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The structure of fractional spaces generated by a two-dimensional elliptic differential operator and its applications

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Abstract

We consider the two-dimensional differential operator $Au(x_1, x_2) = -a_{11}(x_1, x_2)u_{x_1 x_1}(x_1, x_2) - a_{22}(x_1, x_2)u_{x_2 x_2}(x_1, x_2) + \sigma u(x_1, x_2)$ defined on functions on the half-plane $\Omega = \mathbb{R}^+ \times \mathbb{R}$ with the boundary conditions $u(0, x_2) = 0$, $x_2 \in \mathbb{R}$, where $a_{ii}(x_1, x_2)$, $i = 1, 2$, are continuously differentiable and satisfy the uniform ellipticity condition $a_{11}^2(x_1, x_2) + a_{22}^2(x_1, x_2) \geq \delta > 0$, $\sigma > 0$. The structure of the fractional spaces $E_\alpha(A, C^\beta(\overline{\Omega}))$ generated by the operator A is investigated. The positivity of A in Hölder spaces is established. In applications, theorems on well-posedness in a Hölder space of elliptic problems are obtained.

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1 Introduction

It is well known that (see, for example, [1–3] and the references therein) various classical and non-classical boundary value problems for partial differential equations can be considered as an abstract boundary value problem for an ordinary differential equation in a Banach space with a densely defined unbounded operator. The importance of the positivity property of the differential operators in a Banach space in the study of various properties for partial differential equations is well known (see, for example, [4–7] and the references therein). Several authors have investigated the positivity of a wider class of differential and difference operators in Banach spaces (see [8–18] and the references therein).

Let us give the definition of positive operators and introduce the fractional spaces and preliminary facts that will be needed in the sequel.

The operator A is said to be positive in E if its spectrum $\sigma(A)$ lies inside of the sector S of the angle ϕ, $0 < 2\phi < 2\pi$, symmetric with respect to the real axis, and the following estimate (see, for example, [6, 19])

$$\left\| (A - \lambda)^{-1} \right\|_{E \to E} \leq \frac{M(\phi)}{1 + |\lambda|}$$

holds on the edges $S_1(\phi) = \{\rho e^{i\phi} : 0 \leq \rho < \infty\}$, $S_2(\phi) = \{\rho e^{-i\phi} : 0 \leq \rho < \infty\}$ of S, and outside of the sector S. The infimum of all such angles ϕ is called the spectral angle of the positive operator A and is denoted by $\phi(A, E)$. We say that A is a strongly positive operator in E if $\phi(A, E) < \pi/2$.

Throughout the article, M indicates positive constants which may differ from time to time, and we are not interested to precise. If the constant depends only on $\alpha, \beta, \ldots$, then we will write $M(\alpha, \beta, \ldots)$.

With the help of the positive operator A, we introduce the fractional space $E_\beta = E_\beta(E, A)$ $(0 < \beta < 1)$, consisting of all elements $v \in E$ for which the norm

$$\|v\|_{E_\beta} = \sup_{\lambda > 0} \lambda^\beta \left\| A(\lambda + A)^{-1} v \right\|_E + \|v\|_E$$

is finite.

Theorem 1 [20] *Let $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, and let $f, g : \mathbb{R}^+ \to \mathbb{R}$ be any two nonnegative integrable functions such that $0 < \int_0^\infty f^p(x)\,dx < \infty$ and $0 < \int_0^\infty g^q(y)\,dy < \infty$. Then the following Hilbert's inequality holds*:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x + y}\,dx\,dy < \pi \csc\left(\frac{\pi}{p}\right)\left(\int_0^\infty f^p(x)\,dx\right)^{\frac{1}{p}}\left(\int_0^\infty g^q(y)\,dy\right)^{\frac{1}{q}}.$$

Danelich in [12] considered the positivity of a difference analog A_h^x of the $2m$th-order multi-dimensional elliptic operator A^x with dependent coefficients on semi-spaces $\mathbb{R}^+ \times \mathbb{R}^{n-1}$.

The structure of fractional spaces generated by positive multi-dimensional differential and difference operators on the space $\mathbb{R}^n$ in Banach spaces has been well investigated (see [21–23] and the references therein).

In papers [19, 24–27] the structure of fractional spaces generated by positive one-dimensional differential and difference operators in Banach spaces was studied. Note that the structure of fractional spaces generated by positive multi-dimensional differential and difference operators with local and nonlocal conditions on $\Omega \subset \mathbb{R}^n$ in Banach spaces $C(\Omega)$ has not been well studied.

In the present paper, we study the structure of fractional spaces generated by the two-dimensional differential operator

$$Au(x_1, x_2) = -a_{11}(x_1, x_2)u_{x_1 x_1}(x_1, x_2) - a_{22}(x_1, x_2)u_{x_2 x_2}(x_1, x_2) + \sigma u(x_1, x_2), \tag{1}$$

defined over the region $\mathbb{R}_+^2 = \mathbb{R}^+ \times \mathbb{R}$ with the boundary condition $u(0, x_2) = 0$, $x_2 \in \mathbb{R}$. Here, the coefficients $a_{ii}(x_1, x_2)$, $i = 1, 2$, are continuously differentiable and satisfy the uniform ellipticity

$$a_{11}^2(x_1, x_2) + a_{22}^2(x_1, x_2) \geq \delta > 0,$$

and $\sigma > 0$.

Following the paper [12], passing limit when $h \to 0$ in the special case $m = 1$ and $n = 2$, we get that there exists the inverse operator $(A + \lambda)^{-1}$ for all $\lambda \geq 0$ and the following formula

$$(A + \lambda)^{-1} f(x_1, x_2) = \int_0^\infty \int_{-\infty}^\infty G(x_1, x_2, p, s, \lambda) f(p, s)\,ds\,dp \tag{2}$$

holds, where $G(x_1, x_2, p, s, \lambda)$ is the Green function of differential operator (1). Moreover, the following estimates

$$
\begin{aligned}
&\big|G(x_1, x_2, p, s, \lambda)\big| \\
&\quad \le C \exp\big\{-a(\lambda+1)^{1/2}\big(|x_1 - p| + |x_2 - s|\big)\big\} \\
&\quad\quad \times \big(1 + \ln\big\{1 + \big((\lambda+1)^{1/2}\big(|x_1 - p| + |x_2 - s|\big)\big)^{-1}\big\}\big),
\end{aligned}
\tag{3}
$$

and

$$
\begin{aligned}
&\big|G_{x_1}(x_1, x_2, p, s, \lambda)\big|, \big|G_{x_2}(x_1, x_2, p, s, \lambda)\big| \\
&\quad \le C \exp\big\{-a(\lambda+1)^{1/2}\big(|x_1 - p| + |x_2 - s|\big)\big\}\big(|x_1 - p| + |x_2 - s|\big)^{-1}
\end{aligned}
\tag{4}
$$

hold. Here $a = a(\sigma)$.

Next, to formulate our result, we need to introduce the Hölder space $C^\beta = C^\beta(\mathbb{R}_+^2)$ of all continuous bounded functions φ defined on $\mathbb{R}_+^2$ satisfying a Hölder condition with the indicator $\beta \in [0, 1]$ with the norm

$$
\|f\|_{C^\beta(\mathbb{R}_+^2)} = \|f\|_{C(\mathbb{R}_+^2)} + \sup_{\substack{(x_1,x_2),(x_1',x_2')\in\mathbb{R}_+^2 \\ (x_1,x_2)\neq(x_1',x_2')}} \frac{|f(x_1,x_2) - f(x_1', x_2')|}{\big(\sqrt{|x_1 - x_1'|^2 + |x_2 - x_2'|^2}\big)^\beta}.
$$

Here, $C(\mathbb{R}_+^2)$ denotes the Banach space of all continuous bounded functions φ defined on $\mathbb{R}_+^2$ with the norm

$$
\|\varphi\|_{C(\mathbb{R}_+^2)} = \sup_{(x_1,x_2)\in\mathbb{R}_+^2} \big|\varphi(x_1, x_2)\big|.
$$

Clearly, from estimates (3) and (4) it follows that A is a positive operator in $C(\mathbb{R}_+^2)$. Namely, we have the following.

