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Approximate solutions for MHD squeezing fluid flow by a novel method

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Abstract

In this paper, a steady axisymmetric MHD flow of two-dimensional incompressible fluids has been investigated. The reproducing kernel Hilbert space method (RKHSM) has been implemented to obtain a solution of the reduced fourth-order nonlinear boundary value problem. Numerical results have been compared with the results obtained by the Runge-Kutta method (RK-4) and optimal homotopy asymptotic method (OHAM).

MSC: 46E22; 35A24

Keywords: reproducing kernel method; series solutions; squeezing fluid flow; magnetohydrodynamics; reproducing kernel space

1 Introduction

Squeezing flows have many applications in food industry, principally in chemical engineering [1–4]. Some practical examples of squeezing flow include polymer processing, compression and injection molding. Grimm [5] studied numerically the thin Newtonian liquids films being squeezed between two plates. Squeezing flow coupled with magnetic field is widely applied to bearing with liquid-metal lubrication [2, 6–8].

In this paper, we use RKHSM to study the squeezing MHD fluid flow between two infinite planar plates. This problem has been solved by RKHSM and for comparison it has been compared with the OHAM and numerically with the RK-4 by using Maple 16.

The RKHSM, which accurately computes the series solution, is of great interest to applied sciences. The method provides the solution in a rapidly convergent series with components that can be elegantly computed. The efficiency of the method was used by many authors to investigate several scientific applications. Geng and Cui [9] and Zhou *et al.* [10] applied the RKHSM to handle the second-order boundary value problems. Yao and Cui [11] and Wang *et al.* [12] investigated a class of singular boundary value problems by this method and the obtained results were good. Wang and Chao [13], Li and Cui [14], Zhou and Cui [15] independently employed the RKSHSM to variable-coefficient partial differential equations. Du and Cui [16] investigated the approximate solution of the forced Duffing equation with integral boundary conditions by combining the homotopy perturbation method and the RKM. Lv and Cui [17] presented a new algorithm to solve linear fifth-order boundary value problems. Cui and Du [18] obtained the representation of the exact solution for the nonlinear Volterra-Fredholm integral equations by using the RKHSM. Wu and Li [19] applied iterative RKHSM to obtain the analytical approximate solution of a nonlinear oscillator with discontinuities. For more details about RKHSM and the modified forms and its effectiveness, see [9–37] and the references therein.

The paper is organized as follows. We give the problem formulation in Section 2. Section 3 introduces several reproducing kernel spaces. A bounded linear operator is presented in Section 4. In Section 5, we provide the main results, the exact and approximate solutions. An iterative method is developed for the kind of problems in the reproducing kernel space. We prove that the approximate solution converges to the exact solution uniformly. Some numerical experiments are illustrated in Section 6. There are some conclusions in the last section.

2 Problem formulation

Consider a squeezing flow of an incompressible Newtonian fluid in the presence of a magnetic field of a constant density ρ and viscosity μ squeezed between two large planar parallel plates separated by a small distance $2H$ and the plates approaching each other with a low constant velocity V, as illustrated in Figure 1, and the flow can be assumed to quasisteady [1–3, 39]. The Navier-Stokes equations [3, 4] governing such flow in the presence of magnetic field, when inertial terms are retained in the flow, are given as [38]

$$\nabla V \cdot u = 0 \tag{2.1}$$

and

$$\rho\left[\frac{\partial u}{\partial t} + (u \cdot \nabla)u\right] = \nabla \cdot T + J \times B + \rho f, \tag{2.2}$$

where u is the velocity vector, ∇ denotes the material time derivative, T is the Cauchy stress tensor,

$$T = -pI + \mu A_1$$

and

$$A_1 = \nabla u + u^T,$$

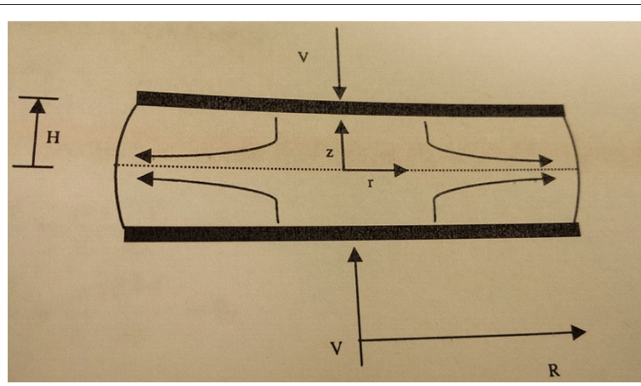

Figure 1 A steady squeezing axisymmetric fluid flow between two parallel plates [38].

J is the electric current density, B is the total magnetic field and

$$B = B_0 + b,$$

B_0 represents the imposed magnetic field and b denotes the induced magnetic field. In the absence of displacement currents, the modified Ohm law and Maxwell's equations (see [40] and the references therein) are given by [38]

$$J = \sigma[E + u \times B] \tag{2.3}$$

and

$$\operatorname{div} B = 0, \qquad \nabla \times B = \mu_m J, \qquad \operatorname{curl} E = \frac{\partial B}{\partial t}, \tag{2.4}$$

in which σ is the electrical conductivity, E is the electric field and μ_m is the magnetic permeability.

The following assumptions are needed [38].

(a) The density ρ, magnetic permeability μ_m and electric field conductivity σ are assumed to be constant throughout the flow field region.

(b) The electrical conductivity σ of the fluid is considered to be finite.

(c) Total magnetic field B is perpendicular to the velocity field V and the induced magnetic field b is negligible compared with the applied magnetic field B_0 so that the magnetic Reynolds number is small (see [40] and the references therein).

