Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

On the numerical solution of hyperbolic IBVP with high-order stable finite difference schemes

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Abstract

The abstract Cauchy problem for the hyperbolic equation in a Hilbert space *H* with self-adjoint positive definite operator *A* is considered. The third and fourth orders of accuracy difference schemes for the approximate solution of this problem are presented. The stability estimates for the solutions of these difference schemes are established. A finite difference method and some results of numerical experiments are presented in order to support theoretical statements.

MSC: 65J10; 65M12; 65N12; 35L30

Keywords: abstract hyperbolic equation; stability; initial boundary value problem

1 Introduction

Partial differential equations of the hyperbolic type play an important role in many branches of science and engineering. For example, acoustics, electromagnetics, hydrodynamics, elasticity, fluid mechanics, and other areas of physics lead to partial differential equations of the hyperbolic type (see, *e.g.*, [1–5] and the references given therein). The stability has been an important research topic in the development of numerical techniques for solving these equations (see [6–27]). Particularly, a convenient model for analyzing the stability is provided by a proper unconditionally absolutely stable difference scheme with an unbounded operator.

A large cycle of works on difference schemes for hyperbolic partial differential equations, in which stability was established under the assumption that the magnitude of the grid steps τ and h with respect to time and space variables are connected (see, *e.g.*, [5–7] and the references therein). Of great interest is the study of absolute stable difference schemes of a high order of accuracy for hyperbolic partial differential equations, in which stability was established without any assumptions in respect of the grid steps τ and h. Such type stability inequalities for the solutions of the first order of accuracy difference scheme for the differential equations of hyperbolic type were established for the first time in [20].

It is known (see [25, 26]) that various initial boundary value problems for a hyperbolic equation can be reduced to the initial value problem

$$\begin{cases} \frac{d^2u(t)}{dt^2} + Au(t) = f(t), \quad 0 < t < T, \\ u(0) = \varphi, \qquad u'(0) = \psi, \end{cases} \tag{1}$$

where A is a self-adjoint positive definite linear operator with the domain $D(A)$ in a Hilbert space H.

A function $u(t)$ is called a solution of problem (1) if the following conditions are satisfied:

(i) $u(t)$ is twice continuously differentiable on the segment $[0,1]$. The derivatives at the endpoints of the segment are understood as the appropriate unilateral derivatives.

(ii) The element $u(t)$ belongs to $D(A)$ for all $t \in [0,1]$ and the function $Au(t)$ is continuous on the segment $[0,1]$.

(iii) $u(t)$ satisfies the equations and the initial conditions (1).

In recent decades, many scientists have worked in the field of a finite difference method for the numerical solutions of hyperbolic PDEs and have published many scientific papers. For problem (1), the first and two types of second order difference schemes were presented and the stability estimates for the solution of these difference schemes and for the first- and second-order difference derivatives were obtained in [11]. The high order of accuracy two-step difference schemes generated by an exact difference scheme or by the Taylor decomposition on the three points for the numerical solution of the same problem were presented and the stability estimates for approximate solution of these difference schemes were obtained in [10]. However, the difference methods developed in these references are generated by square roots of A. This action is very difficult for the realization. Therefore, in spite of theoretical results, the role of their application to a numerical solution for an initial value problem is not great. In this paper, the third order of accuracy difference scheme

$$\begin{cases} \tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + \frac{2}{3}Au_k + \frac{1}{6}A(u_{k+1} + u_{k-1}) \\ \quad + \frac{1}{12}\tau^2 A^2 u_{k+1} = f_k, \\ f_k = \frac{2}{3}f(t_k) + \frac{1}{6}(f(t_{k+1}) + f(t_{k-1})) \\ \quad - \frac{1}{12}\tau^2(-Af(t_{k+1}) + f''(t_{k+1})), \quad 1 \le k \le N-1, \\ u_0 = \varphi, \qquad (I + \frac{\tau^2}{12}A + \frac{\tau^4}{144}A^2)\tau^{-1}(u_1 - u_0) = -\frac{\tau}{2}A\varphi + (I - \frac{\tau^2}{12}A)\psi + f_{1,1}\tau, \end{cases} \tag{2}$$

and the fourth order of accuracy difference scheme

$$\begin{cases} \tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + \frac{5}{6}Au_k + \frac{1}{12}A(u_{k+1} + u_{k-1}) \\ \quad - \frac{\tau^2}{72}A^2 u_k + \frac{\tau^2}{144}A^2(u_{k+1} + u_{k-1}) = f_k, \\ f_k = \frac{5}{6}f(t_k) + \frac{1}{12}(f(t_{k+1}) + f(t_{k-1})) + \frac{\tau^2}{72}(-Af(t_k) + f''(t_k)) \\ \quad - \frac{1}{144}\tau^2(-A(f(t_{k+1}) + f(t_{k-1})) + f''(t_{k+1}) + f''(t_{k-1})), \\ \quad 1 \le k \le N-1, t_k = k\tau, N\tau = 1, \\ u_0 = \varphi, \qquad (I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144})\tau^{-1}(u_1 - u_0) = -\frac{\tau}{2}A\varphi + (I - \frac{\tau^2 A}{12})\psi + f_{2,2}\tau \end{cases} \tag{3}$$

for the approximate solution of initial value problem (1) are constructed using the integer powers of the operator A, and stability estimates for the solution of these difference schemes are obtained. Here,

$$\begin{aligned} f_{1,1} &= \left(I + \frac{\tau^2}{12}A + \frac{\tau^4}{144}A^2\right)\tau^{-1}f_{1,1}^3 \\ &= \left\{f(0) + \left(-f(0) + \tau f'(0)\right)\frac{1}{2} - 2f'(0)\frac{\tau}{6}\right\} \end{aligned} \tag{4}$$

and

$$
\begin{aligned}
f_{2,2} &= \left(I + \frac{\tau^2}{12}A + \frac{\tau^4}{144}A^2\right)\tau^{-1}f_{1,1}^4 \\
&= \left\{\left(I - \frac{\tau^2 A}{12}\right)f(0) + \left(-\left(I - \frac{5\tau^2 A}{12}\right)f(0) + \tau f'(0)\right)\frac{1}{2}\right. \\
&\quad \left. + \left(-A\tau f(0) - 2f'(0) + \tau f''(0)\right)\frac{\tau}{6} + \left(Af(0) - 3f''(0)\right)\frac{\tau^2}{24}\right\}. \qquad (5)
\end{aligned}
$$

Some results of this paper without proof are accepted and will be published in 2013 (see [14]).

Note that boundary value problems for parabolic equations, elliptic equations, and equations of mixed type have been studied extensively by many scientists (see, *e.g.*, [17–27] and the references therein).

2 The stability estimates

In this section, construction of difference schemes (2), (3) and stability estimates for the solutions of these difference schemes are presented.

Let us obtain the third and fourth orders of approximation formulas for the solution of problem (1). If the function $f(t)$ is not only continuous, but also continuously differentiable on $[0,T]$, $\varphi \in D(A)$ and $\psi \in D(A^{\frac{1}{2}})$, it is easy to show that (see [25]) the formula

$$
u(t) = c(t)\varphi + s(t)\psi + \int_0^t s(t-\lambda)f(\lambda)\,d\lambda \qquad (6)
$$

gives a solution of problem (1). Throughout this paper, $\{c(t), t \geq 0\}$ is a strongly continuous cosine operator-function defined by the formula

$$
c(t) = \frac{e^{itA^{1/2}} + e^{-itA^{1/2}}}{2}. \qquad (7)
$$

Then, from the definition of the sine operator-function

$$
s(t)u = \int_0^t c(s)u\,ds, \qquad (8)
$$

it follows that

$$
s(t) = A^{-1/2}\frac{e^{itA^{1/2}} - e^{-itA^{1/2}}}{2i}. \qquad (9)
$$

For the theory of cosine operator-function, we refer to [25] and [27].

A uniform grid is considered on the segment $[0,T]$

$$
[0,T]_\tau = \{t_k = k\tau, k = 0,1,\ldots,N, N\tau = T\}. \qquad (10)
$$

In the construction of two-step difference schemes for the solution of initial value problem (1), it is necessary to approximate differential equation (1) and the derivative $u'(0)$.

