Senlik and Hamal *Boundary Value Problems* 2014, 2014:44
http://www.boundaryvalueproblems.com/content/2014/1/44

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Existence and multiplicity of symmetric positive solutions for nonlinear boundary-value problems with p-Laplacian operator

Tugba Senlik and Nuket Aykut Hamal*

*Correspondence:
nuket.aykut@ege.edu.tr
Department of Mathematics, Ege University, Bornova, Izmir, 35100, Turkey

Abstract

In this paper, we establish the existence and multiplicity of symmetric positive solutions for a class of p-Laplacian fourth-order differential equations with integral boundary conditions. Our proofs use the Leray-Schauder nonlinear alternative and Krasnoselkii's fixed-point theorem in cones.

MSC: 34B10; 39A10

Keywords: boundary-value problem; symmetric positive solution; fixed-point theorem

1 Introduction

In this paper, we are concerned with the existence of symmetric positive solutions of the following fourth-order boundary-value problem with integral boundary conditions:

$$\bigl(\phi_p(u'')\bigr)''(t) = \lambda w(t) f\bigl(t, u(t), u'(t)\bigr), \quad t \in (0,1), \tag{1.1}$$

$$u(0) = u(1) = \int_0^1 g(s)u(s)\,ds,$$

$$\phi_p\bigl(u''(0)\bigr) = \phi_p\bigl(u''(1)\bigr) = \int_0^1 h(s)\phi_p\bigl(u''(s)\bigr)\,ds, \tag{1.2}$$

where $\lambda > 0$, ϕ_p is the p-Laplacian operator, *i.e.*, $\phi_p(s) = |s|^{p-2}s$, $p > 1$ and $(\phi_p)^{-1} = \phi_q$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Now, let us list the following conditions which are to be used in our theorems:

(H1) $w \in C([0,1],[0,+\infty))$ is symmetric on $[0,1]$ and $w(t) \not\equiv 0$ on any subinterval of $[0,1]$;

(H2) $f \in C([0,1] \times [0,+\infty) \times \mathbb{R}, [0,+\infty))$ and for $(t,u,v) \in [0,1] \times [0,+\infty) \times \mathbb{R}$, $f(t,u,v)$ is symmetric in t and even v, *i.e.*, f satisfies $f(1-t,u,v) = f(t,u,v)$ and $f(t,u,v) = f(t,u,-v)$;

(H3) $g, h \in C([0,1],[0,+\infty))$ are symmetric functions on $[0,1]$ and $\mu \in (0,\frac{3}{4}]$, $\nu \in (0,1)$, where

$$\mu = \int_0^1 g(s)\,ds, \qquad \nu = \int_0^1 h(s)\,ds.$$

Boundary-value problems with integral boundary conditions for ordinary differential equations represent a very interesting and important class of problems. They include two-, three-, multi-point and nonlocal boundary-value problems as special cases. For an overview of the literature on integral boundary-value problems and symmetric solutions, see [1–7] and the references therein.

We would like to mention the results of Zhang and Liu [8], Zhang and Ge [9], Ma [10].

In [8], Zhang and Liu considered the following fourth-order boundary-value problems with p-Laplacian operator:

$$\begin{aligned}
&(\phi_p(u''(t))'' = f(t, u(t)), \quad 0 < t < 1, \\
&u(0) = 0, \qquad u(1) = au(\xi), \\
&u''(0) = 0, \qquad u''(1) = bu''(\eta),
\end{aligned}$$

where $\phi_p(t) = |t|^{p-2}t$, $p > 1$, $0 < \xi, \eta < 1$, $f \in C((0,1) \times (0,+\infty), [0,+\infty))$ may be singular at $t = 0$ and/or 1 and $u = 0$.

In [9], Zhang and Ge considered the existence and nonexistence of positive solutions of the following fourth-order boundary-value problems with integral boundary conditions:

$$\begin{aligned}
&x^4(t) = w(t)f(t, x(t), x''(t)), \quad 0 < t < 1, \\
&x(0) = \int_0^1 g(s)x(s)\,ds, \qquad x(1) = 0, \\
&x''(0) = \int_0^1 h(s)x''(s)\,ds, \qquad x''(1) = 0,
\end{aligned}$$

where w may be singular at $t = 0$ and (or) $t = 1$, $f \in C([0,1] \times [0,+\infty) \times (-\infty, 0], [0,+\infty))$, and $g, h \in L^1[0,1]$ are nonnegative.

In [10], Ma considered the existence of a symmetric positive solution for the fourth-order nonlocal boundary-value problem (BVP). The author obtained at least one symmetric positive solution by using the fixed-point index in cones. We have

$$\begin{aligned}
&u''''(t) = h(t)f(t, u), \quad 0 < t < 1, \\
&u(0) = u(1) = \int_0^1 p(s)u(s)\,ds, \\
&u''(0) = u''(1) = \int_0^1 q(s)u(s)\,ds,
\end{aligned}$$

where $p, q \in L^1[0,1]$, $h : (0,1) \to [0,+\infty)$ is continuous, symmetric on $(0,1)$, and maybe singular at $t = 0$ and $t = 1$. $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous and $f(\cdot, u)$ is symmetric on $[0,1]$, for all $u \in [0,+\infty)$.