(d) We assume a situation where no energy is added or extracted from the fluid by the electric field, which implies that there is no electric field present in the fluid flow region.

Under these assumptions, the magnetohydrodynamic force involved in Eq. (2.2) can be put into the form

$$J \times B = -\sigma B_0^2 u. \tag{2.5}$$

An axisymmetric flow in cylindrical coordinates r, θ, z with z-axis perpendicular to plates and $z = \pm H$ at the plates. Since we have axial symmetry, u is represented by

$$u = \big(u_r(r,z), 0, u_z(r,z)\big),$$

when body forces are negligible, Navier-Stokes Eqs. (2.1)-(2.2) in cylindrical coordinates, where there is no tangential velocity ($u_\theta = 0$), are given as [38]

$$\rho\left(u_r\frac{\partial u_r}{\partial r} + u_z\frac{\partial u_r}{\partial z}\right) = -\frac{\partial p}{\partial r} + \left(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} + \frac{\partial^2 u_r}{\partial z^2}\right) + \sigma B_0^2 u \tag{2.6}$$

and

$$\rho\left(u_z\frac{\partial u_z}{\partial r} + u_z\frac{\partial u_z}{\partial z}\right) = -\frac{\partial p}{\partial r} + \left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} + \frac{\partial^2 u_z}{\partial z^2}\right), \tag{2.7}$$

where p is the pressure, and the equation of continuity is given by [38]

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_z}{\partial z} = 0. \tag{2.8}$$

The boundary conditions require

$$\begin{aligned} &u_r = 0, \qquad u_z = -V \quad \text{at } z = H, \\ &\frac{\partial u_r}{\partial z} = 0, \qquad u_z = 0 \quad \text{at } z = 0. \end{aligned} \tag{2.9}$$

Let us introduce the axisymmetric Stokes stream function Ψ as

$$u_r = \frac{1}{r}\frac{\partial \Psi}{\partial z}, \qquad u_z = -\frac{1}{r}\frac{\partial \Psi}{\partial r}. \tag{2.10}$$

The continuity equation is satisfied using Eq. (2.10). Substituting Eqs. (2.3)-(2.5) and Eq. (2.10) into Eqs. (2.7)-(2.8), we obtain

$$-\frac{\rho}{r^2}\frac{\partial \Psi}{\partial r}E^2\Psi = -\frac{\partial p}{\partial r} + \frac{\mu}{r}\frac{\partial E^2\Psi}{\partial z} - \frac{\sigma B_0^2}{r}\frac{\partial \Psi}{\partial z} \tag{2.11}$$

and

$$-\frac{\rho}{r^2}\frac{\partial \Psi}{\partial z}E^2\Psi = -\frac{\partial p}{\partial z} + \frac{\mu}{r}\frac{\partial E^2\Psi}{\partial r}. \tag{2.12}$$

Eliminating the pressure from Eqs. (2.11) and (2.12) by the integrability condition, we get the compatibility equation as [38]

$$-\rho\left[\frac{\partial(\Psi, \frac{E^2\Psi}{r^2})}{\partial(r,z)}\right] = \frac{\mu}{r}E^2\Psi - \frac{\sigma B_0^2}{r}\frac{\partial^{2\Psi}}{\partial z^2}, \tag{2.13}$$

where

$$E^2 = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}.$$

The stream function can be expressed as [1, 3]

$$\Psi(r,z) = r^2F(z). \tag{2.14}$$

In view of Eq. (2.14), the compatibility equation (2.13) and the boundary conditions (2.9) take the form

$$F^{(iv)}(z) - \frac{\sigma B_0^2}{r}F''(z) + 2\frac{\rho}{\mu}F(z)F'''(z) = 0, \tag{2.15}$$

subject to

$$\begin{aligned} &F(0) = 0, \qquad F''(0) = 0, \\ &F(H) = \frac{V}{2}, \qquad F'(H) = 0. \end{aligned} \tag{2.16}$$

Non-dimensional parameters are given as [38]

$$F^* = 2\frac{F}{V}, \qquad z^* = \frac{z}{H}, \qquad \mathrm{Re} = \frac{\rho HV}{\mu}, \qquad m = B_0 H\sqrt{\frac{\sigma}{\mu}}.$$

For simplicity omitting the $*$, the boundary value problem (2.15)-(2.16) becomes [38]

$$F^{(iv)}(z) - m^2 F''(z) + \mathrm{Re}\, F(z)F'''(z) = 0, \tag{2.17}$$

with the boundary conditions

$$\begin{aligned} &F(0) = 0, \qquad F''(0) = 0, \\ &F(1) = 1, \qquad F'(1) = 0, \end{aligned} \tag{2.18}$$

where Re is the Reynolds number and m is the Hartmann number.

3 Reproducing kernel spaces

In this section, we define some useful reproducing kernel spaces.

Definition 3.1 (Reproducing kernel) Let E be a nonempty abstract set. A function $K : E \times E \longrightarrow C$ is a reproducing kernel of the Hilbert space H if and only if

$$\begin{cases} \forall t \in E, \quad K(\cdot,t) \in H, \\ \forall t \in E, \forall \varphi \in H, \quad \langle \varphi(\cdot), K(\cdot,t)\rangle = \varphi(t). \end{cases} \tag{3.1}$$

The last condition is called 'the reproducing property': the value of the function φ at the point t is reproduced by the inner product of φ with $K(\cdot,t)$.