In the first step, the approximation of differential equation (1) is considered. Using Taylor's decomposition on three points, the following formulas for the third order of approximation and the fourth order of approximation of (1) are obtained respectively:

$$\begin{aligned} & u(t_{k+1}) - 2u(t_k) + u(t_{k-1}) - \frac{2}{3}\tau^2 u''(t_k) \\ & \quad - \frac{1}{6}\tau^2\big(u''(t_{k+1}) + u''(t_{k-1})\big) \\ & \quad + \frac{1}{12}\tau^4 u^{(4)}(t_{k+1}) = o\big(\tau^5\big), \end{aligned} \tag{11}$$

$$\begin{aligned} & u(t_{k+1}) - 2u(t_k) + u(t_{k-1}) - \frac{5}{6}\tau^2 u''(t_k) \\ & \quad - \frac{1}{12}\tau^2\big(u''(t_{k+1}) + u''(t_{k-1})\big) \\ & \quad - \frac{1}{72}\tau^4 u^{(4)}(t_k) + \frac{1}{144}\tau^4\big(u^{(4)}(t_{k+1}) + u^{(4)}(t_{k-1})\big) = o\big(\tau^6\big). \end{aligned} \tag{12}$$

Applying equation (1), one can write

$$u''(t_k) = -Au(t) + f(t), \qquad u^{(4)}(t) = -Au''(t) + f''(t) = A^2u(t) - Af(t) + f''(t). \tag{13}$$

Using (11) and (13), the following formula:

$$\begin{aligned} & \frac{u(t_{k+1}) - 2u(t_k) + u(t_{k-1})}{\tau^2} - \frac{2}{3}\big(-Au(t_k) + f(t_k)\big) \\ & \quad - \frac{1}{6}\big(-A\big(u(t_{k+1}) + u(t_{k-1})\big) + f(t_{k+1}) + f(t_{k-1})\big) \\ & \quad + \frac{1}{12}\tau^2\big(A^2u(t_{k+1}) - Af(t_{k+1}) + f''(t_{k+1})\big) = o\big(\tau^3\big) \end{aligned} \tag{14}$$

for the third order of approximation of (1), and using (12), (13), the following formula

$$\begin{aligned} & \frac{u(t_{k+1}) - 2u(t_k) + u(t_{k-1})}{\tau^2} - \frac{5}{6}\big(-Au(t_k) + f(t_k)\big) \\ & \quad - \frac{1}{12}\big(-A\big(u(t_{k+1}) + u(t_{k-1})\big) + f(t_{k+1}) + f(t_{k-1})\big) \\ & \quad - \frac{1}{72}\tau^2\big(A^2u(t_k) - Af(t_k) + f''(t_k)\big) + \frac{1}{144}\tau^2\big(A^2\big(u(t_{k+1}) + u(t_{k-1})\big) \\ & \quad - A\big(f(t_{k+1}) + f(t_{k-1})\big) + f''(t_{k+1}) + f''(t_{k-1})\big) = o\big(\tau^4\big) \end{aligned} \tag{15}$$

for the fourth order of approximation of (1) are obtained.

Neglecting the last small term, we get

$$\begin{aligned} & \tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + \frac{2}{3}Au_k \\ & \quad + \frac{1}{6}A(u_{k+1} + u_{k-1}) + \frac{1}{12}\tau^2A^2u_{k+1} = f_k, \\ & f_k = \frac{2}{3}f(t_k) + \frac{1}{6}\big(f(t_{k+1}) + f(t_{k-1})\big) \\ & \quad - \frac{1}{12}\tau^2\big(-Af(t_{k+1}) + f''(t_{k+1})\big) \end{aligned} \tag{16}$$

and

$$\begin{aligned} &\tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + \frac{5}{6}Au_k + \frac{1}{12}A(u_{k+1} + u_{k-1}) \\ &\quad - \frac{1}{72}\tau^2 A^2 u_k + \frac{1}{144}\tau^2 A^2 (u_{k+1} + u_{k-1}) = f_k, \\ &f_k = \frac{5}{6}f(t_k) + \frac{1}{12}\bigl(f(t_{k+1}) + f(t_{k-1})\bigr) \\ &\quad + \frac{1}{72}\tau^2\bigl(-Af(t_k) + f''(t_k)\bigr) - \frac{1}{144}\tau^2 \\ &\quad \times \bigl(-A\bigl(f(t_{k+1}) + f(t_{k-1})\bigr) + f''(t_{k+1}) + f''(t_{k-1})\bigr) \end{aligned} \tag{17}$$

for the third and fourth orders of approximations of (1), respectively.

In the second step, the approximation of $u'(0)$ is considered. Applying (6), we can write

$$\frac{u(\tau) - u(0)}{\tau} = \frac{c(\tau) - I}{\tau}\varphi + \frac{s(\tau)}{\tau}\psi + \frac{1}{\tau}\int_0^\tau s(\tau - \lambda)f(\lambda)\,d\lambda. \tag{18}$$

From (18) it is obvious that for the approximation of $u'(0)$, it is necessary to approximate the expressions

$$s(\tau), \qquad c(\tau), \quad \text{and} \quad \frac{1}{\tau}\int_0^\tau s(\tau - \lambda)f(\lambda)\,d\lambda. \tag{19}$$

Using the definitions of $s(\tau)$, $c(\tau)$, and Padé fractions for the function e^{-z} (see [9]), the following approximation formulas are obtained:

$$c(\tau) = \frac{R(i\tau B) + R(-i\tau B)}{2} + o\bigl(\tau^3\bigr), \qquad s(\tau) = B^{-1}\frac{R(i\tau B) - R(-i\tau B)}{2i} + o\bigl(\tau^3\bigr) \tag{20}$$

for the third order of approximation of (1), where

$$R(i\tau B) = R = DE, \qquad R(-i\tau B) = \widetilde{R} = \widetilde{D}E, \tag{21}$$

and

$$\begin{aligned} c(\tau) &= \frac{R(i\tau B) + R(-i\tau B)}{2} + o\bigl(\tau^4\bigr), \\ s(\tau) &= B^{-1}\frac{R(i\tau B) - R(-i\tau B)}{2i} + o\bigl(\tau^4\bigr) \end{aligned} \tag{22}$$

for the fourth order of approximation of (1), where

$$R(i\tau B) = R = C\widetilde{C}^{-1}, \qquad R(-i\tau B) = \widetilde{R} = \widetilde{C}C^{-1}. \tag{23}$$

Here $B = A^{1/2}$ and

$$\begin{aligned} &D = \left(I - \frac{1}{3}\tau^2 A + i\tau A^{1/2}\sqrt{I + \frac{1}{72}\tau^4 A^2}\right), \qquad \widetilde{D} = \left(I - \frac{1}{3}\tau^2 A - i\tau A^{1/2}\sqrt{I + \frac{1}{72}\tau^4 A^2}\right), \\ &E = \left(I + \frac{1}{6}\tau^2 A + \frac{1}{12}\tau^4 A^2\right)^{-1}, \end{aligned}$$

$$C = \left(I + \frac{i\tau A^{1/2}}{2} - \frac{\tau^2 A}{12}\right), \qquad \widetilde{C} = \left(I - \frac{i\tau A^{1/2}}{2} - \frac{\tau^2 A}{12}\right).$$

Using (22), (20), (21), (23), we obtain the third order of approximations of $s(\tau)$ and $c(\tau)$

$$c_\tau(\tau) = \left(I - \frac{1}{3}\tau^2 A\right)\left(I + \frac{1}{6}\tau^2 A + \frac{1}{12}\tau^4 A^2\right)^{-1}, \tag{24}$$

$$s_\tau(\tau) = \tau\sqrt{I + \frac{1}{72}\tau^4 A^2}\left(I + \frac{1}{6}\tau^2 A + \frac{1}{12}\tau^4 A^2\right)^{-1} \tag{25}$$

and the fourth order of approximations of $s(\tau)$ and $c(\tau)$

$$c_\tau(\tau) = \left(I - \frac{5}{12}\tau^2 A + \frac{1}{144}\tau^4 A^2\right)\left(I + \frac{1}{12}\tau^2 A + \frac{1}{144}\tau^4 A^2\right)^{-1}, \tag{26}$$

$$s_\tau(\tau) = \tau\left(I - \frac{1}{12}\tau^2 A\right)\left(I + \frac{1}{12}\tau^2 A + \frac{1}{144}\tau^4 A^2\right)^{-1}. \tag{27}$$

