpha_{n-2}\alpha_{n-1}), \\ &\beta_n = -F_n^{-1}A, \qquad \alpha_n = -F_n^{-1}(B + D\beta_{n-1} + E\alpha_{n-2}\beta_{n-1}), \\ &\gamma_n = -F_n^{-1}(I\theta_n - D\gamma_{n-1} - E\alpha_{n-2}\gamma_{n-1} - E\gamma_{n-2}), \quad n = 2, \ldots, M-2 \end{aligned}$$

with $\gamma_0 = \gamma_1 = \vec{0}$, and $\alpha_0 = \beta_0$ are the $(N+1) \times (N+1)$ zero matrix, $\alpha_1 = -4$, $\beta_1 = \frac{4}{5}I$. V_M and V_{M-1} are defined by formulas

$$\begin{aligned} &V_M = \vec{0}, \qquad D_M = (\beta_{M-2} + 5I) - (4I - \alpha_{M-2})\alpha_{M-1}, \\ &V_{M-1} = D_M^{-1}\big[(4I - \alpha_{M-2})\gamma_{M-1} - \gamma_{M-2}\big]. \end{aligned}$$

We rewrite the difference scheme (4.5) in matrix form,

$$\begin{aligned} &AW_{n+2} + BW_{n+1} + CW_n + DW_{n-1} + EW_{n-2} = I\eta_n, \quad n = 2, \ldots, M-2, \\ &W_0 = \vec{0}, \qquad W_M = \vec{0}, \\ &W_1 = \frac{4}{5}W_2 - \frac{1}{5}W_3, \qquad W_{M-1} = \frac{4}{5}W_{M-2} - \frac{1}{5}W_{M-3}. \end{aligned} \tag{4.12}$$

Here, η_n is an $(N+1) \times 1$ column matrix, A, B, D, E are defined by formulas (4.8) and (4.9). We use $(N+1) \times (N+1)$ square matrices, and C is the following matrix:

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ r & c & r & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & r & c & r & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & r & c & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & c & r & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & r & c & r & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & r & c & r \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix}_{(N+1)\times(N+1)}, \tag{4.13}$$

$$\eta_n = \begin{bmatrix} \eta_n^0 \\ \vdots \\ \eta_n^N \end{bmatrix},$$

$$\begin{aligned} \eta_n^0 &= \bigl(\exp(-t_l) + t_l + 1\bigr)\sin(x_n) \\ &\quad - \left(\frac{1}{2}\left(\frac{\lambda}{\tau} - l\right) + \frac{1}{2}\left(\frac{\lambda}{\tau} - l\right)^2\right)v_n^{l-1} - \left(1 - \frac{1}{2}\left(\frac{\lambda}{\tau} - l\right)^2\right)v_n^l, \quad n = 1, \dots, M-1, \\ \eta_n^N &= \bigl(\exp(-t_l) + t_l + 1\bigr)\sin(x_n) - V_n^l, \quad n = 1, \dots, M-1, \end{aligned}$$

$$\eta_n^k = p(x_n) + \frac{\tau^2}{6}p(x_n), \quad n = 1, \dots, N-1, n = 1, \dots, M-1,$$

$$W_s = \begin{bmatrix} w_s^0 \\ \vdots \\ w_s^N \end{bmatrix}_{(N+1)\times 1}, \quad s = n-1, n, n+1.$$

We can write the difference scheme (4.3) in matrix form (4.12), where A, B, D, E are defined by formulas (4.8) and (4.9), θ_n is defined by equation (4.10), C is defined by

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & d & e & g & y & z & 0 & \cdots & 0 & 0 & 0 & 0 \\ r & c & r & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & r & c & r & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & r & c & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & c & r & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & r & c & r & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & r & c & r \\ 0 & 0 & 0 & 0 & \cdots & 0 & d & e & g & y & z & 0 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$\begin{aligned} d &= -\left(\frac{1}{12}\left(\frac{\lambda}{\tau} - l\right) - \frac{1}{12}\left(\frac{\lambda}{\tau} - l\right)^3\right), \\ e &= -\left(-\frac{8}{12}\left(\frac{\lambda}{\tau} - l\right) + \frac{1}{2}\left(\frac{\lambda}{\tau} - l\right)^2 + \frac{1}{6}\left(\frac{\lambda}{\tau} - l\right)^3\right), \\ g &= -\left(1 - \left(\frac{\lambda}{\tau} - l\right)^2\right), \qquad y = -\left(\frac{8}{12}\left(\frac{\lambda}{\tau} - l\right) + \frac{1}{2}\left(\frac{\lambda}{\tau} - l\right)^2 - \frac{1}{6}\left(\frac{\lambda}{\tau} - l\right)^3\right), \\ z &= -\left(-\frac{1}{12}\left(\frac{\lambda}{\tau} - l\right) + \frac{1}{12}\left(\frac{\lambda}{\tau} - l\right)^3\right). \end{aligned}$$