Motivated by the above works, we consider the existence of one and multiple symmetric positive solutions for the BVP (1.1)-(1.2).

The organization of the paper is as follows. In Section 2, we present some necessary lemmas that will be used to prove our main results. In Section 3, we use the Leray-Schauder nonlinear alternative to get the existence of at least one symmetric positive solution for the nonlinear BVP (1.1)-(1.2). In Section 4, we use the Krasnoselkii fixed-point theorem to get the existence of multiple symmetric positive solutions for the nonlinear BVP (1.1)-(1.2).

In this paper, a symmetric positive solution u of (1.1)-(1.2) means a solution of (1.1)-(1.2) satisfying $u > 0$ and $u(t) = u(1-t)$, $t \in [0,1]$.

2 Preliminaries

To state and prove the main results of this paper, we will make use of the following lemmas.

Lemma 2.1 *Assume that* (H3) *holds. Then for any* $v \in \mathcal{C}^1[0,1]$, *the BVP*

$$\phi_p\bigl(u''(t)\bigr) = \upsilon(t), \quad t \in (0,1), \tag{2.1}$$

$$u(0) = u(1) = \int_0^1 g(s)u(s)\,ds, \tag{2.2}$$

has a unique solution u and u can be expressed in the form

$$u(t) = -\int_0^1 H(t,s)\phi_q\bigl(\upsilon(s)\bigr)\,ds, \tag{2.3}$$

where

$$\begin{aligned} H(t,s) &= G(t,s) + \frac{1}{1-\mu}\int_0^1 G(s,\tau)g(\tau)\,d\tau, \\ G(t,s) &= \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1. \end{cases} \end{aligned} \tag{2.4}$$

Proof First suppose that $u \in \mathcal{C}^1[0,1]$ is a solution of the BVP (2.1)-(2.2). We have

$$u''(t) = \phi_q\bigl(\upsilon(t)\bigr).$$

It is easy to see by integration of both sides of (2.1) on $[0,t]$ that

$$u'(t) - u'(0) = \int_0^t \phi_q\bigl(\upsilon(s)\bigr)\,ds.$$

Integrating again, we get

$$u(t) = u(0) + u'(0)t + \int_0^t (t-s)\phi_q\bigl(\upsilon(s)\bigr)\,ds. \tag{2.5}$$

Letting $t = 1$ in (2.5), we find

$$u'(0) = -\int_0^1 (1-s)\phi_q\bigl(\upsilon(s)\bigr)\,ds. \tag{2.6}$$

Substituting $u(0) = \int_0^1 g(s)u(s)\,ds$, we obtain

$$\begin{aligned} u(t) &= \int_0^1 g(s)u(s)\,ds - \int_0^1 t(1-s)\phi_q\bigl(\upsilon(s)\bigr)\,ds + \int_0^t (t-s)\phi_q\bigl(\upsilon(s)\bigr)\,ds \\ &= -\int_0^1 G(t,s)\phi_q\bigl(\upsilon(s)\bigr)\,ds + \int_0^1 g(s)u(s)\,ds, \end{aligned} \tag{2.7}$$

where

$$\int_0^1 g(s)u(s)\,ds = \int_0^1 g(s)\left[-\int_0^1 G(s,\tau)\phi_q\big(\upsilon(\tau)\big)\,d\tau + \int_0^1 g(\tau)u(\tau)\,d\tau\right]ds,$$

and so

$$\int_0^1 g(s)u(s)\,ds = \frac{-1}{1-\int_0^1 g(s)\,ds}\int_0^1 g(s)\left[\int_0^1 G(s,\tau)\phi_q\big(\upsilon(\tau)\big)\,d\tau\right]ds. \tag{2.8}$$

Substituting (2.8) into (2.7) we have

$$\begin{aligned} u(t) &= -\int_0^1 G(t,s)\phi_q\big(\upsilon(s)\big)\,ds - \frac{1}{1-\mu}\int_0^1 g(s)\left[\int_0^1 G(s,\tau)\phi_q\big(\upsilon(\tau)\big)\,d\tau\right]ds \\ &= -\int_0^1 H(t,s)\phi_q\big(\upsilon(s)\big)\,ds, \end{aligned} \tag{2.9}$$

where $H(t,s)$ is defined in (2.4).