Theorem 2 *Let $\lambda \ge 0$. Then the following estimate*

$$
\big\|(A+\lambda)^{-1}\big\|_{C(\mathbb{R}_+^2)\to C(\mathbb{R}_+^2)} \le \frac{M}{1+\lambda}
$$

is valid.

Here, the structure of fractional spaces generated by the operator A is investigated. The positivity of A in Hölder spaces is studied. The organization of the present paper is as follows. In Section 2, the positivity of A in Hölder spaces is established. In Section 3, the main theorem on the structure of fractional spaces $E_\alpha(A, C^\beta(\mathbb{R}_+^2))$ generated by A is investigated. In Section 4, applications on theorems on well-posedness in a Hölder space of parabolic and elliptic problems are presented. Finally, the conclusion is given.

2 Positivity of A in Hölder spaces $C^\beta(\mathbb{R}_+^2)$

Theorem 3 *Let $\beta \in (0, 1)$. For $\lambda \ge 0$, we have the following estimate*:

$$
\big\|(A+\lambda)^{-1}\big\|_{C^\beta(\mathbb{R}_+^2)\to C^\beta(\mathbb{R}_+^2)} \le \frac{M(a)}{1+\lambda}.
$$

Proof Applying formula (2), the triangle inequality, the definition of C^β-norm, estimate (3), and Hilbert's inequality, we get

$$\left| (A + \lambda)^{-1} f(x_1, x_2) \right|$$

$$\leq \int_0^\infty \int_{-\infty}^\infty \left| G(x_1, x_2, p, s, \lambda) \right| \left| f(p, s) \right| ds\, dp$$

$$\leq M \|f\|_{C^\beta} \int_0^\infty \int_{-\infty}^\infty e^{-a(1+\lambda)^{1/2}(|x_1-p|+|x_2-s|)}$$

$$\times \left(1 + \ln\left\{ 1 + \left((1+\lambda)^{1/2} \left(|x_1 - p| + |x_2 - s| \right) \right)^{-1} \right\} \right) ds\, dp$$

$$\leq M \|f\|_{C^\beta} \left[\int_0^\infty \int_{-\infty}^\infty e^{-a(1+\lambda)^{1/2}(|x_1-p|+|x_2-s|)} \, ds\, dp \right.$$

$$\left. + \int_0^\infty \int_{-\infty}^\infty e^{-a(1+\lambda)^{1/2}(|x_1-p|+|x_2-s|)} \left((1+\lambda)^{1/2} \left(|x_1 - p| + |x_2 - s| \right) \right)^{-1} ds\, dp \right]$$

$$\leq \frac{M_1(a)}{1+\lambda} \|f\|_{C^\beta} + \frac{M_2(a)}{(1+\lambda)^{1/2}} \|f\|_{C^\beta} \int_0^\infty \int_0^\infty \frac{e^{-a(1+\lambda)^{1/2}(p+s)}}{p+s} \, ds\, dp$$

$$\leq \frac{M_3(a)}{1+\lambda} \|f\|_{C^\beta} \quad \text{for } (x_1, x_2) \in \mathbb{R}_+^2.$$

Then from that it follows

$$\sup_{(x_1,x_2)\in\mathbb{R}_+^2} \left| (A + \lambda)^{-1} f(x_1, x_2) \right| \leq \frac{M_3(a)}{1+\lambda} \|f\|_{C^\beta}. \tag{5}$$

Without loss of generality, we can put $\tau, h > 0$. Using formula (2) and the triangle inequality, we get

$$\left| (A + \lambda)^{-1} \left[\frac{f(x_1 + \tau, x_2 + h) - f(x_1, x_2)}{(\tau^2 + h^2)^{\beta/2}} \right] \right|$$

$$\leq \frac{1}{(\tau^2 + h^2)^{\beta/2}} \int_0^\infty \int_{-\infty}^\infty \left| G(x_1 + \tau, x_2 + h, p, s; \lambda) - G(x_1, x_2, p, s, \lambda) \right|$$

$$\times \left| f(p, s) \right| ds\, dp \tag{6}$$

for $(x_1, x_2), (x_1 + \tau, x_2 + h) \in \mathbb{R}_+^2$. Now, we will estimate the right-hand side of inequality (6). Let us consider two cases $\tau^2 + h^2 < 1$ and $\tau^2 + h^2 \geq 1$ separately. First, we consider the case $\tau^2 + h^2 < 1$. Using the triangle inequality, estimate (4), the definition of C^β-norm, Hilbert's inequality, and the Lagrange theorem, it follows that for some x_1^* between x_1, $x_1 + \tau$, and x_2^* between x_2, $x_2 + h$,

$$\left| (A + \lambda)^{-1} \left[\frac{f(x_1 + \tau, x_2 + h) - f(x_1, x_2)}{(\tau^2 + h^2)^{\beta/2}} \right] \right|$$

$$\leq M_4 \|f\|_{C^\beta} \left[\frac{\tau}{(\tau^2 + h^2)^{\beta/2}} \int_0^\infty \int_{-\infty}^\infty \left| G_{x_1}\left(x_1^*, x_2^*, p, s; \lambda \right) \right| ds\, dp \right.$$

$$\left. + \frac{h}{(\tau^2 + h^2)^{\beta/2}} \int_0^\infty \int_{-\infty}^\infty \left| G_{x_2}\left(x_1^*, x_2^*, p, s; \lambda \right) \right| ds\, dp \right]$$

$$\leq M_4 \|f\|_{C^\beta} \left[\int_0^\infty \int_{-\infty}^\infty \left| G_{x_1}\left(x_1^*, x_2^*, p, s; \lambda \right) \right| ds\, dp \right.$$

$$+ \int_0^\infty \int_{-\infty}^\infty \left| G_{x_2}\left(x_1^*, x_2^*, p, s; \lambda\right) \right| ds\, dp \Bigg]$$

$$\leq M_5 \|f\|_{C^\beta} \int_0^\infty \int_{-\infty}^\infty \frac{e^{-a(1+\lambda)^{1/2}(|x_1^* - p| + |x_2^* - s|)}}{|x_1^* - p| + |x_2^* - s|} \, ds\, dp \leq \frac{M_6(a)}{1+\lambda} \|f\|_{C^\beta}. \tag{7}$$

Second, we consider the case $\tau^2 + h^2 \geq 1$. Using formula (2), the triangle inequality, estimate (3), the definition of C^β-norm, and estimate (6), we get

$$\left| (A+\lambda)^{-1} \left[\frac{f(x_1+\tau, x_2+h) - f(x_1, x_2)}{(\tau^2 + h^2)^{\beta/2}} \right] \right|$$

$$\leq M_7 \|f\|_{C^\beta} \Bigg[\int_0^\infty \int_{-\infty}^\infty e^{-a(1+\lambda)^{1/2}(|x_1+\tau-p| + |x_2+h-s|)}$$

$$\times \left(1 + \ln\left\{ 1 + \left((\lambda+1)^{1/2} \left(|x_1+\tau-p| + |x_2+h-s| \right) \right)^{-1} \right\} \right) ds\, dp$$

$$+ \int_0^\infty \int_{-\infty}^\infty e^{-a(1+\lambda)^{1/2}(|x_1-p| + |x_2-s|)}$$

$$\times \left(1 + \ln\left\{ 1 + \left((\lambda+1)^{1/2} \left(|x_1-p| + |x_2-s| \right) \right)^{-1} \right\} \right) ds\, dp \Bigg]$$

$$\leq \frac{M_8(a)}{1+\lambda} \|f\|_{C^\beta}. \tag{8}$$

Estimates (7) and (8) yield that

$$\sup_{\substack{(x_1+\tau, x_2+h),(x_1,x_2) \in \mathbb{R}_+^2 \\ (\tau,h) \neq (0,0)}} \left| (A+\lambda)^{-1} \left[\frac{f(x_1+\tau, x_2+h) - f(x_1, x_2)}{(\tau^2 + h^2)^{\beta/2}} \right] \right| \leq \frac{M_8(a)}{1+\lambda} \|f\|_{C^\beta}. \tag{9}$$

Combining estimates (5) and (9), we obtain

$$\left\| (A+\lambda)^{-1} \right\|_{C^\beta(\mathbb{R}_+^2) \to C^\beta(\mathbb{R}_+^2)} \leq \frac{M(a)}{1+\lambda}.$$

This finishes the proof of Theorem 3. □

Note that from the commutativity of A and its resolvent $(A+\lambda)^{-1}$, and Theorem 3, we have the following theorem.