Let us remark that in constructing the approximation of $u'(0)$, it is important to know how to construct $f_{1,1}^3$ and $f_{1,1}^4$ such that

$$\frac{1}{\tau}\int_0^\tau s(\tau-\lambda)f(\lambda)\,d\lambda - f_{1,1}^3 = o(\tau^3), \tag{28}$$

$$\frac{1}{\tau}\int_0^\tau s(\tau-\lambda)f(\lambda)\,d\lambda - f_{1,1}^4 = o(\tau^4), \tag{29}$$

and formulas of $f_{1,1}^3$ and $f_{1,1}^4$ are sufficiently simple. The choice of $f_{1,1}^3$ and $f_{1,1}^4$ is not unique. Using Taylor's formula and integration, we obtain the following formulas for the third order of approximation:

$$\begin{aligned} f_{1,1}^3 &= \left\{S_\tau(\tau)f(0) + \left(-C_\tau(\tau)f(0) + S_\tau(\tau)f'(0)\right)\frac{\tau}{2} - 2C_\tau(\tau)f'(0)\frac{\tau^2}{6}\right\} \\ &= \left(I + \frac{\tau^2}{12}A + \frac{\tau^4}{144}A^2\right)^{-1}\left\{\tau f(0) + \left(-f(0) + \tau f'(0)\right)\frac{\tau}{2} - 2f'(0)\frac{\tau^2}{6}\right\} \end{aligned} \tag{30}$$

and for the fourth order of approximation

$$\begin{aligned} f_{1,1}^4 &= \left\{S_\tau(\tau)f(0) + \left(-C_\tau(\tau)f(0) + S_\tau(\tau)f'(0)\right)\frac{\tau}{2}\right. \\ &\quad + \left(-AS_\tau(\tau)f(0) - 2C_\tau(\tau)f'(0) + S_\tau(\tau)f''(0)\right)\frac{\tau^2}{6} \\ &\quad \left. + \left(AC_\tau(\tau)f(0) - 3C_\tau(\tau)f''(0)\right)\frac{\tau^3}{24}\right\} \\ &= \left(I + \frac{\tau^2}{12}A + \frac{\tau^4}{144}A^2\right)^{-1}\left\{\tau\left(I - \frac{\tau^2 A}{12}\right)f(0)\right. \\ &\quad + \left(-\left(I - \frac{5\tau^2 A}{12}\right)f(0) + \tau f'(0)\right)\frac{\tau}{2} \\ &\quad \left. + \left(-A\tau f(0) - 2f'(0) + \tau f''(0)\right)\frac{\tau^2}{6} + \left(Af(0) - 3f''(0)\right)\frac{\tau^3}{24}\right\}. \end{aligned} \tag{31}$$

For simplicity, denote

$$f_{1,1} = f(0) + \left(-f(0) + \tau f'(0)\right)\frac{1}{2} - 2f'(0)\frac{\tau}{6}, \tag{32}$$

and

$$f_{2,2} = \left\{\left(I - \frac{\tau^2 A}{12}\right)f(0) + \left(-\left(I - \frac{5\tau^2 A}{12}\right)f(0) + \tau f'(0)\right)\frac{1}{2} + \left(-A\tau f(0) - 2f'(0) + \tau f''(0)\right)\frac{\tau}{6} + \left(Af(0) - 3f''(0)\right)\frac{\tau^2}{24}\right\}, \tag{33}$$

where

$$f_{1,1}^3 = \tau\left(I + \frac{\tau^2}{12}A + \frac{\tau^4}{144}A^2\right)^{-1} f_{1,1} \tag{34}$$

and

$$f_{1,1}^4 = \tau\left(I + \frac{\tau^2}{12}A + \frac{\tau^4}{144}A^2\right)^{-1} f_{2,2}. \tag{35}$$

Thus, we have the following formula for the approximation of $u'(0)$:

$$\frac{u_1 - u_0}{\tau} = \frac{c_\tau(\tau) - I}{\tau}\varphi + \frac{s_\tau(\tau)}{\tau}\psi + f_{1,1}^m, \quad m = 3, 4. \tag{36}$$

Using the approximation formulas above, difference schemes (2) and (3) are constructed.

Now, we will obtain the stability estimates for the solution of difference scheme (2). We consider operators R, $\widetilde{R}$ by (21) and the following operators:

$$R_1 = \left(-\frac{5\tau^4}{144}A^2 + \frac{7\tau^6}{216}A^3 - i\tau A^{1/2}\left(I + \frac{\tau^2}{12}A + \frac{\tau^4}{144}A^2\right)\sqrt{I + \frac{1}{72}\tau^4 A^2}\right) \times \left(-i\tau A^{1/2}\left(\sqrt{I + \frac{1}{72}\tau^4 A^2}\right)\left(I + \frac{\tau^2}{12}A + \frac{\tau^4}{144}A^2\right)\right)^{-1}, \tag{37}$$

and its conjugate $\widetilde{R}_1$

$$R_2 = \left(I - \frac{\tau^2}{12}A\right)\left(I + \frac{\tau^2}{6}A + \frac{\tau^4}{12}A^2\right) \times \left(-iA^{1/2}\left(I + \frac{\tau^2}{12}A + \frac{\tau^4}{144}A^2\right)\sqrt{I + \frac{1}{72}\tau^4 A^2}\right)^{-1}, \tag{38}$$

$$R_3 = \left(I + \frac{\tau^2}{6}A + \frac{\tau^4}{12}A^2\right)\left(\left(I + \frac{\tau^2}{12}A + \frac{\tau^4}{144}A^2\right)\left(-i\tau A^{1/2}\sqrt{I + \frac{1}{72}\tau^4 A^2}\right)\right)^{-1}, \tag{39}$$

$$R_4 = \left(I + \frac{\tau^2}{3}A + \frac{\tau^4}{9}A^2 + \frac{\tau^6}{72}A^3\right) \times \left(-iA^{1/2}\left(\sqrt{I + \frac{1}{72}\tau^4 A^2}\right)\left(I + \frac{\tau^2}{6}A + \frac{\tau^4}{12}A^2\right)\left(I + \frac{\tau^2}{6}A\right)\right)^{-1}, \tag{40}$$

$$R_5 = \left(-\frac{\tau^2}{2}A - \frac{\tau^4}{12}A^2 + i\tau A^{1/2}\sqrt{I + \frac{1}{72}\tau^4 A^2}\right)\left(I + \frac{\tau^2}{6}A + \frac{\tau^4}{12}A^2\right)^{-1}, \tag{41}$$

and its conjugate $\widetilde{R}_5$,

$$\begin{aligned} R_6 = {} & \left(I - \frac{1}{3}\tau^2 A + i\tau A^{1/2}\sqrt{I + \frac{1}{72}\tau^4 A^2}\right) \\ & \times \left(\frac{\tau^2}{2}A + \frac{\tau^4}{12}A^2 - i\tau A^{1/2}\sqrt{I + \frac{1}{72}\tau^4 A^2}\right)^{-1}, \end{aligned} \tag{42}$$

and its conjugate $\widetilde{R}_6$.

Let us give one lemma, without proof, that will be needed below.

Lemma 1 *The following estimates hold*:

$$\begin{cases} \|R\|_{H\to H} \le 1, \qquad \|\widetilde{R}\|_{H\to H} \le 1, \\ \|R_1\|_{H\to H} \le 1, \qquad \|\widetilde{R}_1\|_{H\to H} \le 1, \\ \|A^{1/2}R_2\|_{H\to H} \le 1, \qquad \|\tau A^{1/2}R_3\|_{H\to H} \le 1, \\ \|A^{1/2}R_4\|_{H\to H} \le 1, \qquad \|A^{-1/2}R_5\|_{H\to H} \le \tau, \\ \|A^{-1/2}\widetilde{R}_5\|_{H\to H} \le \tau, \qquad \|\tau A^{1/2}R_6\|_{H\to H} \le 1, \\ \|\tau A^{1/2}\widetilde{R}_6\|_{H\to H} \le 1. \end{cases} \tag{43}$$

Now, we will give the first main theorem of the present paper on the stability of difference scheme (2).

Theorem 1 *Let $\varphi \in D(A)$, $\psi \in D(A^{\frac{1}{2}})$, $f_{1,1} \in D(A^{\frac{1}{2}})$. Then, for the solution of difference scheme* (2), *the following stability estimates hold*:

$$\max_{1\le k\le N} \|u_k\|_H \le M\left\{\sum_{s=1}^{N-1} \left\|A^{-1/2}f_s\right\|_H \tau + \|\varphi\|_H + \left\|A^{-1/2}\psi\right\|_H + \tau\left\|A^{-1/2}f_{1,1}\right\|_H\right\}, \tag{44}$$

$$\begin{aligned} & \max_{1\le k\le N} \left\|A^{1/2}u_k\right\|_H + \max_{1\le k\le N}\left\|\frac{u_k - u_{k-1}}{\tau}\right\|_H \\ & \quad \le M\left\{\sum_{s=1}^{N-1} \|f_s\|_H \tau + \left\|A^{1/2}\varphi\right\|_H + \|\psi\|_H + \tau\|f_{1,1}\|_H\right\}, \end{aligned} \tag{45}$$

$$\begin{aligned} & \max_{1\le k\le N} \|Au_k\|_H + \max_{1\le k\le N}\left\|A^{1/2}\frac{u_k - u_{k-1}}{\tau}\right\|_H + \max_{1\le k\le N-1}\left\|\frac{u_{k+1} - 2u_k + u_{k-1}}{\tau^2}\right\|_H \\ & \quad \le M\left\{\sum_{s=2}^{N-1} \|f_s - f_{s-1}\|_H + \|f_1\|_H + \|A\varphi\|_H + \left\|A^{1/2}\psi\right\|_H + \tau\left\|A^{1/2}f_{1,1}\right\|_H\right\}, \end{aligned} \tag{46}$$

where M does not depend on τ, φ, ψ, $f_{1,1}$, and f_s, $1 \le s \le N-1$.