Next let u be as in (2.9), then

$$\begin{aligned} u(t) = {}& -\int_0^t s(1-t)\phi_q\big(\upsilon(s)\big)\,ds - \int_t^1 t(1-s)\phi_q\big(\upsilon(s)\big)\,ds \\ & - \frac{1}{1-\mu}\int_0^1 g(s)\left[\int_0^1 G(s,\tau)\phi_q\big(\upsilon(\tau)\big)\,d\tau\right]ds. \end{aligned} \tag{2.10}$$

Taking the derivative of (2.10), we get

$$u'(t) = \int_0^t s\phi_q\big(v(s)\big)\,ds - \int_t^1 (1-s)\phi_q\big(\upsilon(s)\big)\,ds,$$

and

$$u'' = \phi_q\big(\upsilon(t)\big),$$

and it is easy to verify that $u(0) = u(1) = \int_0^1 g(s)u(s)\,ds$. The proof is complete. □

Lemma 2.2 *Assume that* (H3) *is satisfied. Then for any* $u \in \mathcal{C}^1[0,1]$, *the BVP*

$$v''(t) = \lambda w(t) f\big(t, u(t), u'(s)\big), \quad t \in (0,1), \tag{2.11}$$

$$\upsilon(0) = \upsilon(1) = \int_0^1 h(s)v(s)\,ds \tag{2.12}$$

has a unique solution v

$$v(t) = -\lambda\int_0^1 H_1(t,s)w(s)f\big(s,u(s),u'(s)\big)\,ds, \tag{2.13}$$

where

$$H_1(t,s) = G(t,s) + \frac{1}{1-\nu}\int_0^1 G(s,\tau)h(\tau)\,d\tau. \tag{2.14}$$

Lemma 2.3 [9] *If* (H3) *holds, then, for all* $t,s \in [0,1]$, *the following results are true.*
(i) $G(t,s) \geq 0$, $H(t,s) \geq 0$, $H_1(t,s) \geq 0$;
(ii) $G(1-t,1-s) = G(t,s)$, $H(1-t,1-s) = H(t,s)$, $H_1(1-t,1-s) = H_1(t,s)$;
(iii) $\rho e(s) \leq H(t,s) \leq \gamma e(s)$, $\rho_1 e(s) \leq H_1(t,s) \leq \gamma_1 e(s)$
where

$$e(s) = s(1-s), \qquad \rho = \frac{\int_o^1 e(s)g(s)\,ds}{1-\mu}, \qquad \rho_1 = \frac{\int_o^1 e(s)h(s)\,ds}{1-\nu},$$
$$\gamma = \frac{1}{1-\mu}, \qquad \gamma_1 = \frac{1}{1-\nu};$$

(iv) $e(t)e(s) \leq G(t,s) \leq G(t,t) = t(1-t) = e(t) \leq e^* = \max_{t\in[0,1]} e(t) = \frac{1}{4}$,
where $H(t,s)$, $G(t,s)$, $H_1(t,s)$ *are defined by* (2.4) *and* (2.14), *respectively.*

Lemma 2.4 *If* $u \geq 0$ *and* $\upsilon \geq 0$ *then*

$$\phi_p(u+\upsilon) = \begin{cases} \phi_p(u) + \phi_p(\upsilon), & 1 < p < 2; \\ 2^{p-2}(\phi_p(u) + \phi_p(\upsilon)), & p \geq 2. \end{cases}$$

To obtain the existence of symmetric positive solutions of the BVP (1.1)-(1.2), the following Leray-Schauder nonlinear alternative and Krasnoselkii fixed-point theorem are useful.

Lemma 2.5 [11] *Let* E *be a Banach space with* $P \subseteq E$ *closed and convex. Assume* U *is an open subset of* P *with* $0 \in U$ *and* $T : \overline{U} \to P$ *is a continuous and compact map. Then either*
(i) T *has a fixed point in* $\overline{U}$, *or*
(ii) *there exist* $u \in \partial U$ *and* $\lambda \in (0,1)$ *such that* $u = \lambda Tu$.

Lemma 2.6 [11] *Let* P *be a cone of a real Banach space* E, Ω_1 *and* Ω_2 *be two bounded open sets in* E *such that* $0 \in \Omega_1 \subseteq \overline{\Omega_1} \subseteq \Omega_2$. *Let operator* $T : P \cap (\overline{\Omega_2} \backslash \Omega_1) \to P$ *be completely continuous. Suppose that one of the two conditions*
(i) $\|Tu\| \leq \|u\|$ *for* $u \in P \cap \partial\Omega_1$, $\|Tu\| \geq \|u\|$ *for* $u \in P \cap \partial\Omega_2$,
(ii) $\|Tu\| \geq \|u\|$ *for* $u \in P \cap \partial\Omega_1$, $\|Tu\| \leq \|u\|$ *for* $u \in P \cap \partial\Omega_2$,
is satisfied. Then T *has at least one fixed point in* $P \cap (\overline{\Omega_2} \backslash \Omega_1)$.