Theorem 4 *Let $\lambda \geq 0$. Then the following estimate holds*:

$$\left\| (A+\lambda)^{-1} \right\|_{E_\alpha(A, C^\beta(\mathbb{R}_+^2)) \to E_\alpha(A, C^\beta(\mathbb{R}_+^2))} \leq \frac{M(a)}{1+\lambda}.$$

3 The structure of fractional spaces $E_\alpha(A, C^\beta(\mathbb{R}_+^2))$

Suppose $\beta, 2\alpha + \beta \in (0,1)$. Consider the fractional space $E_\alpha(A, C^\beta(\mathbb{R}_+^2))$ and the Hölder space $C^{2\alpha+\beta}(\mathbb{R}_+^2)$. In this section, we prove the following structure theorem.

Theorem 5 *The norms of the spaces $E_\alpha(A, C^\beta(\mathbb{R}_+^2))$ and $C^{2\alpha+\beta}(\mathbb{R}_+^2)$ are equivalent.*

Proof Assume that $f \in C^{2\alpha+\beta}(\mathbb{R}_+^2)$. Let $(x_1, x_2) \in \mathbb{R}_+^2$ and $\lambda > 0$ be fixed. From formula (2) it follows that

$$A(A + \lambda)^{-1} f(x_1, x_2)$$
$$= \frac{1}{\lambda + 1} f(x_1, x_2) + \lambda \int_0^\infty \int_{-\infty}^\infty G(x_1, x_2, p, s, \lambda)\big(f(x_1, x_2) - f(p, s)\big) \, ds \, dp. \tag{10}$$

Using equation (10), the triangle inequality, the following inequalities

$$(a + b)^p \leq 2^p \big(a^p + b^p\big), \quad 0 < p < 1, a, b \geq 0, \tag{11}$$

$$(at)^\theta e^{-at} \leq M, \quad a > 0, \theta \in [0, 1], \tag{12}$$

estimates (10), (11), (12), and the definition of $C^{2\alpha+\beta}$-norm, we obtain

$$\big|\lambda^\alpha A(A + \lambda)^{-1} f(x_1, x_2)\big|$$

$$\leq \frac{\lambda^\alpha}{\lambda + 1} \big|f(x_1, x_2)\big| + \lambda^{\alpha+1} \int_0^\infty \int_{-\infty}^\infty \big|G(x_1, x_2, p, s, \lambda)\big| \big|f(x_1, x_2) - f(p, s)\big| \, ds \, dp$$

$$\leq M_9 \|f\|_{C^{2\alpha+\beta}} \left[\frac{\lambda^\alpha}{\lambda + 1} + \lambda^{\alpha+1} \int_0^\infty \int_{-\infty}^\infty \big|G(x_1, x_2, p, s, \lambda)\big| \right.$$

$$\left. \times \big(|x_1 - p|^2 + |x_2 - s|^2\big)^{\alpha+\beta/2} \, ds \, dp \right]$$

$$\leq M_{10} \|f\|_{C^{2\alpha+\beta}} \left[\frac{\lambda^\alpha}{\lambda + 1} + \lambda^{\alpha+1} \int_0^\infty \int_{-\infty}^\infty e^{-a(1+\lambda)^{1/2}(|x_1-p|+|x_2-s|)} \right.$$

$$\left. \times \big(1 + \ln\big\{1 + \big((1+\lambda)^{1/2}\big(|x_1 - p| + |x_2 - s|\big)\big)^{-1}\big\}\big)\big(|x_1 - p|^2 + |x_2 - s|^2\big)^{\alpha+\beta/2} \, ds \, dp \right]$$

$$\leq M_{11}(a) \|f\|_{C^{2\alpha+\beta}} \left[\frac{\lambda^\alpha}{\lambda + 1} + \frac{\lambda^{\alpha+1}}{(1+\lambda)^{1+\alpha+\beta/2}} \right] \leq M_{12}(a) \|f\|_{C^{2\alpha+\beta}}$$

$$\text{for } (x_1, x_2) \in \mathbb{R}_+^2. \tag{13}$$

Thus, it follows from estimate (13) that

$$\sup_{\lambda > 0} \sup_{(x_1, x_2) \in \mathbb{R}_+^2} \big|\lambda^\alpha A(A + \lambda)^{-1} f(x_1, x_2)\big| \leq M_{12}(a) \|f\|_{C^{2\alpha+\beta}}. \tag{14}$$

Let $\lambda > 0$ and $(x_1, x_2) \in \mathbb{R}_+^2$ be fixed. Using equation (10), we can write

$$\lambda^\alpha A(A + \lambda)^{-1} \left[\frac{f(x_1 + \tau, x_2 + h) - f(x_1, x_2)}{(\tau^2 + h^2)^{\beta/2}} \right]$$

$$= \frac{\lambda^\alpha}{\lambda + 1} \left[\frac{f(x_1 + \tau, x_2 + h) - f(x_1, x_2)}{(\tau^2 + h^2)^{\beta/2}} \right]$$

$$+ \lambda^{\alpha+1} \int_0^\infty \int_{-\infty}^\infty G(x_1 + \tau, x_2 + h, p, s; \lambda) \left[\frac{f(x_1 + \tau, x_2 + h) - f(p, s)}{(\tau^2 + h^2)^{\beta/2}} \right] ds \, dp$$

$$- \lambda^{\alpha+1} \int_0^\infty \int_{-\infty}^\infty G(x_1, x_2, p, s; \lambda) \left[\frac{f(x_1, x_2) - f(p, s)}{(\tau^2 + h^2)^{\beta/2}} \right] ds \, dp. \tag{15}$$

Now, we will estimate the right-hand side of equation (15). We consider two cases $\tau^2 + h^2 < 1$ and $\tau^2 + h^2 \geq 1$, respectively. Let us first assume that $\tau^2 + h^2 < 1$. Furthermore, this situation will be considered in two cases: $(1 + \lambda)(\tau^2 + h^2) \geq 1$ and $(1 + \lambda)(\tau^2 + h^2) \leq 1$. Let $(1 + \lambda)(\tau^2 + h^2) \geq 1$. From equation (15), the triangle inequality, the definition of $C^{2\alpha+\beta}$-norm, the assumptions $\tau^2 + h^2 < 1$ and $(1 + \lambda)(\tau^2 + h^2) \geq 1$, it follows that

$$\left| \lambda^\alpha A(A + \lambda)^{-1} \left[\frac{f(x_1 + \tau, x_2 + h) - f(x_1, x_2)}{(\tau^2 + h^2)^{\beta/2}} \right] \right|$$

$$\leq M_{13} \|f\|_{C^{2\alpha+\beta}} \left[\frac{\lambda^\alpha}{(1 + \lambda)(\tau^2 + h^2)^{\beta/2}} \right.$$

$$+ \frac{\lambda^{\alpha+1}}{(\tau^2 + h^2)^{\beta/2}} \int_0^\infty \int_{-\infty}^\infty |G(x_1 + \tau, x_2 + h, p, s; \lambda)|$$

$$\times \left(|x_1 + \tau - p|^2 + |x_2 + h - s|^2 \right)^{\alpha+\beta/2} ds\,dp$$

$$\left. + \frac{\lambda^{\alpha+1}}{(\tau^2 + h^2)^{\beta/2}} \int_0^\infty \int_{-\infty}^\infty |G(x_1, x_2, p, s; \lambda)| \left(|x_1 - p|^2 + |x_2 - s|^2 \right)^{\alpha+\beta/2} ds\,dp \right]$$

$$= M_{13} \|f\|_{C^{2\alpha+\beta}} [J_1 + J_2 + J_3]. \tag{16}$$

We will estimate J_i, $i = 1, 2, 3$, separately.