Proof First, we will obtain the formula for the solution of problem (1). It is clear that there exists a unique solution of the initial value problem

$$au_{k-1} - cu_k + bu_{k+1} = \varphi_k, \quad 1 \le k \le N-1, \qquad u_0 = \varphi, \qquad u_1 = \psi \tag{47}$$

and for the solution of (47), the following formula is satisfied (see [11]):

$$\begin{aligned}
&u_0 = \varphi, \qquad u_1 = \psi, \\
&u_k = R\widetilde{R}(\widetilde{R} - R)^{-1}\left[R^{k-1} - \widetilde{R}^{k-1}\right]\varphi + (\widetilde{R} - R)^{-1}\left(\widetilde{R}^k - R^k\right)\psi \\
&\qquad + \sum_{j=1}^{k-1} R\widetilde{R}\left((\widetilde{R} - R)a\right)^{-1}\left[\widetilde{R}^{k-j} - R^{k-j}\right]\varphi_j,
\end{aligned} \tag{48}$$

where $R = q_1 = \frac{c+\sqrt{c^2-4ab}}{2b}$, $\widetilde{R} = q_2 = \frac{c-\sqrt{c^2-4ab}}{2b}$. Here q_1 and q_2 are roots of equation (47), and $a \neq 0$, $b \neq 0$. We can rewrite (47) into the following difference problem:

$$\begin{aligned}
&\left(I + \frac{\tau^2}{6}A\right)u_{k-1} - \left(2 - \frac{2\tau^2}{3}A\right)u_k + \left(I + \frac{\tau^2}{6}A + \frac{\tau^4}{12}A^2\right)u_{k+1} = \tau^2 f_k, \quad 1 \le k \le N-1, \\
&u_0 = \varphi, \qquad u_1 = \varphi + \tau\left(I + \frac{\tau^2}{12}A + \frac{\tau^4}{144}A^2\right)^{-1}\left(-\frac{\tau}{2}A\varphi + \left(I - \frac{\tau^2}{12}A\right)\psi + \tau f_{1,1}\right).
\end{aligned}$$

Replacing a with $(I + \frac{\tau^2}{6}A)$, b with $(I + \frac{\tau^2}{6}A + \frac{\tau^4}{12}A^2)$, c with $(2 - \frac{2\tau^2}{3}A)$, and ψ with

$$\varphi + \tau\left(I + \frac{\tau^2}{12}A + \frac{\tau^4}{144}A^2\right)^{-1}\left(-\frac{\tau}{2}A\varphi + \left(I - \frac{\tau^2}{12}A\right)\psi + \tau f_{1,1}\right),$$

φ_k with $\tau^2 f_k$ and applying formula (48), we obtain the following formulas for u_k, $2 \le k \le N$:

$$\begin{aligned}
u_k &= \frac{1}{2}\left[\widetilde{R}_1 R^k - R_1\widetilde{R}^k\right]\varphi + \frac{1}{2}\left[\widetilde{R}^k - R^k\right]R_2\psi \\
&\quad + \frac{1}{2}\left[\widetilde{R}^k - R^k\right]R_3\tau^2 f_{1,1} + \frac{1}{2}R_4\sum_{s=1}^{k-1}\left[\widetilde{R}^{k-s} - R^{k-s}\right]f_s\tau^2.
\end{aligned} \tag{49}$$

Second, we will prove estimates (44), (45), and (46). Using (43) and the formula for u_1, and the following simple estimates:

$$\left\|\left(I - \frac{5}{12}\tau^2 A + \frac{\tau^4}{144}A^2\right)\left(I + \frac{\tau^2}{12}A + \frac{\tau^4}{144}A^2\right)^{-1}\right\|_{H\to H} \le 1, \tag{50}$$

$$\left\|\tau A^{1/2}\left(I - \frac{\tau^2}{12}A\right)\left(I + \frac{\tau^2}{12}A + \frac{\tau^4}{144}A^2\right)^{-1}\right\|_{H\to H} \le 12, \tag{51}$$

$$\left\|\tau A^{1/2}\left(I + \frac{1}{12}\tau^2 A + \frac{1}{144}\tau^4 A^2\right)^{-1}\right\|_{H\to H} \le \frac{12\sqrt{11}}{12 + \sqrt{11}}, \tag{52}$$

we get

$$\begin{aligned}
\|u_1\|_H &\le \left\|\left(I - \frac{5}{12}\tau^2 A + \frac{\tau^4}{144}A^2\right)\left(I + \frac{\tau^2}{12}A + \frac{\tau^4}{144}A^2\right)^{-1}\right\|_{H\to H}\|\varphi\|_H \\
&\quad + \left\|\tau A^{1/2}\left(I - \frac{\tau^2}{12}A\right)\left(I + \frac{\tau^2}{12}A + \frac{\tau^4}{144}A^2\right)^{-1}\right\|_{H\to H}\left\|A^{-1/2}\psi\right\|_H \\
&\quad + \left\|\tau A^{1/2}\left(I + \frac{\tau^2}{12}A + \frac{\tau^4}{144}A^2\right)^{-1}\right\|_{H\to H}\tau\left\|A^{-1/2}f_{1,1}\right\|_H
\end{aligned}$$

$$\leq \|\varphi\|_H + 12\left\|A^{-1/2}\psi\right\|_H + \frac{12\sqrt{11}}{12+\sqrt{11}}\tau\left\|A^{-1/2}f_{1,1}\right\|_H$$
$$\leq M\left\{\sum_{s=1}^{N-1}\left\|A^{-1/2}f_s\right\|_H\tau + \|\varphi\|_H + \left\|A^{-1/2}\psi\right\|_H + \tau\left\|A^{-1/2}f_{1,1}\right\|_H\right\}.$$

Applying $A^{1/2}$ to the formula for u_1 and using estimates (43), (50), (51), and (52), we get

$$\begin{aligned}\left\|A^{1/2}u_1\right\|_H &\leq \left\|\left(I-\frac{5\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\left\|A^{1/2}\varphi\right\|_H \\ &\quad + \left\|\tau A^{1/2}\left(I-\frac{\tau^2}{12}A\right)\left(I+\frac{\tau^2}{12}A+\frac{\tau^4}{144}A^2\right)^{-1}\right\|_{H\to H}\|\psi\|_H \\ &\quad + \left\|\tau A^{1/2}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\tau\|f_{1,1}\|_H \\ &\leq \left\|A^{1/2}\varphi\right\|_H + 12\|\psi\|_H + \frac{12\sqrt{11}}{12+\sqrt{11}}\tau\|f_{1,1}\|_H \\ &\leq M\left\{\sum_{s=1}^{N-1}\|f_s\|_H\tau + \left\|A^{1/2}\varphi\right\|_H + \|\psi\|_H + \tau\|f_{1,1}\|_H\right\}.\end{aligned}$$

Using the formula

$$\begin{aligned}\frac{u_1-u_0}{\tau} &= \left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1} \\ &\quad \times\left(-\frac{\tau}{2}A\varphi + \left(I-\frac{\tau^2}{12}A\right)\psi + \tau f_{1,1}\right)\end{aligned} \tag{53}$$

and estimates (43), (52), and the following simple estimates:

$$\left\|\left(I-\frac{\tau^2A}{12}\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H} \leq 1, \tag{54}$$

$$\left\|\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H} \leq 1, \tag{55}$$

we get

$$\begin{aligned}\left\|\frac{u_1-u_0}{\tau}\right\|_H &\leq \left\|\frac{1}{2}\tau A^{1/2}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\left\|A^{1/2}\varphi\right\|_H \\ &\quad + \left\|\left(I-\frac{\tau^2}{12}A\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\|\psi\|_H \\ &\quad + \left\|\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\tau\|f_{1,1}\|_H \\ &\leq \frac{6\sqrt{11}}{12+\sqrt{11}}\left\|A^{1/2}\varphi\right\|_H + \|\psi\|_H + \tau\|f_{1,1}\|_H \\ &\leq M\left\{\sum_{s=1}^{N-1}\|f_s\|_H\tau + \left\|A^{1/2}\varphi\right\|_H + \|\psi\|_H + \tau\|f_{1,1}\|_H\right\}.\end{aligned} \tag{56}$$

Applying $A^{1/2}$ to (53) and using estimates (43), (52), (54), (55), we obtain

$$\begin{aligned}
\left\|A^{1/2}\frac{u_1-u_0}{\tau}\right\|_H &\le \left\|\frac{\tau}{2}A^{1/2}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\|A\varphi\|_H \\
&\quad+\left\|\left(I-\frac{\tau^2}{12}A\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\left\|A^{1/2}\psi\right\|_H \\
&\quad+\left\|\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\tau\left\|A^{1/2}f_{1,1}\right\|_H \\
&\le \frac{6\sqrt{11}}{12+\sqrt{11}}\|A\varphi\|_H+\left\|A^{1/2}\psi\right\|_H+\tau\left\|A^{1/2}f_{1,1}\right\|_H \\
&\le M\left\{\sum_{s=2}^{N-1}\|f_s-f_{s-1}\|_H+\|f_1\|_H+\|A\varphi\|_H\right. \\
&\quad\left.+\left\|A^{1/2}\psi\right\|_H+\tau\left\|A^{1/2}f_{1,1}\right\|_H\right\}. \qquad (57)
\end{aligned}$$