Let the space $E = \mathcal{C}^1[0,1]$ equipped with the norm $\|u\|_0 = \|u\| + \|u'\| = \max_{t\in[0,1]} |u(t)| + \max_{t\in[0,1]} |u'(t)|$ be our Banach space. Define P to be cone in E by

$$P = \{u \in E : u(t) \geq 0, u \text{ concave, symmetric on } [0,1]\}.$$

Assume that u is a solution of the BVP (1.1)-(1.2). Then from Lemma 2.1, we get

$$u(t) = -\int_0^1 H(t,s)\phi_q\big(\upsilon(s)\big)\,ds.$$

From Lemma 2.2, we have

$$u(t) = -\int_0^1 H(t,s)\phi_q\left(-\lambda \int_0^1 H_1(s,\tau)w(\tau)f\big(\tau,u(\tau),u'(\tau)\big)\,d\tau\right)ds.$$

3 The existence of one symmetric positive solution

In order to state the following results we need to introduce the notation:

$$A = \phi_q\left(\int_0^1 w(\tau)r(\tau)\,d\tau\right), \qquad B = \phi_q\left(\int_0^1 w(\tau)\big(p(\tau)+q(\tau)\big)\,d\tau\right),$$
$$A' = \left(\frac{\gamma\gamma_1^{q-1}+4\gamma_1^{q-1}}{4^q}\right)\max\{1,2^{q-2}\}A, \qquad B' = \left(\frac{\gamma\gamma_1^{q-1}+4\gamma_1^{q-1}}{4^q}\right)\max\{1,2^{q-2}\}B.$$

Theorem 3.1 *Assume that* (H1)-(H3) *are satisfied and* $f:[0,1]\times[0,+\infty)\times\mathbb{R}\to[0,+\infty)$ *is continuous,* $f(t,0,0)\not\equiv 0$, $t\in[0,1]$ *and there exist nonnegative functions* $q_1,q_2,q_3\in L^1$ *such that*

$$f(t,u,\upsilon)\le q_1(t)\big|u(t)\big|^{p-1}+q_2(t)\big|\upsilon(t)\big|^{p-1}+q_3(t), \quad a.e.\ (t,u,\upsilon)\in[0,1]\times[0,+\infty)\times\mathbb{R},$$

and there exist $t_o\in[0,1]$ *such that* $q_1(t_o)\neq 0$ *or* $q_2(t_o)\neq 0$. *Then there exists a constant* $\lambda^*>0$ *such that for any* $0<\lambda\le\lambda^*$, *the BVP* (1.1)-(1.2) *has at least one nontrivial symmetric positive solution* $u\in P$.

Proof It is easy to see that the BVP (1.1)-(1.2) has a solution $u=u(t)$ if and only if u is a fixed point of the operator equation

$$Tu(t) = \int_0^1 H(t,s)\phi_q\left(\lambda\int_0^1 H_1(s,\tau)w(\tau)f\big(\tau,u(\tau),u'(\tau)\big)\,d\tau\right)ds. \tag{3.1}$$

For all $u\in P$, we have by

$$\begin{aligned}(Tu)'(t) &= \int_t^1 (1-s)\phi_q\left(\lambda\int_0^1 H_1(s,\tau)w(\tau)f\big(\tau,u(\tau),u'(\tau)\big)\,d\tau\right)ds\\ &\quad -\int_0^t s\phi_q\left(\lambda\int_0^1 H_1(s,\tau)w(\tau)f\big(\tau,u(\tau),u'(\tau)\big)\,d\tau\right)ds,\\ (Tu)''(t) &= -\phi_q\left(\lambda\int_0^1 H_1(s,\tau)w(\tau)f\big(\tau,u(\tau),u'(\tau)\big)\,d\tau\right)ds,\\ &\le 0,\end{aligned}$$

which implies Tu is concave on $[0,1]$.

On the other hand, using (H1)-(H3) and Lemma 2.3 we have

$$(Tu)(0) = \int_0^1 H(0,s)\phi_q\left(\lambda\int_0^1 H_1(s,\tau)w(\tau)f\big(\tau,u(\tau),u'(\tau)\big)\,d\tau\right)ds \ge 0,$$

for all $t\in[0,1]$. In a similar way $(Tu)(1)\ge 0$.

It follows that $Tu(t)\ge 0$ for $t\in[0,1]$. Noticing that $w(t)$ is symmetric on $[0,1]$, $u(t)$ is symmetric on $[0,1]$ and $f(t,u,u')$ is symmetric on $[0,1]$ and even in υ we have

$$\begin{aligned}Tu(1-t) &= \int_0^1 H(1-t,s)\phi_q\left(\lambda\int_0^1 H_1(s,\tau)w(\tau)f\big(\tau,u(\tau),u'(\tau)\big)\,d\tau\right)ds\\ &= \int_0^1 H(1-t,1-s)\phi_q\left(\lambda\int_0^1 H_1(1-s,\tau)w(\tau)f\big(\tau,u(\tau),u'(\tau)\big)\,d\tau\right)ds\end{aligned}$$