First, let us estimate J_1. Clearly, by the assumption $(1 + \lambda)(\tau^2 + h^2) \geq 1$, we have

$$J_1 \leq 1. \tag{17}$$

From estimates (3), (11), (12), and the assumption $(1 + \lambda)(\tau^2 + h^2) \geq 1$, it follows that

$$J_2 \leq M_{14} \frac{\lambda^{\alpha+1}}{(\tau^2 + h^2)^{\beta/2}} \int_0^\infty \int_{-\infty}^\infty e^{-a(1+\lambda)^{1/2}(|x_1+\tau-p|+|x_2+h-s|)}$$

$$\times \left(1 + \ln\left\{ 1 + \left((\lambda + 1)^{1/2} (|x_1 + \tau - p| + |x_2 + h - s|) \right)^{-1} \right\} \right)$$

$$\times \left(|x_1 + \tau - p|^2 + |x_2 + h - s|^2 \right)^{\alpha+\beta/2} ds\,dp$$

$$\leq M_{15}(a) \frac{\lambda^{\alpha+1}}{(1 + \lambda)^{1+\alpha+\beta/2}(\tau^2 + h^2)^{\beta/2}} \leq M_{15}(a). \tag{18}$$

Estimates (3), (11), (12), and the assumption $(1 + \lambda)(\tau^2 + h^2) \geq 1$ yield that

$$J_3 \leq M_{16} \frac{\lambda^{\alpha+1}}{(\tau^2 + h^2)^{\beta/2}} \int_0^\infty \int_{-\infty}^\infty e^{-a(1+\lambda)^{1/2}(|x_1-p|+|x_2-s|)}$$

$$\times \left(1 + \ln\left\{ 1 + \left((\lambda + 1)^{1/2} (|x_1 - p| + |x_2 - s|) \right)^{-1} \right\} \right) \left(|x_1 - p|^2 + |x_2 - s|^2 \right)^{\alpha+\beta/2} ds\,dp$$

$$\leq \frac{M_{17}(a)\lambda^{\alpha+1}}{(1 + \lambda)^{1+\alpha+\beta/2}(\tau^2 + h^2)^{\beta/2}} \leq M_{17}(a). \tag{19}$$

Combining estimates (17)-(19), we get

$$\left| \lambda^\alpha A(A + \lambda)^{-1} \left[\frac{f(x_1 + \tau, x_2 + h) - f(x_1, x_2)}{(\tau^2 + h^2)^{\beta/2}} \right] \right| \leq M_{18}(a) \|f\|_{C^{2\alpha+\beta}}. \tag{20}$$

Now, let us consider the case $(1 + \lambda)(\tau^2 + h^2) \leq 1$. Then, using equation (10), we can write

$$
\begin{aligned}
\lambda^\alpha A(A + \lambda)^{-1} & \left[\frac{f(x_1 + \tau, x_2 + h) - f(x_1, x_2)}{(\tau^2 + h^2)^{\beta/2}} \right] \\
= & \frac{\lambda^\alpha}{\lambda + 1} \left[\frac{f(x_1 + \tau, x_2 + h) - f(x_1, x_2)}{(\tau^2 + h^2)^{\beta/2}} \right] \\
& + \lambda^{\alpha+1} \int_0^\infty \int_{-\infty}^\infty G(x_1 + \tau, x_2 + h, p, s; \lambda) \left[\frac{f(x_1 + \tau, x_2 + h) - f(x_1, x_2)}{(\tau^2 + h^2)^{\beta/2}} \right] ds\, dp \\
& + \lambda^{\alpha+1} \int_0^\infty \int_{-\infty}^\infty \left[G(x_1 + \tau, x_2 + h, p, s; \lambda) - G(x_1, x_2, p, s; \lambda) \right] \\
& \times \left[\frac{f(x_1, x_2) - f(p, s)}{(\tau^2 + h^2)^{\beta/2}} \right] ds\, dp.
\end{aligned}
\tag{21}
$$

From equation (21), the triangle inequality, the Lagrange theorem, the definition of $C^{2\alpha+\beta}$-norm, and the assumption $\tau^2 + h^2 < 1$, it follows that for some x_1^* between x_1, $x_2 + \tau$, and x_2^* between x_2, $x_2 + h$,

$$
\begin{aligned}
& \left| \lambda^\alpha A(A + \lambda)^{-1} \left[\frac{f(x_1 + \tau, x_2 + h) - f(x_1, x_2)}{(\tau^2 + h^2)^{\beta/2}} \right] \right| \\
& \leq M_{19} \|f\|_{C^{2\alpha+\beta}} \left[\frac{\lambda^\alpha}{\lambda + 1} \left(\tau^2 + h^2\right)^\alpha + \lambda^{\alpha+1} \left(\tau^2 + h^2\right)^\alpha \right. \\
& \quad \times \int_0^\infty \int_{-\infty}^\infty \left| G(x_1 + \tau, x_2 + h, p, s; \lambda) \right| ds\, dp \\
& \quad + \frac{\lambda^{\alpha+1}\tau}{(\tau^2 + h^2)^{\beta/2}} \int_0^\infty \int_{-\infty}^\infty \left| G_{x_1}\left(x_1^*, x_2^*, p, s; \lambda\right) \right| \left(|x_1 - p|^2 + |x_2 - s|^2\right)^{\alpha+\beta/2} ds\, dp \\
& \quad \left. + \frac{\lambda^{\alpha+1}h}{(\tau^2 + h^2)^{\beta/2}} \int_0^\infty \int_{-\infty}^\infty \left| G_{x_2}\left(x_1^*, x_2^*, p, s; \lambda\right) \right| \left(|x_1 - p|^2 + |x_2 - s|^2\right)^{\alpha+\beta/2} ds\, dp \right] \\
& = M_{19} \|f\|_{C^{2\alpha+\beta}} [L_1 + L_2 + L_3 + L_4].
\end{aligned}
$$

We will estimate L_i, $i = 1, \ldots, 4$, separately. Let us start with L_1. Clearly, we have

$$
L_1 \leq 1.
\tag{22}
$$

Using estimates (3), (4), (11), (12), we obtain

$$
\begin{aligned}
L_2 \leq M_{20}\lambda \int_0^\infty \int_{-\infty}^\infty & e^{-a(1+\lambda)^{1/2}(|x_1+\tau-p|+|x_2+h-s|)} \\
& \times \left(1 + \ln\left\{1 + \left((\lambda + 1)^{1/2}\left(|x_1 + \tau - p| + |x_2 + h - s|\right)\right)^{-1}\right\}\right) ds\, dp \leq M.
\end{aligned}
\tag{23}
$$

Note that by the triangle inequality and the fact that x_1^* between x_1, $x_1 + \tau$, we get

$$
\left| x_1^* - p \right| \geq |x_1 - p| - \left| x_1^* - x_1 \right| \geq |x_1 - p| - \tau,
\tag{24}
$$

$$
\left| x_2^* - s \right| \geq |x_2 - s| - \left| x_2^* - x_2 \right| \geq |x_2 - s| - h.
\tag{25}
$$

By inequalities (11), (12), estimates (4), (24), (25), and Hilbert's inequality, we have

$$
\begin{aligned}
L_3 &\leq M_{21} \frac{\lambda^{\alpha+1}\tau}{(\tau^2 + h^2)^{\beta/2}} \int_0^\infty \int_{-\infty}^\infty \frac{e^{-a(1+\lambda)^{1/2}(|x_1^* - p| + |x_2^* - s|)}}{|x_1^* - p| + |x_2^* - s|} \left(|x_1 - p| + |x_2 - s|\right)^{2\alpha+\beta} \, ds \, dp \\
&\leq M_{22} \frac{\lambda^{\alpha+1}\tau}{(\tau^2 + h^2)^{\beta/2}} \int_0^\infty \int_{-\infty}^\infty e^{-a(1+\lambda)^{1/2}(|x_1^* - p| + |x_2^* - s|)} \left(|x_1^* - p| + |x_2^* - s|\right)^{2\alpha+\beta-1} \, ds \, dp \\
&\quad + M_{22} \frac{\lambda^{\alpha+1}\tau}{(\tau^2 + h^2)^{\beta/2}} \int_0^\infty \int_{-\infty}^\infty e^{-a(1+\lambda)^{1/2}(|x_1^* - p| + |x_2^* - s|)} \\
&\quad \times \left(|x_1^* - p| + |x_2^* - s|\right)^{-1} (\tau + h)^{2\alpha+\beta} \, ds \, dp \\
&\leq M_{23}(a) \left[\frac{\lambda^{\alpha+1}\tau}{(1+\lambda)^{1+\alpha+\beta/2}(\tau^2 + h^2)^{\beta/2}} + \frac{\lambda^{\alpha+1}\tau(\tau + h)^{2\alpha+\beta}}{(1+\lambda)(\tau^2 + h^2)^{\beta/2}} \right] \leq M_{24}(a).
\end{aligned}
\tag{26}
$$