Applying A to the formula for u_1 and using estimates (43), (50), (51), (52), we get

$$\begin{aligned}
\|Au_1\|_H &\le \left\|\left(I-\frac{5\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\|A\varphi\|_H \\
&\quad+\left\|\tau A^{1/2}\left(I-\frac{\tau^2}{12}A\right)\left(I+\frac{\tau^2}{12}A+\frac{\tau^4}{144}A^2\right)^{-1}\right\|_{H\to H}\left\|A^{1/2}\psi\right\|_H \\
&\quad+\left\|\tau A^{1/2}\left(I+\frac{1}{12}\tau^2A+\frac{1}{144}\tau^4A^2\right)^{-1}\right\|_{H\to H}\tau\left\|A^{1/2}f_{1,1}\right\|_H \\
&\le \|A\varphi\|_H+12\left\|A^{1/2}\psi\right\|_H+\frac{12\sqrt{11}}{12+\sqrt{11}}\tau\left\|A^{1/2}f_{1,1}\right\|_H \\
&\le M\left\{\sum_{s=2}^{N-1}\|f_s-f_{s-1}\|_H+\|f_1\|_H+\|A\varphi\|_H\right. \\
&\quad\left.+\left\|A^{1/2}\psi\right\|_H+\tau\left\|A^{1/2}f_{1,1}\right\|_H\right\}. \qquad (58)
\end{aligned}$$

Using difference scheme (2) and formula (53), we obtain

$$\begin{aligned}
\frac{u_2-2u_1+u_0}{\tau^2} &= \left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1} \\
&\quad\times\left\{-\left(I-\frac{\tau^2A}{3}-\frac{5\tau^4A^2}{72}+\frac{\tau^6A^3}{1{,}728}\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}A\varphi\right. \\
&\quad-\tau\left(A+\frac{1}{6}\tau^2A^2\right)\left(I-\frac{\tau^2}{12}A\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\psi \\
&\quad\left.-\left(A+\frac{\tau^2A^2}{6}\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\tau^2f_{1,1}+f_1\right\}. \qquad (59)
\end{aligned}$$

Using formula (59), estimates (43), (51), (52), (55), and the following simple estimates:

$$\left\| \left(I - \frac{\tau^2 A}{3} - \frac{5\tau^4 A^2}{72} + \frac{\tau^6 A^3}{1{,}728}\right)\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-2} \right\|_{H\to H} \le 2, \tag{60}$$

$$\left\| \left(I + \frac{\tau^2 A}{6}\right)\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1} \right\|_{H\to H} \le 2, \tag{61}$$

we have

$$\begin{aligned}
&\left\| \frac{u_2 - 2u_1 + u_0}{\tau^2} \right\|_H \\
&\quad \le \left\| \left(I - \frac{\tau^2 A}{3} - \frac{5\tau^4 A^2}{72} + \frac{\tau^6 A^3}{1{,}728}\right)\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-2} \right\|_{H\to H} \|A\varphi\|_H \\
&\qquad + \left\| \left(I + \frac{\tau^2 A}{6}\right)\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1} \right\|_{H\to H} \\
&\qquad \times \left\| \tau A^{1/2}\left(I - \frac{\tau^2}{12}A\right)\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1} \right\|_{H\to H} \left\|A^{1/2}\psi\right\|_H \\
&\qquad + \left\| \left(I + \frac{\tau^2 A}{6}\right)\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1} \right\|_{H\to H} \\
&\qquad \times \left\| \tau A^{1/2}\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1} \right\|_{H\to H} \left\|A^{1/2} f_{1,1}\right\|_H \tau \\
&\qquad + \left\| \left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1} \right\|_{H\to H} \|f_1\|_H \\
&\quad \le 2\|A\varphi\|_H + \|f_1\|_H + 24\left\|A^{1/2}\psi\right\|_H + \frac{12\sqrt{11}}{12+\sqrt{11}}\tau\left\|A^{1/2}f_{1,1}\right\|_H \\
&\quad \le M\left\{\sum_{s=2}^{N-1}\|f_s - f_{s-1}\|_H + \|f_1\|_H + \|A\varphi\|_H + \left\|A^{1/2}\psi\right\|_H + \tau\left\|A^{1/2}f_{1,1}\right\|_H\right\}.
\end{aligned} \tag{62}$$

So, for $k = 1$, the following estimates are proved:

$$\|u_k\|_H \le M\left\{\sum_{s=1}^{N-1}\left\|A^{-1/2}f_s\right\|_H\tau + \|\varphi\|_H + \left\|A^{-1/2}\psi\right\|_H + \tau\left\|A^{-1/2}f_{1,1}\right\|_H\right\},$$

$$\left\|A^{1/2}u_k\right\|_H \le M\left\{\sum_{s=1}^{N-1}\|f_s\|_H\tau + \left\|A^{1/2}\varphi\right\|_H + \|\psi\|_H + \tau\|f_{1,1}\|_H\right\},$$

$$\left\|\frac{u_k - u_{k-1}}{\tau}\right\|_H \le M\left\{\sum_{s=1}^{N-1}\|f_s\|_H\tau + \left\|A^{1/2}\varphi\right\|_H + \|\psi\|_H + \tau\|f_{1,1}\|_H\right\},$$

$$\begin{aligned}
\left\|A^{1/2}\frac{u_k - u_{k-1}}{\tau}\right\|_H \le M\Bigg\{&\sum_{s=2}^{N-1}\|f_s - f_{s-1}\|_H + \|f_1\|_H + \|A\varphi\|_H \\
&+ \left\|A^{1/2}\psi\right\|_H + \tau\left\|A^{1/2}f_{1,1}\right\|_H\Bigg\},
\end{aligned}$$

$$\|Au_k\|_H \le M\left\{\sum_{s=2}^{N-1}\|f_s-f_{s-1}\|_H+\|f_1\|_H+\|A\varphi\|_H+\|A^{1/2}\psi\|_H+\tau\|A^{1/2}f_{1,1}\|_H\right\},$$

$$\left\|\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}\right\|_H \le M\left\{\sum_{s=2}^{N-1}\|f_s-f_{s-1}\|_H+\|f_1\|_H+\|A\varphi\|_H\right.$$
$$\left.+\|A^{1/2}\psi\|_H+\tau\|A^{1/2}f_{1,1}\|_H\right\}.$$

Now, we will establish these estimates for any $k\ge 2$. Using formula (49), estimate (43), and the triangle inequality, we obtain

$$\begin{aligned}\|u_k\|_H &\le \frac{1}{2}\left(\|\widetilde{R}_1\|_{H\to H}\|R^k\|_{H\to H}+\|R_1\|_{H\to H}\|\widetilde{R}^k\|_{H\to H}\right)\|\varphi\|_H\\ &\quad+\frac{1}{2}\left(\|A^{1/2}R_2\|_{H\to H}\|\widetilde{R}^k\|_{H\to H}+\|A^{1/2}R_2\|_{H\to H}\|R^k\|_{H\to H}\right)\|A^{-1/2}\psi\|_H\\ &\quad+\frac{1}{2}\left(\|\tau A^{1/2}R_3\|_{H\to H}\|\widetilde{R}^k\|_{H\to H}\right.\\ &\quad\left.+\|\tau A^{1/2}R_3\|_{H\to H}\|R^k\|_{H\to H}\right)\tau\|A^{-1/2}f_{1,1}\|_H\\ &\quad+\frac{1}{2}\|\tau A^{1/2}R_4\|_{H\to H}\sum_{s=1}^{k-1}\left[\|\widetilde{R}^{k-s}\|_{H\to H}+\|R^{k-s}\|_{H\to H}\right]\|A^{-1/2}f_s\|_H\tau\\ &\le M\left\{\sum_{s=1}^{N-1}\|A^{-1/2}f_s\|_H\tau+\|\varphi\|_H+\|A^{-1/2}\psi\|_H+\tau\|A^{-1/2}f_{1,1}\|_H\right\}\end{aligned}\tag{63}$$

for any $k\ge 2$. Combining the estimates $\|u_k\|_H$ for any k, we obtain (44).