$$\begin{aligned}
&= \int_0^1 H(t,s)\phi_q\left(\lambda \int_0^1 H_1(1-s,\tau)w(\tau)f\big(\tau,u(\tau),u'(\tau)\big)\,d\tau\right)ds \\
&= \int_0^1 H(t,s)\phi_q\left(\lambda \int_1^0 H_1(1-s,1-\tau)w(1-\tau)\right. \\
&\quad \left.\times f\big(1-\tau,u(1-\tau),u'(1-\tau)\big)d(1-\tau)\right)ds \\
&= \int_0^1 H(t,s)\phi_q\left(\lambda \int_0^1 H_1(s,\tau)w(\tau)f\big(\tau,u(\tau),u'(\tau)\big)\,d\tau\right)ds \\
&= (Tu)(t),
\end{aligned}$$

i.e., $(Tu)(1-t) = (Tu)(t)$, $t \in [0,1]$. Therefore, $(Tu)(t)$ is symmetric on $[0,1]$. So $Tu \in P$ and $T(P) \subset P$. By applying the Arzela-Ascoli theorem, we can see that $T(P)$ is relatively compact. In view of Lebesgue convergence theorem, it is obvious that T is a continuous operator. Hence, $T : P \mapsto P$ is completely continuous operator. By a similar argument in [9] we may proceed; we omit the details here.

$f(t,0,0) \le q_3(t)$, for all $t \in [0,1]$, and $w(t) \not\equiv 0$, $t \in [0,1]$, we know that $A > 0$, $B > 0$. Thus $A' > 0$, $B' > 0$. Let

$$r = \frac{A'}{B'}, \qquad \Omega = \big\{u \in P : \|u\| < r\big\}.$$

Suppose $u \in \partial\Omega$, $0 < \mu < 1$ such that $u = \mu Tu$. Then

$$r = \mu\|Tu\|_0 = \mu\big(\|Tu\| + \big\|(Tu)'\big\|\big).$$

By Lemma 2.3

$$\begin{aligned}
\|Tu\| &= \max_{t\in[0,1]}\big|Tu(t)\big| \\
&\le \int_0^1 \gamma e(s)\phi_q\left(\lambda \int_0^1 \gamma_1 e(\tau)w(\tau)f\big(\tau,u(\tau),u'(\tau)\big)\,d\tau\right)ds \\
&\le \gamma\gamma_1^{q-1}\frac{1}{4}\left(\frac{1}{4}\right)^{q-1}\phi_q\left(\lambda \int_0^1 w(\tau)\big[q_1(\tau)\big|u(\tau)\big|^{p-1}\right. \\
&\quad \left. + q_2(\tau)\big|u'(\tau)\big|^{p-1} + q_3(\tau)\big]\,d\tau\right) \\
&\le \gamma\gamma_1^{q-1}\left(\frac{1}{4}\right)^{q}\phi_q\left(\lambda \int_0^1 w(\tau)\big[\|u\|_0^{p-1}\big(q_1(\tau)+q_2(\tau)\big) + q_3(\tau)\big]\,d\tau\right) \\
&\le \gamma\gamma_1^{q-1}\left(\frac{1}{4}\right)^{q}\phi_q\left(\lambda\left[\|u\|_0^{p-1}\int_0^1 w(\tau)\big(q_1(\tau)+q_2(\tau)\big)\,d\tau + \int_0^1 w(\tau)q_3(\tau)\,d\tau\right]\right) \\
&\le \left(\frac{\gamma\gamma_1^{q-1}}{4^q}\right)\max\big\{1,2^{q-2}\big\}\phi_q(\lambda)\|u\|_0\phi_q\left(\int_0^1 w(\tau)\big(q_1(\tau)+q_2(\tau)\big)\,d\tau\right) \\
&\quad + \left(\frac{\gamma\gamma_1^{q-1}}{4^q}\right)\max\big\{1,2^{q-2}\big\}\phi_q(\lambda)\phi_q\left(\int_0^1 w(\tau)q_3(\tau)\,d\tau\right) \\
&\le \gamma_1^{q-1}\left(\frac{1}{4}\right)^{q-1}\phi_q\left(\lambda\int_0^1 w(\tau)\big[q_1(\tau)\big|u(\tau)\big|^{p-1} + q_2(\tau)\big|u'(\tau)\big|^{p-1} + q_3(\tau)\big]\,d\tau\right)
\end{aligned}$$

$$\begin{aligned}&\le \left(\frac{\gamma_1}{4}\right)^{q-1}\max\{1,2^{q-2}\}\phi_q(\lambda)\|u\|_0\phi_q\left(\int_0^1 w(\tau)\big(q_1(\tau)+q_2(\tau)\big)\,d\tau\right)\\&\quad+\left(\frac{\gamma_1}{4}\right)^{q-1}\max\{1,2^{q-2}\}\phi_q(\lambda)\phi_q\left(\int_0^1 w(\tau)q_3(\tau)\,d\tau\right).\end{aligned}$$

So,

$$\|Tu\|_0 \le \phi_q(\lambda)\|u\|_0 B' + \phi_q(\lambda)A'.$$

Choose $\lambda^* = (\frac{1}{2B'})^{p-1}$. Then when $0 < \lambda < \lambda^*$, we have

$$\begin{aligned}&r = \|u\|_0 = \mu\|Tu\|_0 \le \mu\left(\frac{1}{2B'}B'\|u\|_0 + \frac{A'}{2B'}\right),\\&r \le \mu\left(\frac{1}{2}r + \frac{1}{2}r\right).\end{aligned}$$