Similarly, we get

$$
L_4 \leq M_{24}(a).
\tag{27}
$$

Combining estimates (22), (23), (26), (27), we obtain that for $\tau^2 + h^2 < 1$, $(1 + \lambda)^{1/2}(\tau^2 + h^2) \leq 1$,

$$
\left| \lambda^\alpha A(A + \lambda)^{-1} \left[\frac{f(x_1 + \tau, x_2 + h) - f(x_1, x_2)}{(\tau^2 + h^2)^{\beta/2}} \right] \right| \leq M_{25}(a) \|f\|_{C^{2\alpha+\beta}}.
\tag{28}
$$

It follows from estimates (20) and (28) that for $\tau^2 + h^2 < 1$, $\lambda > 0$,

$$
\left| \lambda^\alpha A(A + \lambda)^{-1} \left[\frac{f(x_1 + \tau, x_2 + h) - f(x_1, x_2)}{(\tau^2 + h^2)^{\beta/2}} \right] \right| \leq M_{26}(a) \|f\|_{C^{2\alpha+\beta}}.
\tag{29}
$$

Next, let us assume that $\tau^2 + h^2 \geq 1$. By equation (10), we have

$$
\begin{aligned}
&\lambda^\alpha A(A + \lambda)^{-1} \left[\frac{f(x_1 + \tau, x_2 + h) - f(x_1, x_2)}{(\tau^2 + h^2)^{\beta/2}} \right] \\
&= \frac{\lambda^\alpha}{\lambda + 1} \left[\frac{f(x_1 + \tau, x_2 + h) - f(x_1, x_2)}{(\tau^2 + h^2)^{\beta/2}} \right] \\
&\quad + \lambda^{\alpha+1} \int_0^\infty \int_{-\infty}^\infty G(x_1 + \tau, x_2 + h, p, s; \lambda) \left[\frac{f(x_1 + \tau, x_2 + h) - f(p, s)}{(\tau^2 + h^2)^{\beta/2}} \right] \, ds \, dp \\
&\quad - \lambda^{\alpha+1} \int_0^\infty \int_{-\infty}^\infty G(x_1, x_2, p, s; \lambda) \left[\frac{f(x_1, x_2) - f(p, s)}{(\tau^2 + h^2)^{\beta/2}} \right] \, ds \, dp.
\end{aligned}
\tag{30}
$$

Equation (30), estimates (3), (11), Hilbert's inequality, the triangle inequality, the definition of $C^{2\alpha+\beta}$-norm, and the assumption $\tau^2 + h^2 \geq 1$ yield that for $\lambda > 0$,

$$
\begin{aligned}
&\left| \lambda^\alpha A(A + \lambda)^{-1} \left[\frac{f(x_1 + \tau, x_2 + h) - f(x_1, x_2)}{(\tau^2 + h^2)^{\beta/2}} \right] \right| \\
&\leq M_{27} \|f\|_{C^{2\alpha+\beta}} \left[1 + \lambda^{\alpha+1} \int_0^\infty \int_{-\infty}^\infty \left| G(x_1 + \tau, x_2 + h, p, s; \lambda) \right| \right. \\
&\quad \times \left(|x_1 + \tau - p|^2 + |x_2 + h - s|^2 \right)^{\alpha+\beta/2} \, ds \, dp
\end{aligned}
$$

$$+ \lambda^{\alpha+1} \int_0^\infty \int_{-\infty}^\infty \left| G(x_1, x_2, p, s; \lambda) \right| \left(|x_1 - p|^2 + |x_2 - s|^2 \right)^{\alpha+\beta/2} ds \, dp$$

$$\leq M_{28}(a) \|f\|_{C^{2\alpha+\beta}} \left[1 + \frac{\lambda^{\alpha+1}}{(1+\lambda)^{1+\alpha+\beta/2}} + \frac{\lambda^{\alpha+1}}{(1+\lambda)^{1+\alpha+\beta/2}} \right]$$

$$\leq M_{29}(a) \|f\|_{C^{2\alpha+\beta}}. \tag{31}$$

From estimates (29) and (31) it follows that

$$\sup_{\substack{\lambda > 0 \\ }} \sup_{\substack{(x_1+\tau, x_2+h),(x_1,x_2) \in \mathbb{R}_+^2 \\ (\tau, h) \neq (0,0)}} \left| \lambda^\alpha A(A+\lambda)^{-1} \left[\frac{f(x_1+\tau, x_2+h) - f(x_1, x_2)}{(\tau^2 + h^2)^{\beta/2}} \right] \right|$$

$$\leq M_{30}(a) \|f\|_{C^{2\alpha+\beta}}. \tag{32}$$

Combining estimates (14) and (32), we obtain

$$C^{2\alpha+\beta}\left(\mathbb{R}_+^2\right) \subset E_\alpha\left(A, C^\beta\left(\mathbb{R}_+^2\right)\right).$$

Now, we will prove that $E_\alpha = E_\alpha(A, C^\beta(\mathbb{R}_+^2)) \subset C^{2\alpha+\beta}(\mathbb{R}_+^2)$. By Theorem 4, A is a positive operator in the Banach space E_α. Hence, for $V \in E_\alpha$, we have

$$V = \int_0^\infty A(\lambda + A)^{-2} V \, d\lambda. \tag{33}$$

Let $f \in E_\alpha$. It follows from formula (2) and equation (33) that

$$f(x_1, x_2) = \int_0^\infty A(\lambda + A)^{-2} f(x_1, x_2) \, d\lambda$$

$$= \int_0^\infty \int_0^\infty \int_{-\infty}^\infty G(x_1, x_2, p, s; \lambda) A(\lambda + A)^{-1} f(p, s) \, ds \, dp \, d\lambda. \tag{34}$$

Using the triangle inequality, equation (34), estimate (3), and the definition of E_α-norm, we obtain

$$\left| f(x_1, x_2) \right| \leq M_{31} \|f\|_{E_\alpha} \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \frac{1}{\lambda^\alpha} e^{-a(1+\lambda)^{1/2}(|x_1-p|+|x_2-s|)}$$

$$\times \left(1 + \ln\left\{ 1 + \left((1+\lambda)^{1/2} \left(|x_1 - p| + |x_2 - s| \right) \right)^{-1} \right\} \right) ds \, dp \, d\lambda$$

$$\leq M_{32}(a) \|f\|_{E_\alpha} \int_0^\infty \frac{1}{\lambda^\alpha(1+\lambda)} \, d\lambda \leq \frac{M_{33}(a)}{\alpha(1-\alpha)} \|f\|_{E_\alpha}.$$

Thus,

$$\sup_{(x_1, x_2) \in \mathbb{R}_+^2} \left| f(x_1, x_2) \right| \leq \frac{M_{33}(a)}{\alpha(1-\alpha)} \|f\|_{E_\alpha}. \tag{35}$$

Let $(x_1, x_2) \in \mathbb{R}_+^2$ be fixed. From equation (34) it follows that

$$\frac{f(x_1+\tau, x_2+h) - f(x_1, x_2)}{(\tau^2 + h^2)^{\alpha+\beta/2}}$$

$$= \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \frac{G(x_1+\tau, x_2+h, p, s; \lambda)}{(\tau^2 + h^2)^{\alpha+\beta/2} \lambda^\alpha} \lambda^\alpha A(A+\lambda)^{-1}$$

$$\times \big(f(p,s) - f(x_1,x_2)\big)\,ds\,dp\,d\lambda$$

$$- \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \frac{G(x_1,x_2,p,s;\lambda)}{(\tau^2+h^2)^{\alpha+\beta/2}\lambda^\alpha}\lambda^\alpha A(A+\lambda)^{-1}\big(f(p,s) - f(x_1,x_2)\big)\,ds\,dp\,d\lambda. \quad (36)$$