Definition 3.2 We define the space $W_2^5[0,1]$ by

$$W_2^5[0,1] = \left\{ \begin{array}{l} u|u, u', u'', u''', u^{(4)} \text{ are absolutely continuous in } [0,1], \\ u^{(5)} \in L^2[0,1], x \in [0,1], u(0) = u(1) = u'(1) = u''(0) = 0 \end{array} \right\}.$$

The fifth derivative of u exists almost everywhere since $u^{(4)}$ is absolutely continuous. The inner product and the norm in $W_2^5[0,1]$ are defined respectively by

$$\langle u,v\rangle_{W_2^5} = \sum_{i=0}^{4} u^{(i)}(0)v^{(i)}(0) + \int_0^1 u^{(5)}(x)v^{(5)}(x)\,dx, \quad u,v \in W_2^5[0,1]$$

and

$$\|u\|_{W_2^5} = \sqrt{\langle u,u\rangle_{W_2^5}}, \quad u \in W_2^5[0,1].$$

The space $W_2^5[0,1]$ is a reproducing kernel space, *i.e.*, for each fixed $y \in [0,1]$ and any $u \in W_2^5[0,1]$, there exists a function R_y such that

$$u = \langle u, R_y\rangle_{W_2^5}.$$

Definition 3.3 We define the space $W_2^4[0,1]$ by

$$W_2^4[0,1] = \left\{ \begin{array}{l} u|u, u', u'', u''' \text{ are absolutely continuous in } [0,1], \\ u^{(4)} \in L^2[0,1], x \in [0,1] \end{array} \right\}.$$

The fourth derivative of u exists almost everywhere since $u^{(3)}$ is absolutely continuous. The inner product and the norm in $W_2^4[0,1]$ are defined respectively by

$$\langle u, v\rangle_{W_2^4} = \sum_{i=0}^{3} u^{(i)}(0)v^{(i)}(0) + \int_0^1 u^{(4)}(x)v^{(4)}(x)\,dx, \quad u, v \in W_2^4[0,1]$$

and

$$\|u\|_{W_2^4} = \sqrt{\langle u, u\rangle_{W_2^4}}, \quad u \in W_2^4[0,1].$$

The space $W_2^4[0,1]$ is a reproducing kernel space, *i.e.*, for each fixed $y \in [0,1]$ and any $u \in W_2^4[0,1]$, there exists a function r_y such that

$$u = \langle u, r_y\rangle_{W_2^4}.$$

Theorem 3.1 *The space* $W_2^5[0,1]$ *is a reproducing kernel Hilbert space whose reproducing kernel function is given by*

$$R_y(x) = \begin{cases} \sum_{i=1}^{10} c_i(y)x^{i-1}, & x \le y, \\ \sum_{i=1}^{10} d_i(y)x^{i-1}, & x > y, \end{cases}$$

where $c_i(y)$ *and* $d_i(y)$ *can be obtained easily by using Maple* 16 *and the proof of Theorem* 3.1 *is given in Appendix*.

Remark 3.1 The reproducing kernel function r_y of $W_2^4[0,1]$ is given as

$$r_y(x) = \begin{cases} 1 + xy + \frac{1}{4}y^2x^2 + \frac{1}{36}y^3x^3 + \frac{1}{144}y^3x^4 - \frac{1}{240}y^2x^5 + \frac{1}{720}yx^6 - \frac{1}{5{,}040}x^7, & x \le y, \\ 1 + yx + \frac{1}{4}y^2x^2 + \frac{1}{36}y^3x^3 + \frac{1}{144}x^3y^4 - \frac{1}{240}x^2y^5 + \frac{1}{720}xy^6 - \frac{1}{5{,}040}y^7, & x > y. \end{cases}$$

This can be proved easily as the proof of Theorem 3.1.

4 Bounded linear operator in $W_2^5[0,1]$

In this section, the solution of Eq. (2.17) is given in the reproducing kernel space $W_2^5[0,1]$.

On defining the linear operator $L : W_2^5[0,1] \to W_2^4[0,1]$ as

$$Lu = u^{(4)}(x) + \operatorname{Re}\frac{e^x}{e}\left(x^3 - 4x^2 + 4x\right)u^{(3)}(x) - m^2u''(x) + \operatorname{Re}\frac{e^x}{e}\left(x^3 + 5x^2 - 2x - 6\right)u(x).$$

Model problem (2.17)-(2.18) changes the following problem:

$$\begin{cases} Lu = M(x, u, u^{(3)}), & x \in [0,1], \\ u(0) = 0, \quad u(1) = 0, \quad u'(1) = 0, \quad u''(0) = 0, \end{cases} \tag{4.1}$$

where

$$F(x) = u(x) + \frac{e^x}{e}\left(x^3 - 4x^2 + 4x\right)$$

and

$$M\left(x, u, u^{(3)}\right) = -\operatorname{Re} u^{(3)}(x)u(x) - \operatorname{Re}\left(\frac{e^x}{e}\right)^2\left(x^3 - 4x^2 + 4x\right)\left(x^3 + 5x^2 - 2x - 6\right)$$
$$- \frac{e^x}{e}\left(x^3 + 8x^2 + 8x - 2\right) + m^2\frac{e^x}{e}\left(x^3 + 2x^2 - 6x\right).$$

Theorem 4.1 *The operator L defined by* (4.1) *is a bounded linear operator.*

Proof We only need to prove

$$\|Lu\|^2_{W_2^4} \le P\|Lu\|^2_{W_2^5},$$

where P is a positive constant. By Definition 3.3, we have

$$\|u\|^2_{W_2^4} = \langle u, u\rangle_{W_2^4} = \sum_{i=0}^{3}\left[u^{(i)}(0)\right]^2 + \int_0^1 \left[u^{(4)}(x)\right]^2 dx, \quad u \in W_2^4[0,1],$$

and

$$\|Lu\|^2_{W_2^4} = \langle Lu, Lu\rangle_{W_2^4} = \left[(Lu)(0)\right]^2 + \left[(Lu)'(0)\right]^2 + \left[(Lu)''(0)\right]^2$$
$$+ \left[(Lu)^{(3)}(0)\right]^2 + \int_0^1 \left[(Lu)^{(4)}(x)\right]^2 dx.$$