Consequently, $\mu \ge 1$. This contradicts $0 < \mu < 1$ period, by (i) of Lemma 2.5, T has a fixed point $u \in \overline{\Omega}$, since $f(t,0,0) \not\equiv 0$, then when $0 < \lambda \le \lambda^*$, the BVP (1.1)-(1.2) has a nontrivial symmetric positive solution $u \in P$. The proof is complete. □

Theorem 3.2 *Assume that* (H1)-(H3) *are satisfied and* $f : [0,1] \times [0,+\infty) \times \mathbb{R} \to [0,+\infty)$ *is continuous*, $f(t,0,0) \not\equiv 0$, $t \in [0,1]$ *and*

$$0 \le L = \limsup_{|u|+|\upsilon|\to+\infty} \max_{t\in[0,1]} \frac{f(t,u,\upsilon)}{|u|^{p-1}+|\upsilon|^{p-1}} < +\infty. \tag{3.2}$$

Then there exists a constant $\lambda^* > 0$ *such that for any* $0 < \lambda \le \lambda^*$, *the BVP* (1.1)-(1.2) *has at least one nontrivial symmetric positive solution* $u \in P$.

Proof Let $\epsilon > 0$ such that $L + \epsilon > 0$. By (3.2), there exists $H > 0$ such that

$$f(t,u,\upsilon) \le (L+\epsilon)\big(|u|^{p-1} + |\upsilon|^{p-1}\big), \quad |u| + |\upsilon| \ge H, 0 \le t \le 1.$$

Let $K = \max_{t\in[0,1],|u|+|\upsilon|\le H} f(t,u,\upsilon)$. Then for any $(t,u,\upsilon) \in [0,1] \times \mathbb{R}^+ \times \mathbb{R}$, we have

$$f(t,u,\upsilon) \le (L+\epsilon)\big(|u|^{p-1} + |\upsilon|^{p-1}\big) + K.$$

From Theorem 3.1, we know that the BVP (1.1)-(1.2) has at least one nontrivial symmetric positive solution $u \in P$. The proof is complete. □

Corollary 3.1 *Assume that* (H1)-(H3) *are satisfied and* $f : [0,1] \times [0,+\infty) \times \mathbb{R} \to [0,+\infty)$ *is continuous*, $f(t,0,0) \not\equiv 0$, $t \in [0,1]$ *and*

$$\begin{aligned}&0 \le L = \limsup_{|u|+|\upsilon|\to+\infty} \max_{t\in[0,1]} \frac{f(t,u,\upsilon)}{|u|^{p-1}} < +\infty, \quad \textit{or}\\&0 \le L = \limsup_{|u|+|\upsilon|\to+\infty} \max_{t\in[0,1]} \frac{f(t,u,\upsilon)}{|\upsilon|^{p-1}} < +\infty.\end{aligned}$$

Then there exists a constant $\lambda^* > 0$, *such that for any* $0 < \lambda \le \lambda^*$, *the BVP* (1.1)-(1.2) *has at least one nontrivial solution* $u \in P$.

Example 3.1 We consider the following fourth-order BVP.

Let $p = 4$, $w(t) = 1$ in (1.1) and $h(t) = g(t) = \frac{1}{2}$. Then

$$\big(\phi_4(u'')\big)''(t) = \lambda f\big(t, u(t), u'(t)\big), \quad t \in (0,1), \tag{3.3}$$

$$u(0) = u(1) = \frac{1}{2}\int_0^1 u(s)\,ds,$$
$$\phi_4\big(u''(0)\big) = \phi_4\big(u''(1)\big) = \frac{1}{2}\int_0^1 \phi_p\big(u''(s)\big)\,ds, \tag{3.4}$$

where

$$f(t,u,\upsilon) = u^3 + \left(t - \frac{1}{2}\right)^2 |\sin\upsilon|^3 + t(1-t) + 1$$
$$\text{for } (t,u,\upsilon) \in [0,1] \times [0,+\infty) \times (-\infty,+\infty).$$

It is obvious that $f : [0,1] \times [0,+\infty) \times (-\infty,+\infty) \to [0,+\infty)$ is continuous, symmetric on the interval $[0,1]$ and even υ, we have

$$u^3 + \left(t - \frac{1}{2}\right)^2 |\sin\upsilon|^3 + t(1-t) + 1 \le |u|^3 + \big(t^2+1\big)|\upsilon|^3 + t + 1 \quad \text{and}$$
$$f(t,0,0) \not\equiv 0, \quad t \in [0,1].$$