Applying $A^{1/2}$ to (49), using estimate (43) and the triangle inequality, we get

$$\begin{aligned}\|A^{1/2}u_k\|_H &\le \frac{1}{2}\left(\|\widetilde{R}_1\|_{H\to H}\|R^k\|_{H\to H}+\|R_1\|_{H\to H}\|\widetilde{R}^k\|_{H\to H}\right)\|A^{1/2}\varphi\|_H\\ &\quad+\frac{1}{2}\left(\|A^{1/2}R_2\|_{H\to H}\|\widetilde{R}^k\|_{H\to H}+\|A^{1/2}R_2\|_{H\to H}\|R^k\|_{H\to H}\right)\|\psi\|_H\\ &\quad+\frac{1}{2}\left(\|\tau A^{1/2}R_3\|_{H\to H}\|\widetilde{R}^k\|_{H\to H}\right.\\ &\quad\left.+\|\tau A^{1/2}R_3\|_{H\to H}\|R^k\|_{H\to H}\right)\tau\|f_{1,1}\|_H\\ &\quad+\frac{1}{2}\|\tau A^{1/2}R_4\|_{H\to H}\sum_{s=1}^{k-1}\left(\|\widetilde{R}^{k-s}\|_{H\to H}+\|R^{k-s}\|_{H\to H}\right)\|f_s\|_H\tau\\ &\le M\left\{\sum_{s=1}^{N-1}\|f_s\|_H\tau+\|A^{1/2}\varphi\|_H+\|\psi\|_H+\tau\|f_{1,1}\|_H\right\}\end{aligned}$$

for $k\ge 2$. Combining the estimates for $\|A^{1/2}u_k\|_H$ for any k, we obtain

$$\max_{1\le k\le N}\|A^{1/2}u_k\|_H \le M\left\{\sum_{s=1}^{N-1}\|f_s\|_H\tau+\|A^{1/2}\varphi\|_H+\|\psi\|_H+\tau\|f_{1,1}\|_H\right\}.\tag{64}$$

Applying formula (49), we get

$$\begin{aligned}\frac{u_k - u_{k-1}}{\tau} = \frac{1}{\tau}\Bigg\{&\frac{1}{2}\big[\widetilde{R}_1 R_5 R^{k-1} - R_1 \widetilde{R}_5 \widetilde{R}^{k-1}\big]\varphi \\ &+ \frac{1}{2}\big[\widetilde{R}_5 \widetilde{R}^{k-1} - R_5 R^{k-1}\big] R_2 \psi + \frac{1}{2}\big[\widetilde{R}_5 \widetilde{R}^{k-1} - R_5 R^{k-1}\big] R_3 \tau^2 f_{1,1} \\ &+ \frac{1}{2}[\widetilde{R} - R] R_4 \tau^2 f_{k-1} + \frac{1}{2} R_4 \sum_{s=1}^{k-2} \big[\widetilde{R}_5 \widetilde{R}^{k-1-s} - R_5 R^{k-1-s}\big] f_s \tau^2 \Bigg\}. \end{aligned} \tag{65}$$

Using (65), estimate (43), and the triangle inequality, we obtain

$$\begin{aligned}\left\|\frac{u_k - u_{k-1}}{\tau}\right\|_H \le \frac{1}{\tau}\Bigg\{&\frac{1}{2}\big(\big\|A^{-1/2} R_5\big\|_{H\to H} \|\widetilde{R}_1\|_{H\to H} \big\|R^{k-1}\big\|_{H\to H} \\ &+ \big\|A^{-1/2}\widetilde{R}5\big\|_{H\to H} \|R_1\|_{H\to H} \big\|\widetilde{R}^{k-1}\big\|_{H\to H}\big) \big\|A^{1/2}\varphi\big\|_H \\ &+ \frac{1}{2}\big(\big\|A^{1/2} R_2\big\|_{H\to H} \big\|A^{-1/2}\widetilde{R}_5\big\|_{H\to H} \big\|\widetilde{R}^{k-1}\big\|_{H\to H} \\ &+ \big\|A^{1/2} R_2\big\|_{H\to H} \big\|A^{-1/2} R_5\big\|_{H\to H} \big\|R^{k-1}\big\|_{H\to H}\big) \|\psi\|_H \\ &+ \frac{1}{2}\big(\big\|\tau A^{1/2} R_3\big\|_{H\to H} \big\|A^{-1/2}\widetilde{R}_5\big\|_{H\to H} \big\|\widetilde{R}^{k-1}\big\|_{H\to H} \\ &+ \big\|\tau A^{1/2} R_3\big\|_{H\to H} \big\|A^{-1/2} R_5\big\|_{H\to H} \big\|R^{k-1}\big\|_{H\to H}\big) \\ &\times \tau \|f_{1,1}\|_H + \frac{1}{2}\big(\big\|\tau A^{1/2} R_4\big\|_{H\to H} \|\widetilde{R}\|_{H\to H} \\ &+ \big\|\tau A^{1/2} R_4\big\|_{H\to H} \|R\|_{H\to H}\big) \big\|A^{-1/2} f_{k-1}\big\|_H \tau + \frac{1}{2}\big\|\tau A^{1/2} R_4\big\|_{H\to H} \\ &\times \sum_{s=1}^{k-2} \big(\big\|A^{-1/2}\widetilde{R}_5\big\|_{H\to H} \big\|\widetilde{R}^{k-1-s}\big\|_{H\to H} \\ &+ \big\|A^{-1/2} R_5\big\|_{H\to H} \big\|R^{k-1-s}\big\|_{H\to H}\big) \|f_s\|_H \tau \Bigg\} \\ \le M\Bigg\{&\sum_{s=1}^{N-1} \|f_s\|_H \tau + \big\|A^{1/2}\varphi\big\|_H + \|\psi\|_H + \tau\|f_{1,1}\|_H\Bigg\}. \end{aligned} \tag{66}$$

Combining the estimates for $\|\tau^{-1}(u_k - u_{k-1})\|_H$ for any k, we obtain

$$\max_{1\le k\le N-1} \left\|\frac{u_k - u_{k-1}}{\tau}\right\|_H \le M\Bigg\{\sum_{s=1}^{N-1} \|f_s\|_H \tau + \big\|A^{1/2}\varphi\big\|_H + \|\psi\|_H + \tau\|f_{1,1}\|_H\Bigg\}. \tag{67}$$

From estimates (64), (67), estimate (45) follows.

Now, applying Abel's formula to (65), we obtain

$$\begin{aligned}\frac{u_k - u_{k-1}}{\tau} = \frac{1}{\tau}\Bigg\{&\frac{1}{2}\big[\widetilde{R}_1 R_5 R^{k-1} - R_1 \widetilde{R}_5 \widetilde{R}^{k-1}\big]\varphi \\ &+ \frac{1}{2}\big[\widetilde{R}_5 \widetilde{R}^{k-1} - R_5 R^{k-1}\big] R_2 \psi + \frac{1}{2}\big[\widetilde{R}_5 \widetilde{R}^{k-1} - R_5 R^{k-1}\big] R_3 \tau^2 f_{1,1} \end{aligned}$$

$$+\frac{1}{2}[\widetilde{R}-R]R_4\tau^2 f_{k-1}+\frac{1}{2}R_4\tau^2\Biggl(\sum_{s=2}^{k-2}\bigl[R_5R_6R^{k-1-s}-\widetilde{R}_5\widetilde{R}_6\widetilde{R}^{k-1-s}\bigr](f_s-f_{s-1})$$

$$\times[\widetilde{R}_5\widetilde{R}_6-R_5R_6]f_{k-2}-\bigl[\widetilde{R}_5\widetilde{R}_6\widetilde{R}^{k-2}-R_5R_6R^{k-2}\bigr]f_1\Biggr)\Biggr\},\quad 2\le k\le N. \tag{68}$$

Next, applying $A^{1/2}$ to formula (68) and using estimate (43) and the triangle inequality, we get