By the reproducing property, we have

$$u(x) = \langle u, R_x\rangle_{W_2^5},$$

and

$$(Lu)(x) = \left\langle u, (LR_x)\right\rangle_{W_2^5}, \qquad (Lu)'(x) = \left\langle u, (LR_x)'\right\rangle_{W_2^5},$$
$$(Lu)''(x) = \left\langle u, (LR_x)''\right\rangle_{W_2^5}, \qquad (Lu)^{(3)}(x) = \left\langle u, (LR_x)^{(3)}\right\rangle_{W_2^5},$$
$$(Lu)^{(4)}(x) = \left\langle u, (LR_x)^{(4)}\right\rangle_{W_2^5}.$$

Therefore, by the Cauchy-Schwarz inequality, we get

$$\left|(Lu)(x)\right| \le \|u\|_{W_2^5}\|LR_x\|_{W_2^5} = a_1\|u\|_{W_2^5} \quad \text{(where } a_1 > 0 \text{ is a positive constant)},$$
$$\left|(Lu)'(x)\right| \le \|u\|_{W_2^5}\left\|(LR_x)'\right\|_{W_2^5} = a_2\|u\|_{W_2^5} \quad \text{(where } a_2 > 0 \text{ is a positive constant)},$$
$$\left|(Lu)''(x)\right| \le \|u\|_{W_2^5}\left\|(LR_x)''\right\|_{W_2^5} = a_3\|u\|_{W_2^5} \quad \text{(where } a_3 > 0 \text{ is a positive constant)},$$
$$\left|(Lu)^{(3)}(x)\right| \le \|u\|_{W_2^5}\left\|(LR_x)^{(3)}\right\|_{W_2^5} = a_4\|u\|_{W_2^5} \quad \text{(where } a_4 > 0 \text{ is a positive constant)}.$$

Thus

$$\big[(Lu)(0)\big]^2+\big[(Lu)'(0)\big]^2+\big[(Lu)''(0)\big]^2+\big[(Lu)^{(3)}(0)\big]^2\le\big(a_1^2+a_2^2+a_3^2+a_4^2\big)\|u\|^2_{W_2^5}.$$

Since

$$(Lu)^{(4)}=\big\langle u,(LR_x)^{(4)}\big\rangle_{W_2^5},$$

then

$$\big|(Lu)^{(4)}\big|\le\|u\|_{W_2^5}\big\|(LR_x)^{(4)}\big\|_{W_2^5}=a_5\|u\|_{W_2^4}\quad\text{(where } a_5>0 \text{ is a positive constant)}.$$

Therefore, we have

$$\big[(Lu)^{(4)}\big]^2\le a_5^2\|u\|^2_{W_2^5}$$

and

$$\int_0^1\big[(Lu)^{(4)}(x)\big]^2\,dx\le a_5^2\|u\|^2_{W_2^5},$$

that is,

$$\begin{aligned}\|Lu\|^2_{W_2^4}&=\big[(Lu)(0)\big]^2+\big[(Lu)'(0)\big]^2+\big[(Lu)''(0)\big]^2+\big[(Lu)^{(3)}(0)\big]^2+\int_0^1\big[(Lu)^{(4)}(x)\big]^2\,dx\\&\le\big(a_1^2+a_2^2+a_3^2+a_4^2+a_5^2\big)\|u\|^2_{W_2^5}=P\|u\|^2_{W_2^4},\end{aligned}$$

where $P=(a_1^2+a_2^2+a_3^2+a_4^2+a_5^2)>0$ is a positive constant. This completes the proof. □

5 Analysis of the solution of (2.17)-(2.18)

Let $\{x_i\}_{i=1}^{\infty}$ be any dense set in $[0,1]$ and $\Psi_x(y)=L^*r_x(y)$, where L^* is the adjoint operator of L and r_x is given by Remark 3.1. Furthermore

$$\Psi_i(x)\overset{\text{def}}{=}\Psi_{x_i}(x)=L^*r_{x_i}(x).$$

Lemma 5.1 *$\{\Psi_i(x)\}_{i=1}^{\infty}$ is a complete system of $W_2^5[0,1]$.*

Proof For $u\in W_2^5[0,1]$, let

$$\langle u,\Psi_i\rangle=0\quad(i=1,2,\ldots),$$

that is,

$$\big\langle u,L^*r_{x_i}\big\rangle=(Lu)(x_i)=0.$$

Note that $\{x_i\}_{i=1}^{\infty}$ is the dense set in $[0,1]$. Therefore $(Lu)(x)=0$. Assume that (4.1) has a unique solution. Then L is one-to-one on $W_2^5[0,1]$ and thus $u(x)=0$. This completes the proof. □

Lemma 5.2 *The following formula holds*:

$$\Psi_i(x) = \big(L_\eta R_x(\eta)\big)(x_i),$$

where the subscript η *of the operator* L_η *indicates that the operator* L *applies to a function of* η.