We have the difference scheme (4.6) in the matrix form of equation (4.12), where A, B, D, E are defined by formulas (4.8) and (4.9), θ_n is defined by formula (4.10), C is defined by equation (4.13), η_n is defined by

$$\begin{aligned} \eta_n^0 &= \bigl(\exp(-t_l) + t_l + 1\bigr)\sin(x_n) - \left(-\frac{8}{12}\left(\frac{\lambda}{\tau} - l\right) + \frac{1}{2}\left(\frac{\lambda}{\tau} - l\right)^2 + \frac{1}{6}\left(\frac{\lambda}{\tau} - l\right)^3\right)v_n^{l-2} \\ &\quad - \left(1 - \left(\frac{\lambda}{\tau} - l\right)^2\right)v_n^{l-1} - \left(1 - \frac{1}{2}\left(\frac{\lambda}{\tau} - l\right)^2\right)v_n^l - \left(\frac{8}{12}\left(\frac{\lambda}{\tau} - l\right) + \frac{1}{2}\left(\frac{\lambda}{\tau} - l\right)^2\right. \end{aligned}$$

$$-\frac{1}{6}\left(\frac{\lambda}{\tau}-l\right)^3\bigg)v_n^{l+1}-\left(-\frac{1}{12}\left(\frac{\lambda}{\tau}-l\right)+\frac{1}{12}\left(\frac{\lambda}{\tau}-l\right)^3\right)v_n^{l+2},\quad n=1,\ldots,M-1,$$

$$\eta_n^N=\left(\exp(-t_l)+t_l+1\right)\sin(x_n)-\left(-\frac{8}{12}\left(\frac{\lambda}{\tau}-l\right)+\frac{1}{2}\left(\frac{\lambda}{\tau}-l\right)^2+\frac{1}{6}\left(\frac{\lambda}{\tau}-l\right)^3\right)v_n^{l-2}$$

$$-\left(1-\left(\frac{\lambda}{\tau}-l\right)^2\right)v_n^{l-1}-\left(1-\frac{1}{2}\left(\frac{\lambda}{\tau}-l\right)^2\right)v_n^{l}-\bigg(\frac{8}{12}\left(\frac{\lambda}{\tau}-l\right)+\frac{1}{2}\left(\frac{\lambda}{\tau}-l\right)^2$$

$$-\frac{1}{6}\left(\frac{\lambda}{\tau}-l\right)^3\bigg)v_n^{l+1}-\left(-\frac{1}{12}\left(\frac{\lambda}{\tau}-l\right)+\frac{1}{12}\left(\frac{\lambda}{\tau}-l\right)^3\right)v_n^{l+2},\quad n=1,\ldots,M-1,$$

$$\eta_n^k=p(x_n)+\frac{\tau^2}{6}p(x_n),\quad k=1,\ldots,N-1,n=1,\ldots,M-1.$$

Now we give the results of the numerical analysis using MATLAB programs. The numerical solutions are recorded for different values of N, M; and u_n^k represents the numerical solutions of these difference schemes at the grid points of (t_k,x_n), and p_n represents the numerical solutions at x_n. For comparison with the exact solutions, the errors are computed by

$$\mathrm{E}v_M^N=\max_{1\le k\le N-1}\left(\sum_{n=1}^{M-1}\left|v(t_k,x_n)-v_n^k\right|^2h\right)^{\frac{1}{2}},$$

$$\mathrm{E}u_M^N=\max_{1\le k\le N-1}\left(\sum_{n=1}^{M-1}\left|u(t_k,x_n)-u_n^k\right|^2h\right)^{\frac{1}{2}},$$

$$\mathrm{E}p_M=\left(\sum_{n=1}^{M-1}\left|p(x_n)-p_n\right|^2h\right)^{\frac{1}{2}}.$$

Tables 1-3 are constructed for $N=6$, $M=108$, $N=10$, $M=300$. Hence, the third order and fourth order of the accuracy difference schemes are more accurate than the second order of the accuracy difference schemes (ADS). Table 1 gives the error between the exact solution and solutions derived by difference schemes for the nonlocal problem. Table 2 includes the error between the exact p solution and approximate p derived by the difference schemes. Table 3 gives the error between the exact u solution and solutions derived by the difference schemes.

Table 1 Error $\mathrm{E}v_M^N$

Difference schemes for v	$N=6$, $M=108$	$N=10$, $M=300$
Second order ADS	0.012664	0.003977
Third order ADS	0.0017276	3.79×10^{-4}
Fourth order ADS	1.99×10^{-4}	2.91×10^{-5}

Table 2 Error $\mathrm{E}p_M$

Calculation of p	$N=6$, $M=108$	$N=10$, $M=300$
Second order ADS	0.025416	0.0079655
Third order ADS	0.0033668	7.47×10^{-4}
Fourth order ADS	4.86×10^{-4}	6.98×10^{-5}

Table 3 Error Eu_M^N

Difference schemes for *u*	$N = 6, M = 108$	$N = 10, M = 300$
Second order ADS	0.0055245	0.0016936
Third order ADS	9.83×10^{-4}	2.11×10^{-4}
Fourth order ADS	5.83×10^{-5}	9.3×10^{-6}

5 Conclusion

In this paper, the overdetermination problem for an elliptic differential equation with Dirichlet boundary condition is considered. The third and fourth orders of accuracy difference schemes for approximate solutions of this problem are presented. Theorems on the stability, almost coercive stability, and coercive stability estimates for the solutions of difference schemes for the elliptic equation are proved. As a result of the application of established abstract theorems, we get well-posedness of high order difference schemes of the inverse problem for a multidimensional elliptic equation. Numerical experiments are given. As can be seen from Tables 1-3, the third and fourth orders of the accuracy difference schemes are more accurate than the second order of the accuracy difference scheme.

Competing interests
The author declares that he has no competing interests.

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