Now, we will estimate the right-hand side of equation (36). We consider two cases $\tau^2 + h^2 < 1$ and $\tau^2 + h^2 \geq 1$. Let us first assume that $\tau^2 + h^2 < 1$. By equation (36), the triangle inequality, the Lagrange theorem, the definition of E_α-norm, and the assumption $\tau^2 + h^2 < 1$, we have, for some x_1^* between x_1, $x_1 + \tau$, and x_2^* between x_2, $x_2 + h$, that

$$\left|\frac{f(x_1+\tau, x_2+h) - f(x_1,x_2)}{(\tau^2+h^2)^{\alpha+\beta/2}}\right|$$

$$\leq M_{34}\|f\|_{E_\alpha}\Bigg[\int_0^1 \int_0^\infty \int_{-\infty}^\infty \left[\frac{|G(x_1+\tau, x_2+h,p,s;\lambda) - G(x_1,x_2,p,s;\lambda)|}{(\tau^2+h^2)^{\alpha+\beta/2}\lambda^\alpha}\right]$$

$$\times \big(|x_1-p|^2 + |x_2-s|^2\big)^{\beta/2}\,ds\,dp\,d\lambda$$

$$+ \int_1^{1/(\tau^2+h^2)} \int_0^\infty \int_{-\infty}^\infty \left[\frac{|G(x_1+\tau, x_2+h,p,s;\lambda) - G(x_1,x_2,p,s;\lambda)|}{(\tau^2+h^2)^{\alpha+\beta/2}\lambda^\alpha}\right]$$

$$\times \big(|x_1-p|^2 + |x_2-s|^2\big)^{\beta/2}\,ds\,dp\,d\lambda$$

$$+ \int_{1/(\tau^2+h^2)}^\infty \int_0^\infty \int_{-\infty}^\infty \left[\frac{|G(x_1+\tau, x_2+h,p,s;\lambda) - G(x_1,x_2,p,s;\lambda)|}{(\tau^2+h^2)^{\alpha+\beta/2}\lambda^\alpha}\right]$$

$$\times \big(|x_1-p|^2 + |x_2-s|^2\big)^{\beta/2}\,ds\,dp\,d\lambda\Bigg]$$

$$= M_{34}\|f\|_{E_\alpha}[I_1 + I_2 + I_3].$$

We will estimate I_i, $i = 1, 2, 3$, separately. Using the triangle inequality, estimates (4), (11), (25), (24), (12), the Lagrange theorem, and the assumption $\tau^2 + h^2 < 1$, we obtain

$$I_1 \leq M_{35}\Bigg[\int_0^1 \int_0^\infty \int_{-\infty}^\infty \left[\frac{\tau|G_{x_1}(x_1^*,x_2^*,p,s;\lambda)| + h|G_{x_2}(x_1^*,x_2^*,p,s;\lambda)|}{(\tau^2+h^2)^{\alpha+\beta/2}\lambda^\alpha}\right]$$

$$\times \big(|x_1-p|^2 + |x_2-s|^2\big)^{\beta/2}\,ds\,dp\,d\lambda\Bigg]$$

$$\leq \int_0^1 \int_0^\infty \int_{-\infty}^\infty \left[\frac{e^{-a(1+\lambda)^{1/2}(|x_1^*-p|+|x_2^*-s|)}}{(|x_1^*-p|+|x_2^*-s|)\lambda^\alpha}\right]\big(|x_1-p|^2 + |x_2-s|^2\big)^{\beta/2}\,ds\,dp\,d\lambda$$

$$\leq M_{36}\Bigg[\int_0^1 \int_0^\infty \int_{-\infty}^\infty \frac{e^{-a(1+\lambda)^{1/2}(|x_1^*-p|+|x_2^*-s|)}(|x_1^*-p|+|x_2^*-s|)^\beta}{\lambda^\alpha(|x_1^*-p|+|x_2^*-s|)}\,ds\,dp\,d\lambda$$

$$+ \int_0^1 \int_0^\infty \int_{-\infty}^\infty \frac{e^{-a(1+\lambda)^{1/2}(|x_1^*-p|+|x_2^*-s|)}(\tau+h)^\beta}{\lambda^\alpha(|x_1^*-p|+|x_2^*-s|)}\,ds\,dp\,d\lambda\Bigg]$$

$$\leq M_{37}(a)\Bigg[\int_0^1 \frac{1}{\lambda^{\alpha+\beta/2+1/2}}\,d\lambda + \int_0^1 \frac{1}{\lambda^{\alpha+1/2}}\,d\lambda\Bigg]$$

$$\leq M_{38}(a).$$

From the triangle inequality, estimates (4), (11), (12), (24), (25), the Lagrange theorem, and the assumption $\tau^2 + h^2 < 1$, it follows that

$$
I_2 \le M_{39}\left[\int_1^{1/(\tau^2+h^2)}\int_0^\infty\int_{-\infty}^\infty\left[\frac{\tau|G_{x_1}(x_1^*,x_2^*,p,s;\lambda)| + h|G_{x_2}(x_1^*,x_2^*,p,s;\lambda)|}{(\tau^2+h^2)^{\alpha+\beta/2}\lambda^\alpha}\right]\right.
$$
$$
\left.\times\left(|x_1-p|^2 + |x_2-s|^2\right)^{\beta/2}\,ds\,dp\,d\lambda\right]
$$
$$
\le M_{40}\left[\int_1^{1/(\tau^2+h^2)}\int_0^\infty\int_{-\infty}^\infty\frac{(\tau+h)e^{-a(1+\lambda)^{1/2}(|x_1^*-p|+|x_2^*-s|)}(|x_1-p|+|x_2-s|)^\beta}{(\tau^2+h^2)^{\alpha+\beta/2}\lambda^\alpha(|x_1^*-p|+|x_2^*-s|)}\,ds\,dp\,d\lambda\right]
$$
$$
\le M_{41}\left[\int_1^{1/(\tau^2+h^2)}\int_0^\infty\int_{-\infty}^\infty\frac{(\tau+h)e^{-a(1+\lambda)^{1/2}(|x_1^*-p|+|x_2^*-s|)}(|x_1^*-p|+|x_2^*-s|)^\beta}{(\tau^2+h^2)^{\alpha+\beta/2}\lambda^\alpha(|x_1^*-p|+|x_2^*-s|)}\,ds\,dp\,d\lambda\right.
$$
$$
\left.+\int_1^{1/(\tau^2+h^2)}\int_0^\infty\int_{-\infty}^\infty\frac{(\tau+h)e^{-a(1+\lambda)^{1/2}(|x_1^*-p|+|x_2^*-s|)}(\tau+h)^\beta}{(\tau^2+h^2)^{\alpha+\beta/2}\lambda^\alpha(|x_1^*-p|+|x_2^*-s|)}\,ds\,dp\,d\lambda\right]
$$
$$
\le M_{42}(a)\left[\int_1^{1/(\tau^2+h^2)}\frac{(\tau^2+h^2)^{1/2}}{(\tau^2+h^2)^{\alpha+\beta/2}\lambda^{\alpha+\beta/2+1/2}}\,d\lambda + \int_1^{1/(\tau^2+h^2)}\frac{(\tau^2+h^2)^{1/2+\beta/2}}{(\tau^2+h^2)^{\alpha+\beta/2}\lambda^{\alpha+1/2}}\,d\lambda\right]
$$
$$
\le M_{43}(a).
$$