Proof

$$\begin{aligned}\Psi_i(x) &= \langle \Psi_i(\xi), R_x(\xi)\rangle_{W_2^5[0,1]} \\ &= \langle L^* r_{x_i}(\xi), R_x(\xi)\rangle_{W_2^5[0,1]} \\ &= \langle (r_{x_i})(\xi), \big(L_\eta R_x(\eta)\big)(\xi)\rangle_{W_2^4[0,1]} \\ &= \big(L_\eta R_x(\eta)\big)(x_i).\end{aligned}$$

This completes the proof. □

Remark 5.1 The orthonormal system $\{\overline{\Psi}_i(x)\}_{i=1}^{\infty}$ of $W_2^5[0,1]$ can be derived from the Gram-Schmidt orthogonalization process of $\{\Psi_i(x)\}_{i=1}^{\infty}$ as

$$\overline{\Psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\Psi_k(x) \quad (\beta_{ii} > 0, i = 1, 2, \ldots), \tag{5.1}$$

where β_{ik} are orthogonal coefficients.

In the following, we give the representation of the exact solution of Eq. (2.17) in the reproducing kernel space $W_2^5[0,1]$.

Theorem 5.1 *If* u *is the exact solution of* (4.1), *then*

$$u = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} M\big(x_k, u(x_k), u^{(3)}(x_k)\big)\overline{\Psi}_i(x),$$

where $\{x_i\}_{i=1}^{\infty}$ *is a dense set in* $[0,1]$.

Proof From (5.1) and the uniqueness of solution of (4.1), we have

$$\begin{aligned}u &= \sum_{i=1}^{\infty}\langle u, \overline{\Psi}_i\rangle_{W_2^5}\overline{\Psi}_i = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle u, L^* r_{x_k}\rangle_{W_2^5}\overline{\Psi}_i \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle Lu, r_{x_k}\rangle_{W_2^4}\overline{\Psi}_i = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle M\big(x, u, u^{(3)}\big), r_{x_k}\rangle_{W_2^4}\overline{\Psi}_i \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik} M\big(x_k, u(x_k), u^{(3)}(x_k)\big)\overline{\Psi}_i(x).\end{aligned}$$

This completes the proof. □

Now the approximate solution u_n can be obtained by truncating the n-term of the exact solution u as

$$u_n = \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} M\bigl(x_k, u(x_k), u^{(3)}(x_k)\bigr) \overline{\Psi}_i(x).$$

Lemma 5.3 ([30]) *Assume that u is the solution of* (4.1) *and r_n is the error between the approximate solution u_n and the exact solution u. Then the error sequence r_n is monotone decreasing in the sense of $\|\cdot\|_{W_2^5}$ and $\|r_n(x)\|_{W_2^5} \to 0$.*

6 Numerical results

In this section, comparisons of results are made through different Reynolds numbers Re and magnetic field effect m. All computations are performed by Maple 16. Figure 5.7 shows comparisons of $F(z)$ for a fixed Reynolds number with increasing magnetic field effect $m = 1, 3, 8, 20$. From this figure, the velocity decreases due to an increase in m. Figure 5.8 shows comparisons of $F(z)$ for a fixed magnetic field $m = 1$ with increasing Reynolds numbers Re $= 1, 4, 10$. It is observed that much increase in Reynolds numbers affects the results. The RKHSM does not require discretization of the variables, *i.e.*, time and space, it is not affected by computation round of errors and one is not faced with necessity of large computer memory and time. The accuracy of the RKHSM for the MHD squeezing fluid flow is controllable and absolute errors are small with present choice of x (see Tables 1-6 and Figures 2-7). The numerical results we obtained justify the advantage of this methodology. Generally it is not possible to find the exact solution of these problems.

Table 1 Numerical results at $m = 1$ and Re = 1

x	OHAM	Numerical solution (RK-4)	Approximate solution RKHSM	Absolute error	Relative error	Time (s)
0.1	0.150265	0.150294	0.15029400074386619072	7.43×10^{-10}	4.94×10^{-9}	2.948
0.2	0.297424	0.297481	0.29748099943286204844	5.67×10^{-10}	1.9×10^{-9}	2.980
0.3	0.438387	0.438467	0.43846699936146542481	6.38×10^{-10}	1.45×10^{-9}	2.870
0.4	0.570093	0.570189	0.57018899983086605298	1.69×10^{-10}	2.96×10^{-10}	2.792
0.5	0.68952	0.689624	0.68962399932753349664	6.72×10^{-10}	9.75×10^{-10}	2.824
0.6	0.793695	0.793796	0.79379600052975674440	5.29×10^{-10}	6.67×10^{-10}	2.902
0.7	0.879695	0.879779	0.87977900034152532706	3.41×10^{-10}	3.88×10^{-10}	2.964
0.8	0.944641	0.944696	0.94469600021478585921	2.14×10^{-10}	2.27×10^{-10}	2.808
0.9	0.985687	0.985707	0.98570699945336089741	5.46×10^{-10}	5.54×10^{-10}	2.761
1.0	1.0	1.0	1.0	0.0	0.0	2.902

Table 2 Numerical results at $m = 3$ and Re = 1

x	OHAM	Numerical solution (RK-4)	Approximate solution RKHSM	Absolute error	Relative error	Time (s)
0.1	0.13709	0.137044	0.13704399924397146430	7.56×10^{-10}	5.51×10^{-9}	3.261
0.2	0.272583	0.272494	0.27249400041809657591	4.18×10^{-10}	1.53×10^{-9}	3.542
0.3	0.404759	0.404637	0.40463699937791012358	6.22×10^{-10}	1.53×10^{-9}	2.949
0.4	0.531649	0.531508	0.53150799980699743080	1.93×10^{-10}	3.63×10^{-10}	3.541
0.5	0.650894	0.650756	0.65075599905912100256	9.4×10^{-10}	1.44×10^{-9}	3.089
0.6	0.759591	0.759478	0.75947799979255971384	2.07×10^{-10}	2.73×10^{-10}	2.996
0.7	0.854106	0.854035	0.85403499924057783299	7.59×10^{-10}	8.89×10^{-10}	3.026
0.8	0.929845	0.929817	0.92981700082221438640	8.22×10^{-10}	8.84×10^{-10}	7.582
0.9	0.980966	0.980963	0.98096299961587653980	3.84×10^{-10}	3.91×10^{-10}	3.291
1.0	1.0	1.0	1.0	0.0	0.0	2.902