It follows from a direct calculation that

$$B = \phi_q\left(\int_0^1 \big(s^2+2\big)\,ds\right) = \sqrt[3]{\frac{7}{3}} \quad \text{and} \quad \gamma = \gamma_1 = 2;$$
$$\begin{aligned} B' &= \left(\frac{\gamma\gamma_1^{q-1} + 4\gamma_1^{q-1}}{4^q}\right)\max\big\{1, 2^{q-2}\big\}B \\ &= \left(\frac{2\sqrt[3]{2} + 4\sqrt[3]{2}}{\sqrt[3]{4^4}}\right)\sqrt[3]{\frac{7}{3}} \\ &\simeq 1,1905. \end{aligned}$$

So,

$$\lambda^* = \frac{1}{2B'} \simeq 0.074.$$

Then by Theorem 3.1 we know that the BVP (3.3)-(3.4) has a nontrivial symmetric positive solution $u \in P$ for any $\lambda \in (0,\lambda^*]$.

4 The existence of multiple symmetric positive solutions

In this section, we impose growth conditions on f which allows us to apply Lemma 2.6 to establish the existence of two symmetric positive solutions of the BVP (1.1)-(1.2), and we begin by introducing some notation:

$$\sigma^* = \rho\rho_1^{q-1}\int_0^1 e(s)\,ds\phi_q\left(\int_0^1 e(\tau)w(\tau)\,d\tau\right),$$
$$\sigma = \gamma_1^{q-1}\phi_q\left(\int_0^1 e(\tau)w(\tau)\,d\tau\right).$$

Theorem 4.1 *Assume that* (H1)-(H3) *are satisfied and* f *satisfies the following conditions for* $t \in [0,1]$.

(i) *There exist numbers* $0 < r < R < +\infty$ *such that* $f(t,u,\upsilon) \leq \frac{1}{\lambda}\phi_p(\frac{u+\upsilon}{2\sigma})$ *for* $0 \leq |u| + |\upsilon| \leq r$ *and* $R \leq |u| + |\upsilon|$.

(ii) *There exist numbers* $0 < r < p_1 < R < +\infty$ *such that* $f(t,u,\upsilon) > \frac{1}{\lambda}\phi_p(\frac{p_1}{\sigma^*})$ *for* $0 \leq |u| + |\upsilon| \leq p_1$.

Then the BVP (1.1)-(1.2) *has at least two nontrivial symmetric positive solution* $u \in P$.

Proof Let the operator T be defined by (3.1).

$$\Omega_1 = \{u \in P : \|u\|_0 < r\} \quad \text{and} \quad \Omega_2 = \{u \in P : \|u\|_0 < p_1\}.$$

We first show that

$$\|Tu\|_0 \leq \|u\|_0 \quad \text{for } u \in P \cap \partial\Omega_1.$$

For $u \in P \cap \partial\Omega_1$, we obtain $|u| + |u'| \in [0,r]$, which implies $f(t,u,u') \leq \frac{1}{\lambda}\phi_p(\frac{u+\upsilon}{2\sigma})$. Hence for $t \in [0,1]$, by Lemma 2.3,

$$\begin{aligned} Tu(t) &\leq \frac{\gamma\gamma_1^{q-1}}{4}\phi_q\left(\lambda\int_0^1 e(\tau)w(\tau)f(\tau,u(\tau),u'(\tau))\,d\tau\right)\\ &\leq \frac{\gamma\gamma_1^{q-1}}{4}\phi_q\left(\lambda\int_0^1 e(\tau)w(\tau)\frac{1}{\lambda}\phi_p\left(\frac{u+\upsilon}{2\sigma}\right)d\tau\right)\\ &\leq \frac{\gamma}{8}\|u\|_0. \end{aligned}$$

Note that $1 < \gamma \leq 4$. Thus, $u \in P \cap \partial\Omega_1$ implies

$$\|Tu\| \leq \frac{\|u\|_0}{2}, \tag{4.1}$$

and

$$\begin{aligned} (Tu)'(t) &\leq \phi_q\left(\lambda\int_0^1 \gamma_1 e(\tau)w(\tau)f(\tau,u(\tau),u'(\tau))\,d\tau\right)\\ &\leq \gamma_1^{q-1}\phi_q\left(\lambda\int_0^1 e(\tau)w(\tau)\frac{1}{\lambda}\phi_p\left(\frac{u+\upsilon}{2\sigma}\right)d\tau\right)\\ &\leq \frac{\|u\|_0}{2}. \end{aligned}$$

Thus, $u \in P \cap \partial\Omega_1$ implies

$$\|(Tu)'\| \leq \frac{\|u\|_0}{2}. \tag{4.2}$$

By (4.1) and (4.2), we obtain

$$\|Tu\|_0 = \|Tu\| + \|(Tu)'\| \le \frac{\|u\|_0}{2} + \frac{\|u\|_0}{2} = \|u\|_0 \quad \text{for } u \in P \cap \partial\Omega_1. \tag{4.3}$$