Using the triangle inequality, estimates (3), (24), (25), (12), (11), the assumption $\tau^2 + h^2 < 1$, and the following estimate

$$
|x_1 - p| = |x_1 - p + \tau - \tau| \le |x_1 - p + \tau| + \tau,
$$
$$
|x_2 - s| = |x_2 - s + h - h| \le |x_2 - s + h| + h,
$$

we obtain

$$
I_3 \le M_{44}\left[\int_{1/(\tau^2+h^2)}^\infty\int_0^\infty\int_{-\infty}^\infty\frac{e^{-a(1+\lambda)^{1/2}(|x_1+\tau-p|+|x_2+h-s|)}}{(\tau^2+h^2)^{\alpha+\beta/2}\lambda^\alpha}\right.
$$
$$
\times\left(1+\ln\left\{1+\left((\lambda+1)^{1/2}\left(|x_1+\tau-p|+|x_2+h-s|\right)\right)^{-1}\right\}\right)
$$
$$
\times\left(|x_1-p+\tau|+|x_2-s+h|+\tau+h\right)^\beta\,ds\,dp\,d\lambda
$$
$$
+\int_{1/(\tau^2+h^2)}^\infty\int_0^\infty\int_{-\infty}^\infty\left(e^{-a(1+\lambda)^{1/2}(|x_1-p|+|x_2-s|)}\right.
$$
$$
\times\left(1+\ln\left\{1+\left((\lambda+1)^{1/2}\left(|x_1-p|+|x_2-s|\right)\right)^{-1}\right\}\right))/\left((\tau^2+h^2)^{\alpha+\beta/2}\lambda^\alpha\right)
$$
$$
\left.\times\left(|x_1-p|^2+|x_2-s|^2\right)^{\beta/2}\,ds\,dp\,d\lambda\right]
$$
$$
\le M_{45}(a)\left[\int_{1/(\tau^2+h^2)}^\infty\frac{1}{(\tau^2+h^2)^{\alpha+\beta/2}\lambda^{\alpha+\beta/2+1}}\,d\lambda + \int_{1/(\tau^2+h^2)}^\infty\frac{\tau+h}{(\tau^2+h^2)^{\alpha+\beta/2}\lambda^{\alpha+1}}\,d\lambda\right]
$$
$$
\le M_{46}(a).
$$

Thus, for $\tau^2 + h^2 < 1$, we have

$$
\left|\frac{f(x_1+\tau,x_2+h) - f(x_1,x_2)}{(\tau^2+h^2)^{\alpha+\beta/2}}\right| \le M_{47}(a)\|f\|_{E_\alpha}. \tag{37}
$$

Next, let us assume that $\tau^2 + h^2 \geq 1$. Using formula (36), estimate (3), the triangle inequality, Hilbert's inequality, and the assumption $\tau^2 + h^2 \geq 1$, we get

$$
\left| \frac{f(x_1 + \tau, x_2 + h) - f(x_1, x_2)}{(\tau^2 + h^2)^{\alpha + \beta/2}} \right|
$$

$$
\leq \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \frac{|G(x_1 + \tau, x_2 + h, p, s; \lambda)|}{\lambda^\alpha} |\lambda^\alpha A(A + \lambda)^{-1} f(p, s)| \, ds \, dp \, d\lambda
$$

$$
+ \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \frac{|G(x_1, x_2, p, s; \lambda)|}{\lambda^\alpha} |\lambda^\alpha A(A + \lambda)^{-1} f(p, s)| \, ds \, dp \, d\lambda
$$

$$
\leq M_{48} \|f\|_{E_\alpha} \left[\int_0^\infty \int_0^\infty \int_{-\infty}^\infty \frac{1}{\lambda^\alpha} e^{-a(1+\lambda)^{1/2}(|x_1 + \tau - p| + |x_2 + h - s|)} \right.
$$

$$
\times \left(1 + \ln\left\{ 1 + \left((1 + \lambda)^{1/2} (|x_1 + \tau - p| + |x_2 + h - s|) \right)^{-1} \right\} \right) ds \, dp \, d\lambda
$$

$$
+ \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \frac{1}{\lambda^\alpha} e^{-a(1+\lambda)^{1/2}(|x_1 - p| + |x_2 - s|)}
$$

$$
\left. \times \left(1 + \ln\left\{ 1 + \left((1 + \lambda)^{1/2} (|x_1 - p| + |x_2 - s|) \right)^{-1} \right\} \right) ds \, dp \, d\lambda \right]
$$

$$
\leq M_{49}(a) \|f\|_{E_\alpha} \int_0^\infty \frac{1}{\lambda^\alpha (1 + \lambda)} \, d\lambda \leq \frac{M_{50}(a)}{\alpha(1 - \alpha)} \|f\|_{E_\alpha}.
$$

Hence, for $\tau^2 + h^2 \geq 1$, we obtain

$$
\left| \frac{f(x_1 + \tau, x_2 + h) - f(t, x)}{(\tau^2 + h^2)^{\alpha + \beta/2}} \right| \leq \frac{M_{50}(a)}{\alpha(1 - \alpha)} \|f\|_{E_\alpha}. \tag{38}
$$

Combining estimates (37) and (38), we get

$$
\sup_{\substack{\lambda > 0 \\ (x_1 + \tau, x_2 + h), (x_1, x_2) \in \mathbb{R}^2_+ \\ (\tau, h) \neq (0, 0)}} \left| \frac{f(x_1 + \tau, x_2 + h) - f(x_1, x_2)}{(\tau^2 + h^2)^{\alpha + \beta/2}} \right| \leq \frac{M_{51}(a)}{\alpha(1 - \alpha)} \|f\|_{E_\alpha}. \tag{39}
$$

Estimates (35) and (39) yield that

$$
E_\alpha = E_\alpha \left(A, C^\beta \left(\mathbb{R}^2_+ \right) \right) \subset C^{2\alpha + \beta} \left(\mathbb{R}^2_+ \right).
$$

This is the end of the proof of Theorem 5. $\qquad \square$

4 Applications

In this section, we consider some applications of Theorem 5. First, we consider the boundary value problem for the elliptic equation

$$
\begin{cases}
-\frac{\partial^2 u(y, x_1, x_2)}{\partial y^2} - a_{11}(x_1, x_2) \frac{\partial^2 u(y, x_1, x_2)}{\partial x_1^2} - a_{22}(x_1, x_2) \frac{\partial^2 u(y, x_1, x_2)}{\partial x_2^2} + \sigma u(y, x_1, x_2) \\
\quad = f(y, x_1, x_2), \quad 0 < y < T, x_1 \in \mathbb{R}^+, x_2 \in \mathbb{R}, \\
u(0, x_1, x_2) = \varphi(x_1, x_2), \qquad u(T, x_1, x_2) = \psi(x_1, x_2), \quad x_1 \in \mathbb{R}^+, x_2 \in \mathbb{R}, \\
u(y, 0, x_2) = 0, \quad 0 \leq y \leq T, x_2 \in \mathbb{R}.
\end{cases} \tag{40}
$$

Here, $a_{11}(x_1, x_2)$, $a_{22}(x_1, x_2)$, $\varphi(x_1, x_2)$, $\psi(x_1, x_2)$, and $f(y, x_1, x_2)$ are given smooth functions and they satisfy every compatibility condition and

$$a_{11}^2(x_1, x_2) + a_{22}^2(x_1, x_2) \geq \delta > 0, \tag{41}$$

and $\sigma > 0$, which guarantees that problem (40) has a smooth solution $u(y, x_1, x_2)$.

Theorem 6 *For the solution of boundary value problem* (40), *we have the following estimate*:

$$\|u_{yy}\|_{C(C^{2\alpha+\beta}(\mathbb{R}_+^2))} + \|u_{x_1 x_1}\|_{C(C^{2\alpha+\beta}(\mathbb{R}_+^2))} + \|u_{x_2 x_2}\|_{C(C^{2\alpha+\beta}(\mathbb{R}_+^2))}$$
$$\leq M(\alpha, \beta)\big[\|\varphi_{x_1 x_1}\|_{C^{2\alpha+\beta}(\mathbb{R}_+^2)} + \|\varphi_{x_2 x_2}\|_{C^{2\alpha+\beta}(\mathbb{R}_+^2)} + \|\psi_{x_1 x_1}\|_{C^{2\alpha+\beta}(\mathbb{R}_+^2)}$$
$$+ \|\psi_{x_2 x_2}\|_{C^{2\alpha+\beta}(\mathbb{R}_+^2)} + \|f\|_{C(C^{2\alpha+\beta}(\mathbb{R}_+^2))}\big],$$

where $M(\alpha, \beta)$ is independent of φ, ψ, and f.