Table 3 Numerical results at $m = 8$ and Re = 1

x	OHAM	Numerical solution (RK-4)	Approximate solution RKHSM	Absolute error	Relative error	Time (s)
0.1	0.11507	0.114976	0.11497599095960418967	9.04×10^{-9}	7.86×10^{-8}	4.290
0.2	0.230068	0.229882	0.22988199268533318687	7.31×10^{-9}	3.18×10^{-8}	4.134
0.3	0.344866	0.344604	0.34460400584434350472	5.84×10^{-9}	1.69×10^{-8}	4.477
0.4	0.459205	0.458904	0.45890399132822355411	8.67×10^{-9}	1.88×10^{-8}	4.275
0.5	0.572545	0.572276	0.5722759999680104400	3.19×10^{-11}	5.58×10^{-11}	3.931
0.6	0.683769	0.683628	0.68362799155831029523	8.44×10^{-9}	1.23×10^{-8}	4.556
0.7	0.790543	0.790607	0.79060700783664672119	7.83×10^{-9}	9.91×10^{-9}	4.461
0.8	0.887936	0.888173	0.88817300466724146312	4.66×10^{-9}	5.25×10^{-9}	3.885
0.9	0.965381	0.965578	0.96557800220185786369	2.2×10^{-9}	2.28×10^{-9}	5.007
1.0	1.0	1.0	1.0	0.0	0.0	2.902

Table 4 Numerical results at $m = 20$ and Re = 1

x	OHAM	Numerical solution (RK-4)	Approximate solution RKHSM	Absolute error	Relative error	Time (s)
0.1	0.105312	0.105391	0.10539098947593257979	1.05×10^{-8}	9.98×10^{-8}	4.134
0.2	0.210625	0.210782	0.2107819933190829	6.68×10^{-9}	3.16×10^{-8}	5.101
0.3	0.315938	0.316173	0.3161729190893567630	8.09×10^{-8}	2.55×10^{-7}	3.010
0.4	0.421249	0.421563	0.4215629919618786430	8.03×10^{-9}	1.9×10^{-8}	3.198
0.5	0.526551	0.526952	0.5269519479728988	5.2×10^{-8}	9.87×10^{-8}	3.042
0.6	0.631824	0.632324	0.632323981769674315	1.82×10^{-8}	2.88×10^{-8}	3.074
0.7	0.736971	0.737586	0.7375860570172070642	5.7×10^{-8}	7.73×10^{-8}	3.089
0.8	0.841352	0.842051	0.84205103495023398982	3.49×10^{-8}	4.15×10^{-8}	3.073
0.9	0.94035	0.940861	0.94086101815219431313	1.81×10^{-8}	1.92×10^{-8}	3.135
1.0	1.0	1.0	1.0	0.0	0.0	2.902

Table 5 Numerical results at $m = 1$ and Re = 4

x	OHAM	Numerical solution (RK-4)	Approximate solution RKHSM	Absolute error	Relative error	Time (s)
0.1	0.156218	0.158104	0.15810400012535311729	1.25×10^{-10}	7.92×10^{-10}	5.304
0.2	0.308363	0.311962	0.31196200057873017887	5.78×10^{-10}	1.85×10^{-9}	7.332
0.3	0.452557	0.457539	0.45753900003164153289	3.16×10^{-11}	6.91×10^{-11}	5.913
0.4	0.585287	0.591193	0.59119300033029000468	3.3×10^{-10}	5.58×10^{-10}	6.272
0.5	0.703518	0.709771	0.70977100026331200670	2.63×10^{-10}	3.7×10^{-10}	5.757
0.6	0.804726	0.810642	0.81064200064720692438	6.47×10^{-10}	7.98×10^{-10}	6.256
0.7	0.886838	0.891666	0.89166599939606220359	6.03×10^{-10}	6.03×10^{-10}	6.396
0.8	0.948051	0.95112	0.95112000044608660232	4.46×10^{-10}	4.69×10^{-10}	5.101
0.9	0.986529	0.987612	0.98761199979328069240	2.06×10^{-10}	2.09×10^{-10}	5.616
1.0	1.0	1.0	1.0	0.0	0.0	2.902

Table 6 Numerical results at $m = 1$ and Re = 10

x	OHAM	Numerical solution (RK-4)	Approximate solution RKHSM	Absolute error	Relative error	Time (s)
0.1	0.175911	0.167616	0.1676160001397322991	1.39×10^{-10}	8.33×10^{-10}	5.569
0.2	0.344336	0.329031	0.32903100221406728329	2.21×10^{-9}	6.72×10^{-9}	6.365
0.3	0.498671	0.478907	0.47890699791462877619	2.08×10^{-9}	4.35×10^{-9}	7.378
0.4	0.633941	0.613252	0.61325199550552162812	4.49×10^{-9}	7.32×10^{-9}	7.254
0.5	0.747277	0.729428	0.72942799845508679063	1.54×10^{-9}	2.11×10^{-9}	6.271
0.6	0.838004	0.825843	0.82584300690485584332	6.9×10^{-9}	8.36×10^{-9}	7.425
0.7	0.907244	0.901576	0.90157600840425340903	8.4×10^{-9}	9.32×10^{-9}	6.162
0.8	0.956954	0.901576	0.90157518382496567601	8.16×10^{-7}	9.05×10^{-7}	7.410
0.9	0.988387	0.988978	0.98897799997420425356	2.57×10^{-11}	2.6×10^{-11}	7.910
1.0	1.0	1.0	1.0	0.0	0.0	2.902