$$\begin{aligned}
&\left\|A^{1/2}\frac{u_k-u_{k-1}}{\tau}\right\|_H\\
&\le\frac{1}{\tau}\Biggl\{\frac{1}{2}\bigl(\|A^{-1/2}R_5\|_{H\to H}\|\widetilde{R}_1\|_{H\to H}\|R^{k-1}\|_{H\to H}\\
&\quad+\|A^{-1/2}\widetilde{R}_5\|_{H\to H}\|R_1\|_{H\to H}\|R^{k-1}\|_{H\to H}\bigr)\|A\varphi\|_H\\
&\quad+\frac{1}{2}\bigl(\|A^{1/2}R_2\|_{H\to H}\|A^{-1/2}\widetilde{R}_5\|_{H\to H}\|\widetilde{R}^{k-1}\|_{H\to H}\\
&\quad+\|A^{1/2}R_2\|_{H\to H}\|A^{-1/2}R_5\|_{H\to H}\|R^{k-1}\|_{H\to H}\bigr)\|A^{1/2}\psi\|_H\\
&\quad+\frac{1}{2}\bigl(\|\tau A^{1/2}R_3\|_{H\to H}\|A^{-1/2}\widetilde{R}_5\|_{H\to H}\|\widetilde{R}^{k-1}\|_{H\to H}\\
&\quad+\|\tau A^{1/2}R_3\|_{H\to H}\|A^{-1/2}R_5\|_{H\to H}\|R^{k-1}\|_{H\to H}\bigr)\tau\|A^{1/2}f_{1,1}\|_H\\
&\quad+\frac{1}{2}\bigl(\|\tau A^{1/2}R_4\|_{H\to H}\|\widetilde{R}\|_{H\to H}+\|\tau A^{1/2}R_4\|_{H\to H}\|R\|_{H\to H}\bigr)\|f_{k-1}\|_H\tau\\
&\quad+\frac{1}{2}\|\tau A^{1/2}R_4\|_{H\to H}\Biggl(\sum_{s=2}^{k-2}\bigl(\|A^{-1/2}R_5\|_{H\to H}\|\tau A^{1/2}R_6\|_{H\to H}\|R^{k-1-s}\|_{H\to H}\\
&\quad+\|A^{-1/2}\widetilde{R}_5\|_{H\to H}\|\tau A^{1/2}\widetilde{R}_6\|_{H\to H}\|\widetilde{R}^{k-1-s}\|_{H\to H}\bigr)\|f_s-f_{s-1}\|_H\\
&\quad+\bigl(\|A^{-1/2}\widetilde{R}_5\|_{H\to H}\|\tau A^{1/2}\widetilde{R}_6\|_{H\to H}+\|A^{-1/2}R_5\|_{H\to H}\|\tau A^{1/2}R_6\|_{H\to H}\bigr)\\
&\quad\times\|f_{k-2}\|_H+\bigl(\|A^{-1/2}\widetilde{R}_5\|_{H\to H}\|\tau A^{1/2}\widetilde{R}_6\|_{H\to H}\|\widetilde{R}^{k-2}\|_{H\to H}\\
&\quad+\|A^{-1/2}R_5\|_{H\to H}\|\tau A^{1/2}R_6\|_{H\to H}\|R^{k-2}\|_{H\to H}\bigr)\|f_1\|_H\Biggr)\Biggr\}\\
&\le M\Biggl\{\sum_{s=2}^{N-1}\|f_s-f_{s-1}\|_H+\|f_1\|_H+\|A\varphi\|_H\\
&\quad+\|A^{1/2}\psi\|_H+\tau\|A^{1/2}f_{1,1}\|_H\Biggr\}.
\end{aligned}$$

Combining the estimates for $\|A^{1/2}\tau^{-1}(u_k-u_{k-1})\|_H$ for any k, we obtain

$$\begin{aligned}
\max_{1\le k\le N-1}\left\|A^{1/2}\frac{u_k-u_{k-1}}{\tau}\right\|_H&\le M\Biggl\{\sum_{s=2}^{N-1}\|f_s-f_{s-1}\|_H+\|f_1\|_H\\
&\quad+\|A^{1/2}\psi\|_H+\|A\varphi\|_H+\tau\|A^{1/2}f_{1,1}\|_H\Biggr\}.
\end{aligned}\tag{69}$$

Now, applying Abel's formula to (49), we have

$$\begin{aligned} u_k &= \frac{1}{2}\left[\widetilde{R}_1 R^k - R_1 \widetilde{R}^k\right]\varphi + \frac{1}{2}\left[\widetilde{R}^k - R^k\right]R_2\psi \\ &\quad + \frac{1}{2}\left[\widetilde{R}^k - R^k\right]R_3\tau^2 f_{1,1} + \frac{1}{2}\tau^2 R_4\left(\sum_{s=2}^{k-1}\left[R_6R^{k-s} - \widetilde{R}_6\widetilde{R}^{k-s}\right](f_s - f_{s-1})\right. \\ &\quad \left. + (\widetilde{R}_6 - R_6)f_{k-1} - \left[\widetilde{R}_6\widetilde{R}^{k-1} - R_6R^{k-1}\right]f_1\right), \quad 2 \le k \le N. \end{aligned} \tag{70}$$

Next, applying A to formula (70) and using (43) and the triangle inequality, we get

$$\begin{aligned} \|Au_k\|_H &\le \frac{1}{2}\left(\|\widetilde{R}_1\|_{H\to H}\left\|R^k\right\|_{H\to H} + \|R_1\|_{H\to H}\left\|\widetilde{R}^k\right\|_{H\to H}\right)\|A\varphi\|_H \\ &\quad + \frac{1}{2}\left(\left\|A^{1/2}R_2\right\|_{H\to H}\left\|\widetilde{R}^k\right\|_{H\to H} + \left\|A^{1/2}R_2\right\|_{H\to H}\left\|R^k\right\|_{H\to H}\right)\left\|A^{1/2}\psi\right\|_H \\ &\quad + \frac{1}{2}\left(\left\|\tau A^{-1/2}R_3\right\|_{H\to H}\left\|\widetilde{R}^k\right\|_{H\to H} + \left\|\tau A^{-1/2}R_3\right\|_{H\to H}\left\|R^k\right\|_{H\to H}\right) \\ &\quad \times \tau\left\|A^{1/2}f_{1,1}\right\|_H + \frac{1}{2}\left\|\tau A^{1/2}R_4\right\|_{H\to H} \\ &\quad \times \left(\sum_{s=2}^{k-1}\left[\left\|\tau A^{1/2}R_6\right\|_{H\to H}\left\|R^{k-s}\right\|_{H\to H} + \left\|\tau A^{1/2}\widetilde{R}_6\right\|_{H\to H}\left\|\widetilde{R}^{k-s}\right\|_{H\to H}\right]\right. \\ &\quad \times \|f_s - f_{s-1}\|_H + \left(\left\|\tau A^{1/2}\widetilde{R}_6\right\|_{H\to H} + \left\|\tau A^{1/2}R_6\right\|_{H\to H}\right)\|f_{k-1}\|_H \\ &\quad \left. + \left[\left\|\tau A^{1/2}\widetilde{R}_6\right\|_{H\to H}\left\|\widetilde{R}^{k-1}\right\|_{H\to H} + \left\|\tau A^{1/2}R_6\right\|_{H\to H}\left\|R^{k-1}\right\|_{H\to H}\right]\|f_1\|_H\right) \\ &\le M\left\{\sum_{s=2}^{N-1}\|f_s - f_{s-1}\|_H + \|f_1\|_H + \|A\varphi\|_H + \left\|A^{1/2}\psi\right\|_H + \tau\left\|A^{1/2}f_{1,1}\right\|_H\right\} \end{aligned}$$

for $k \ge 2$. Combining the estimates for $\|Au_k\|_H$ for any k, we obtain

$$\begin{aligned} \max_{1\le k\le N}\|Au_k\|_H &\le M\left\{\sum_{s=2}^{N-1}\|f_s - f_{s-1}\|_H + \|f_1\|_H\right. \\ &\quad \left. + \|A\varphi\|_H + \left\|A^{1/2}\psi\right\|_H + \tau\left\|A^{1/2}f_{1,1}\right\|_H\right\}. \end{aligned} \tag{71}$$

Now, applying formula (49), we get

$$\begin{aligned} \frac{u_{k+1} - 2u_k + u_{k-1}}{\tau^2} &= \frac{1}{\tau^2}\left\{\frac{1}{2}\left[\widetilde{R}_1R_5^2R^{k-1} - R_1\widetilde{R}_5^2\widetilde{R}^{k-1}\right]\varphi\right. \\ &\quad + \frac{1}{2}\left[\widetilde{R}_5^2\widetilde{R}^{k-1} - R_5^2R^{k-1}\right]R_2\psi + \frac{1}{2}\left[\widetilde{R}_5^2\widetilde{R}^{k-1} - R_5^2R^{k-1}\right]R_3\tau^2 f_{1,1} \\ &\quad + \frac{1}{2}[\widetilde{R} - R]R_4\tau^2 f_k + \frac{1}{2}\left(\widetilde{R}_5^2 - R_5^2\right)R_4\tau^2 f_{k-1} \\ &\quad \left. + \frac{1}{2}R_4\sum_{s=1}^{k-2}\left[\widetilde{R}_5^2\widetilde{R}^{k-1-s} - R_5^2R^{k-1-s}\right]f_s\tau^2\right\}. \end{aligned} \tag{72}$$