Proof We introduce the Banach space $\mathbb{C}([0, T], E)$ of all continuous abstract functions $u(y)$ defined on $[0, T]$ with values in E, equipped with the norm

$$\|u\|_{\mathbb{C}([0,T],E)} = \max_{0 \leq y \leq T} \|u(y)\|_E.$$

Note that problem (40) can be written in the form of the abstract boundary value problem

$$-\frac{d^2 u(y)}{dy^2} + Au(y) = f(y), \quad 0 < y < T, \qquad u(0) = \varphi, \qquad u(T) = \psi \tag{42}$$

in a Banach space $E = C(\mathbb{R}_+^2)$ with a positive operator A defined by (1). Here $f(y) = f(y, x_1, x_2)$ is the given abstract function defined on $[0, T]$ with values in E, $\varphi = \varphi(x_1, x_2)$, $\psi = \psi(x_1, x_2)$ are elements of $D(A)$. Therefore, the proof of Theorem 6 is based on Theorem 5 on the structure of the fractional spaces $E_\alpha(A, C^\beta(\mathbb{R}_+^2))$, Theorem 4 on the positivity of the operator A, on the following theorems on coercive stability of elliptic problems, nonlocal boundary value for the abstract elliptic equation and on the structure of the fractional space $E'_\alpha = E_\alpha(A^{1/2}, E)$. This is the end of the proof of Theorem 6. □

Theorem 7 *Under assumption* (41) *for the solution of elliptic problem*

$$\begin{cases} -a_{11}(x_1, x_2)\frac{\partial^2 u(x_1, x_2)}{\partial x_1^2} - a_{22}(x_1, x_2)\frac{\partial^2 u(x_1, x_2)}{\partial x_2^2} + \sigma u(x_1, x_2) = g(x_1, x_2), \\ x_1 \in \mathbb{R}^+, x_2 \in \mathbb{R}, \\ u(0, x_2) = 0, \quad x_2 \in \mathbb{R}, \end{cases}$$

the following coercive inequality holds:

$$\left\|\frac{\partial^2 u}{\partial x_1^2}\right\|_{C^\mu(\mathbb{R}_+^2)} + \left\|\frac{\partial^2 u}{\partial x_2^2}\right\|_{C^\mu(\mathbb{R}_+^2)} \leq M(\mu)\|g\|_{C^\mu(\mathbb{R}_+^2)},$$

where $M(\mu)$ $(0 < \mu < 1)$ does not depend on g.

The proof of Theorem 7 uses the techniques introduced in [5, Chapter 5] and it is based on estimates (3) and (4).

Theorem 8 ([5, Theorem 5.2.48]) *The spaces $E_\alpha(A, E)$ and $E'_{2\alpha}(A^{1/2}, E)$ coincide for any $0 < \alpha < \frac{1}{2}$, and their norms are equivalent.*

Theorem 9 ([28, Theorem 3.1]) *Let A be a positive operator in a Banach space E and $f \in C([0, T], E'_\alpha)$ ($0 < \alpha < 1$). Then, for the solution of nonlocal boundary value problem* (42), *the coercive inequality*

$$\left\| u'' \right\|_{C([0,T],E'_\alpha)} + \| Au \|_{C([0,T],E'_\alpha)}$$

$$\leq M \left[\| A\varphi \|_{E'_\alpha} + \| A\psi \|_{E'_\alpha} + \frac{M}{\alpha(1 - \alpha)} \| f \|_{C([0,T],E'_\alpha)} \right]$$

holds, where M does not depend on α, φ, ψ, and f.

Second, we consider the nonlocal boundary value problem for the elliptic equation under assumption (41)

$$\begin{cases} -\frac{\partial^2 u(y,x_1,x_2)}{\partial y^2} - a_{11}(x_1, x_2) \frac{\partial^2 u(y,x_1,x_2)}{\partial x_1^2} - a_{22}(x_1, x_2) \frac{\partial^2 u(y,x_1,x_2)}{\partial x_2^2} + \sigma u(y, x_1, x_2) \\ = f(y, x_1, x_2), \quad 0 < y < T, x_1 \in \mathbb{R}^+, x_2 \in \mathbb{R}, \\ u(0, x_1, x_2) = u(T, x_1, x_2), \qquad u_y(0, x_1, x_2) = u_y(T, x_1, x_2), \quad x_1 \in \mathbb{R}^+, x_2 \in \mathbb{R}, \\ u(y, 0, x_2) = 0, \quad 0 \leq y \leq T, x_2 \in \mathbb{R}. \end{cases} \tag{43}$$

Here, $a_{11}(x_1, x_2)$, $a_{22}(x_1, x_2)$, and $f(y, x_1, x_2)$ are given smooth functions and they satisfy every compatibility condition and (41), which guarantees that problem (43) has a smooth solution $u(y, x_1, x_2)$.

Theorem 10 *For the solution of initial boundary value problem* (43), *we have the following estimate*:

$$\| u_{yy} \|_{C(C^{2\alpha+\beta}(\mathbb{R}_+^2))} + \| u_{x_1 x_1} \|_{C(C^{2\alpha+\beta}(\mathbb{R}_+^2))} + \| u_{x_2 x_2} \|_{C(C^{2\alpha+\beta}(\mathbb{R}_+^2))}$$

$$\leq M(\alpha, \beta) \| f \|_{C(C^{2\alpha+\beta}(\mathbb{R}_+^2))},$$

where $M(\alpha, \beta)$ is independent of f.

The proof of Theorem 10 is based on Theorem 5 on the structure of the fractional spaces $E_\alpha(A, C^\beta(\mathbb{R}_+^2))$, Theorem 4 on the positivity of the operator A, Theorem 7 on coercive stability of the elliptic problem, Theorem 8 on the structure of the fractional space $E'_\alpha = E_\alpha(A^{1/2}, E)$, and the following theorem on coercive stability of the nonlocal boundary value for the abstract elliptic equation.

Theorem 11 ([28, Theorem 3.1]) *Let A be a positive operator in a Banach space E and $f \in C([0, T], E'_\alpha)$ ($0 < \alpha < 1$). Then, for the solution of nonlocal boundary value problem* (40)

$$\begin{cases} -u''(y) + Au(y) = f(y), \quad 0 < y < T, \\ u(0) = u(T), \quad u'(0) = u'(T) \end{cases}$$

in a Banach space *E* with a positive operator *A*, the coercive inequality

$$\left\| u'' \right\|_{C([0,T],E'_\alpha)} + \left\| Au \right\|_{C([0,T],E'_\alpha)} \leq \frac{M}{\alpha(1-\alpha)} \left\| f \right\|_{C([0,T],E'_\alpha)}$$

holds, where *M* does not depend on *α* and *f*.

5 Conclusion

In the present article, the structure of the fractional spaces $E_\alpha(A, C^\beta(\mathbb{R}^2_+))$ generated by the two-dimensional elliptic differential operator *A* is investigated. The positivity of this operator *A* in a Hölder space is established. Of course, the Banach fixed point theorem and the method of the present paper enable us to establish the existence and uniqueness results which hold under some sufficient conditions on the nonlinear term for the solution of the mixed problem

$$\begin{cases} -\frac{\partial^2 u(y,x_1,x_2)}{\partial y^2} - a_{11}(x_1,x_2)\frac{\partial^2 u(y,x_1,x_2)}{\partial x_1^2} - a_{22}(x_1,x_2)\frac{\partial^2 u(y,x_1,x_2)}{\partial x_2^2} + \sigma u(y,x_1,x_2) \\ \quad = f(y,x_1,x_2,u,u_{x_1},u_{x_2},u_y), \quad 0 < y < T, x_1 \in \mathbb{R}^+, x_2 \in \mathbb{R}, \\ u(0,x_1,x_2) = \varphi(x_1,x_2), \qquad u(T,x_1,x_2) = \psi(x_1,x_2), \quad x_1 \in \mathbb{R}^+, x_2 \in \mathbb{R}, \\ u(y,0,x_2) = 0, \quad 0 < y < T, x_2 \in \mathbb{R}. \end{cases}$$

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