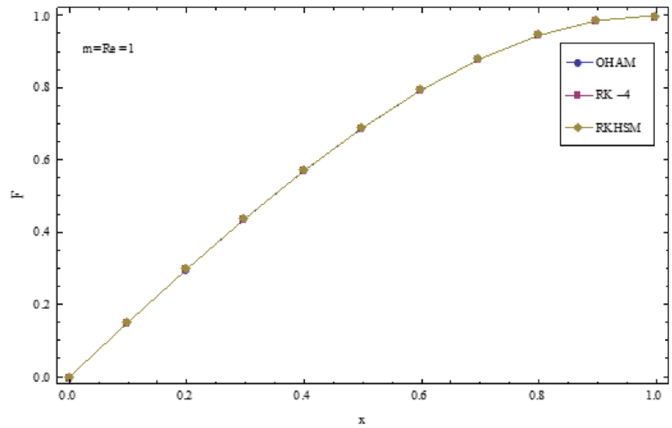

Figure 2 Comparison RKHSM, OHAM and RK-4 solutions for *m* = Re = 1.

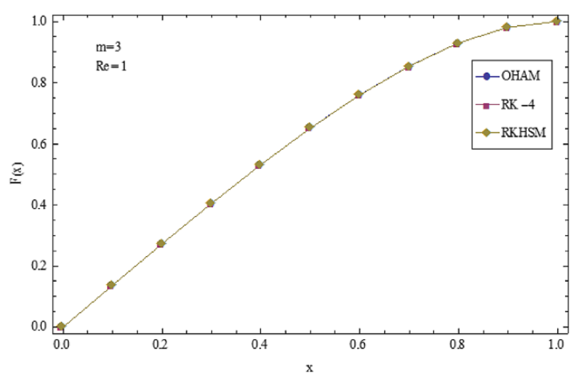

Figure 3 Comparison RKHSM, OHAM and RK-4 solutions for *m* = 3 and Re = 1.

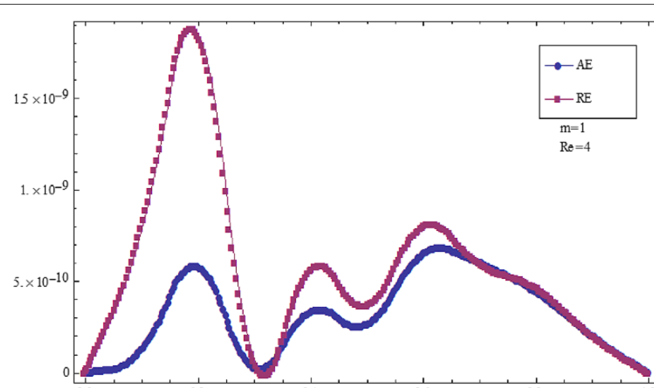

Figure 4 AE and RE for *m* = 1 and Re = 4.

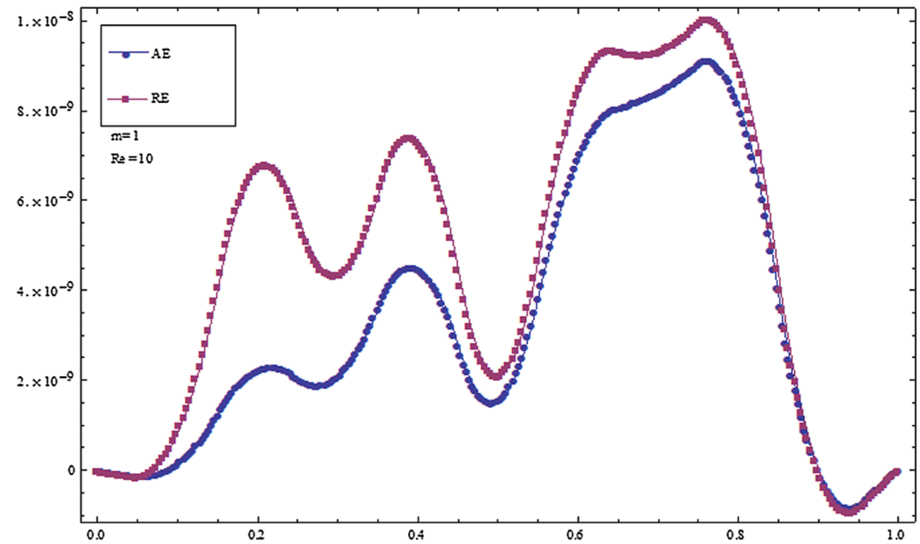

Figure 5 AE and RE for $m = 1$ and Re = 10.

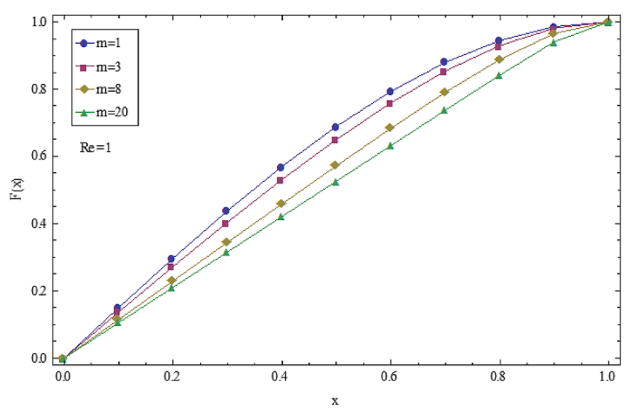

Figure 6 Comparison of squeezing flow for a fixed Reynolds number Re = 1 and increasing magnetic field effect $m = 1, 3, 8, 20$.