Next we show that

$$\|Tu\|_0 \ge \|u\|_0 \quad \text{for } u \in P \cap \partial\Omega_2.$$

For $u \in P \cap \partial\Omega_2$, we obtain $|u| + |u'| \in [0, p_1]$, which implies $f(t,u,u') > \frac{1}{\lambda}\phi_p(\frac{p_1}{\sigma^*})$. Hence for $t \in [0,1]$, by Lemma 2.3,

$$\begin{aligned} Tu(t) &\ge \rho\rho_1^{q-1} \int_0^1 e(s)\,ds\phi_q\left(\lambda \int_0^1 e(\tau)w(\tau)f\big(\tau, u(\tau), u'(\tau)\big)\,d\tau\right) \\ &> \rho\rho_1^{q-1} \int_0^1 e(s)\,ds\phi_q\left(\lambda \int_0^1 e(\tau)w(\tau)\frac{1}{\lambda}\phi_p\left(\frac{p_1}{\sigma^*}\right)d\tau\right) \\ &> p_1 = \|u\|_0. \end{aligned} \tag{4.4}$$

By (4.4), we obtain

$$\|Tu\|_0 = \|Tu\| + \|(Tu)'\| > \|u\|_0 \quad \text{for } u \in P \cap \partial\Omega_2. \tag{4.5}$$

Applying Lemma 2.6 to (4.3) and (4.5) shows that T has a fixed point $u_1 \in P \cap (\overline{\Omega}_2 \backslash \Omega_1)$ with $0 < r \le \|u_1\|_0 < p_1$. Also, let $R^* = \frac{\gamma_1\gamma^{q-1}}{\rho_1\rho^{q-1}}R$ and note that $0 < \frac{\rho_1\rho^{q-1}}{\gamma_1\gamma^{q-1}} < 1$. So $u(t) \ge \frac{\rho_1\rho^{q-1}}{\gamma_1\gamma^{q-1}}u(s)$ for $s \in [0,1]$. Then $\Omega_3 = \{u \in P : \|u\|_0 < R^*\}$ and $u \in P \cap \partial\Omega_3$, and we obtain $u(t) \ge \frac{\rho_1\rho^{q-1}}{\gamma_1\gamma^{q-1}}R^* = R$. Applying Theorem 4.1(i) for $|u| + |u'| \in [R, +\infty)$, we have $\|Tu\|_0 \le \|u\|_0$ for $u \in P \cap \partial\Omega_3$. From Lemma 2.6, T has a fixed point $u_2^* \in P \cap (\overline{\Omega}_3 \backslash \Omega_2)$ with $p_1 < \|u_2\| \le R^*$.

Then the BVP (1.1)-(1.2) has two nontrivial symmetric positive solutions $u_1, u_2 \in P$ with $0 < \|u_1\| < p_1 < \|u_2\| \le R^*$. The proof is complete. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgements
The authors would like to thank the referees for their useful comments and suggestions.

Received: 1 October 2013 Accepted: 7 February 2014 Published: 20 Feb 2014

References

1. Lomtatidze, A, Malaguti, L: On a nonlocal boundary value problems for second order nonlinear singular differential equations. Georgian Math. J. **7**, 133-154 (2000)
2. Gallardo, JM: Second order differential operators with integral boundary conditions and generation of semigroups. Rocky Mt. J. Math. **30**, 1265-1292 (2000)
3. Zhanga, X, Feng, M, Ge, W: Symmetric positive solutions for *p*-Laplacian fourth-order differential equations with integral boundary conditions. J. Comput. Appl. Math. **222**, 561-573 (2008)
4. Zhang, X, Liu, L: Eigenvalue of fourth-order *m*-point boundary value problem with derivatives. Comput. Math. Appl. **56**, 172-185 (2008)
5. Ma, R, Xu, L: Existence of positive solutions of nonlinear fourth-order boundary value problem. Appl. Math. Lett. **23**, 537-543 (2010)
6. Avery, RI, Henderson, J: Three symmetric positive solutions for a second-order boundary value problem. Appl. Math. Lett. **13**, 1-7 (2000)

7. Feng, H, Pang, H, Ge, W: Multiplicity of symmetric positive solutions for a multipoint boundary value problem with a one-dimensional p-Laplacian. Nonlinear Anal. **69**, 3050-3059 (2008)
8. Zhang, X, Liu, L: Positive solutions of fourth-order four-point boundary value problems with ϕ-Laplacian operator. J. Math. Anal. Appl. **336**, 1414-1423 (2007)
9. Zhang, X, Ge, W: Positive solutions for a class of boundary-value problems with integral boundary conditions. Comput. Math. Appl. **58**, 203-215 (2009)
10. Ma, H: Symmetric positive solutions for nonlocal boundary value problems of fourth order. Nonlinear Anal. **68**, 645-651 (2008)
11. Guo, D, Lashmikanthan, V: Nonlinear Problems in Abstract Cones. Academic Press, San Diego (1988)

10.1186/1687-2770-2014-44
Cite this article as: Senlik and Hamal: **Existence and multiplicity of symmetric positive solutions for nonlinear boundary-value problems with p-Laplacian operator.** *Boundary Value Problems* 2014, 2014:44