7 Conclusion

In this paper, we introduced an algorithm for solving the MHD squeezing fluid flow. We applied a new powerful method RKHSM to the reduced nonlinear boundary value problem. The approximate solution obtained by the present method is uniformly convergent. Clearly, the series solution methodology can be applied to much more complicated nonlinear differential equations and boundary value problems. However, if the problem becomes nonlinear, then the RKHSM does not require discretization or perturbation and it does not make closure approximation. Results of numerical examples show that the present method is an accurate and reliable analytical method for this problem.

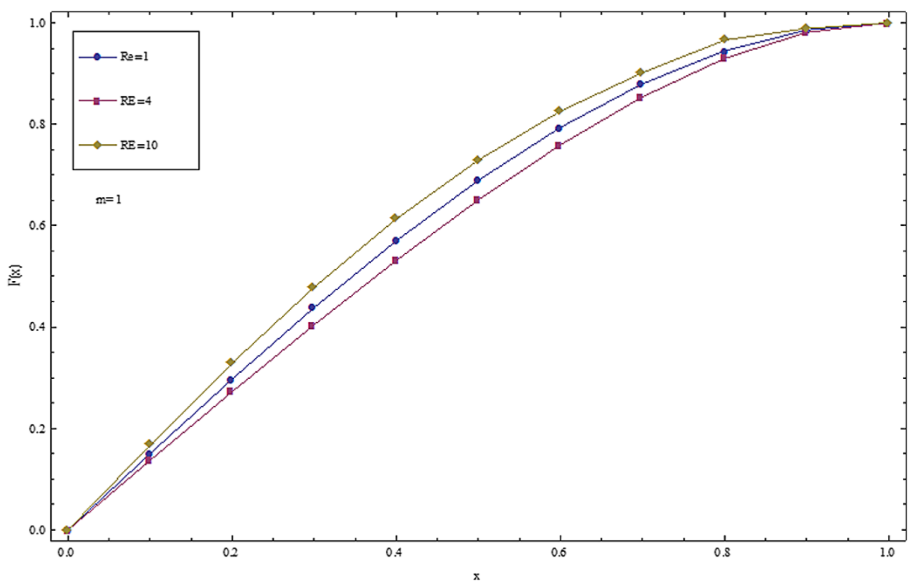

Figure 7 Comparison of squeezing flow for a fixed magnetic field effect $m = 1$ and increasing Reynolds numbers Re = 1, 4, 10.

Appendix

Proof of Theorem 3.1 Let $u \in W_2^5[0,1]$. By Definition 3.2 we have

$$\langle u, R_y \rangle_{W_2^5} = \sum_{i=0}^{4} u^{(i)}(0) R_y^{(i)}(0) + \int_0^1 u^{(5)}(x) R_y^{(5)}(x)\,dx. \tag{A.1}$$

Through several integrations by parts for (A.1), we have

$$\begin{aligned}\langle u, R_y \rangle_{W_2^5} &= \sum_{i=0}^{4} u^{(i)}(0)\big[R_y^{(i)}(0) - (-1)^{(4-i)} R_y^{(9-i)}(0)\big] \\ &\quad + \sum_{i=0}^{4} (-1)^{(4-i)} u^{(i)}(1) R_y^{(9-i)}(1) - \int_0^1 u(x) R_y^{(10)}(x)\,dx.\end{aligned} \tag{A.2}$$

Note the property of the reproducing kernel

$$\langle u, R_y \rangle_{W_2^5} = u(y).$$

Now, if

$$\begin{cases} R_y'(0) + R_y^{(8)}(0) = 0, \\ R_y^{(3)}(0) + R_y^{(6)}(0) = 0, \\ R_y^{(4)}(0) - R_y^{(5)}(0) = 0, \\ R_y^{(5)}(1) = 0, \\ R_y^{(6)}(1) = 0, \\ R_y^{(7)}(1) = 0, \end{cases} \tag{A.3}$$

then (A.2) implies that

$$R_y^{(10)}(x) = -\delta(x-y),$$

when $x \neq y$

$$R_y^{(10)}(x) = 0,$$

and therefore

$$R_y(x) = \begin{cases} \sum_{i=1}^{10} c_i(y)x^{i-1}, & x \leq y, \\ \sum_{i=1}^{10} d_i(y)x^{i-1}, & x > y. \end{cases}$$

Since

$$R_y^{(10)}(x) = \delta(x-y),$$

we have

$$R_{y^+}^{(k)}(y) = R_{y^-}^{(k)}(y), \quad k = 0,1,2,3,4,5,6,7,8, \tag{A.4}$$

and

$$R_{y^+}^{(9)}(y) - R_{y^-}^{(9)}(y) = -1. \tag{A.5}$$

Since $R_y(x) \in W_2^5[0,1]$, it follows that

$$R_y(0) = 0, \qquad R_y(1) = 0, \qquad R_y'(1) = 0, \qquad R_y''(0) = 0. \tag{A.6}$$

From (A.3)-(A.6), the unknown coefficients $c_i(y)$ and $d_i(y)$ $(i = 1,2,\ldots,12)$ can be obtained. This completes the proof. □

Competing interests
The authors declare that they do not have any competing or conflict of interests.

Authors' contributions
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