First, applying Abel's formula to (72), we have

$$\begin{aligned}
&\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}\\
&\quad=\frac{1}{\tau^2}\Biggl\{\frac{1}{2}\bigl[\widetilde{R}_1R_5^2R^{k-1}-R_1\widetilde{R}_5^2\widetilde{R}^{k-1}\bigr]\varphi\\
&\qquad+\frac{1}{2}\bigl[\widetilde{R}_5^2\widetilde{R}^{k-1}-R_5^2R^{k-1}\bigr]R_2\psi+\frac{1}{2}\bigl[\widetilde{R}_5^2\widetilde{R}^{k-1}-R_5^2R^{k-1}\bigr]R_3\tau^2f_{1,1}\\
&\qquad+\frac{1}{2}[\widetilde{R}-R]R_4\tau^2f_k+\frac{1}{2}\bigl(\widetilde{R}_5^2-R_5^2\bigr)R_4\tau^2f_{k-1}\\
&\qquad+\frac{1}{2}R_4\tau^2\Biggl(\sum_{s=2}^{k-2}\bigl[R_6R_5^2R^{k-1-s}-\widetilde{R}_6\widetilde{R}_5^2\widetilde{R}^{k-1-s}\bigr](f_s-f_{s-1})\\
&\qquad+\bigl(\widetilde{R}_5^2\widetilde{R}_6-R_5^2R_6\bigr)f_{k-2}-\bigl[\widetilde{R}_5^2\widetilde{R}_6\widetilde{R}^{k-2}-R_5^2R_6R^{k-2}\bigr]f_1\Biggr)\Biggr\},\quad 2\le k\le N.
\end{aligned}\tag{73}$$

Second, using formulas (73), (43), and the triangle inequality, we get

$$\begin{aligned}
&\left\|\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}\right\|_H\\
&\quad\le\frac{1}{\tau^2}\Biggl\{\frac{1}{2}\Bigl(\bigl\|\bigl(A^{-1/2}R_5\bigr)^2\bigr\|_{H\to H}\|\widetilde{R}_1\|_{H\to H}\bigl\|R^{k-1}\bigr\|_{H\to H}\\
&\qquad+\bigl\|\bigl(A^{-1/2}\widetilde{R}_5\bigr)^2\bigr\|_{H\to H}\|\widetilde{R}_1\|_{H\to H}\bigl\|\widetilde{R}^{k-1}\bigr\|_{H\to H}\Bigr)\|A\varphi\|_H\\
&\qquad+\frac{1}{2}\Bigl(\bigl\|\bigl(A^{-1/2}\widetilde{R}_5\bigr)^2\bigr\|_{H\to H}\bigl\|A^{1/2}R_2\bigr\|_{H\to H}\bigl\|\widetilde{R}^{k-1}\bigr\|_{H\to H}\\
&\qquad+\bigl\|\bigl(A^{-1/2}R_5\bigr)^2\bigr\|_{H\to H}\bigl\|A^{1/2}R_2\bigr\|_{H\to H}\bigl\|R^{k-1}\bigr\|_{H\to H}\Bigr)\bigl\|A^{1/2}\psi\bigr\|_H\\
&\qquad+\frac{1}{2}\Bigl(\bigl\|\bigl(A^{-1/2}\widetilde{R}_5\bigr)^2\bigr\|_{H\to H}\bigl\|\tau A^{1/2}R_3\bigr\|_{H\to H}\bigl\|\widetilde{R}^{k-1}\bigr\|_{H\to H}\\
&\qquad+\bigl\|\bigl(A^{-1/2}R_5\bigr)^2\bigr\|_{H\to H}\bigl\|\tau A^{1/2}R_3\bigr\|_{H\to H}\bigl\|R^{k-1}\bigr\|_{H\to H}\Bigr)\tau\bigl\|A^{1/2}f_{1,1}\bigr\|_H\\
&\qquad+\frac{1}{2}\Bigl(\bigl\|\tau A^{1/2}R_4\bigr\|_{H\to H}\|\widetilde{R}\|_{H\to H}+\bigl\|\tau A^{1/2}R_4\bigr\|_{H\to H}\|R\|_{H\to H}\Bigr)\tau\|f_k\|_H\\
&\qquad+\frac{1}{2}\Bigl(\bigl\|\bigl(A^{-1/2}\widetilde{R}_5\bigr)^2\bigr\|_{H\to H}+\bigl\|\bigl(A^{-1/2}\widetilde{R}_5\bigr)^2\bigr\|_{H\to H}\Bigr)\\
&\qquad\times\bigl\|A^{1/2}R_4\bigr\|_{H\to H}\tau\bigl\|A^{1/2}f_{k-1}\bigr\|_H+\frac{1}{2}\bigl\|\tau A^{1/2}R_4\bigr\|_{H\to H}\\
&\qquad\times\Biggl(\sum_{s=2}^{k-2}\Bigl(\bigl\|\bigl(A^{-1/2}R_5\bigr)^2\bigr\|_{H\to H}\bigl\|\tau A^{1/2}R_6\bigr\|_{H\to H}\bigl\|R^{k-1-s}\bigr\|_{H\to H}\\
&\qquad+\bigl\|\bigl(A^{-1/2}\widetilde{R}_5\bigr)^2\bigr\|_{H\to H}\bigl\|\tau A^{1/2}\widetilde{R}_6\bigr\|_{H\to H}\bigl\|\widetilde{R}^{k-1-s}\bigr\|_{H\to H}\Bigr)\|f_s-f_{s-1}\|_H\\
&\qquad+\Bigl(\bigl\|\bigl(A^{-1/2}\widetilde{R}_5\bigr)^2\bigr\|_{H\to H}\bigl\|\tau A^{1/2}\widetilde{R}_6\bigr\|_{H\to H}+\bigl\|\bigl(A^{-1/2}R_5\bigr)^2\bigr\|_{H\to H}\bigl\|\tau A^{1/2}R_6\bigr\|_{H\to H}\Bigr)\|f_{k-2}\|_H\\
&\qquad+\Bigl(\bigl\|\bigl(A^{-1/2}\widetilde{R}_5\bigr)^2\bigr\|_{H\to H}\bigl\|\tau A^{1/2}\widetilde{R}_6\bigr\|_{H\to H}\bigl\|\widetilde{R}^{k-2}\bigr\|_{H\to H}\\
&\qquad+\bigl\|\bigl(A^{-1/2}R_5\bigr)^2\bigr\|_{H\to H}\bigl\|\tau A^{1/2}R_6\bigr\|_{H\to H}\bigl\|R^{k-2}\bigr\|_{H\to H}\Bigr)\|f_1\|_H\Biggr)\Biggr\}
\end{aligned}$$

$$\leq M\Biggl\{\sum_{s=2}^{N-1}\|f_s-f_{s-1}\|_H+\|f_1\|_H+\|A\varphi\|_H$$

$$+\bigl\|A^{1/2}\psi\bigr\|_H+\tau\bigl\|A^{1/2}f_{1,1}\bigr\|_H\Biggr\}. \tag{74}$$

Combining the estimates for $\|\tau^{-1}(u_{k+1}-2u_k+u_{k-1})\|_H$ for any k, we obtain

$$\max_{1\leq k\leq N-1}\bigl\|\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\bigr\|_H$$

$$\leq M\Biggl\{\sum_{s=2}^{N-1}\|f_s-f_{s-1}\|_H+\|f_1\|_H+\bigl\|A^{1/2}\psi\bigr\|_H+\|A\varphi\|_H+\tau\|Af_{1,1}\|_H\Biggr\}. \tag{75}$$

From estimates (69), (71), and (75), we obtain (46). Theorem 1 is proved. □

Now, we will obtain the stability estimates for the solution of difference scheme (3). We consider operators R, $\widetilde{R}$ by (23) and the following operators:

$$J_1=\left(I-\frac{\tau^2A}{12}\right)\left(I-\frac{i\tau A^{1/2}}{2}-\frac{\tau^2A}{12}\right)^{-1},$$

$$\widetilde{J}_1=\left(I-\frac{\tau^2A}{12}\right)\left(I+\frac{i\tau A^{1/2}}{2}-\frac{\tau^2A}{12}\right)^{-1},$$

$$J_2=\left(-i\tau A^{1/2}\left(I-\frac{i\tau A^{1/2}}{2}-\frac{\tau^2A}{12}\right)\right)^{-1},$$

$$\widetilde{J}_2=\left(-i\tau A^{1/2}\left(I+\frac{i\tau A^{1/2}}{2}-\frac{\tau^2A}{12}\right)\right)^{-1},$$

$$J_3=i\tau A^{1/2}\left(I-\frac{\tau^2A}{12}\right)\left(I-\frac{i\tau A^{1/2}}{2}-\frac{\tau^2A}{12}\right)^{-2},$$

$$\widetilde{J}_3=-i\tau A^{1/2}\left(I-\frac{\tau^2A}{12}\right)\left(I+\frac{i\tau A^{1/2}}{2}-\frac{\tau^2A}{12}\right)^{-2},$$

$$J_4=\left(I-\frac{\tau^2A}{12}\right)\left(i\tau A^{1/2}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)\right)^{-1},$$

$$\widetilde{J}_4=\left(I-\frac{\tau^2A}{12}\right)\left(-i\tau A^{1/2}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)\right)^{-1},$$

which will be used in the sequel. It is obvious that $J_1,\widetilde{J}_1$, and $J_2,\widetilde{J}_2$, and $J_3,\widetilde{J}_3$, and $J_4,\widetilde{J}_4$ are conjugates.

Let us give one lemma, without proof, that will be needed below.

Lemma 2 *The following estimates hold*:

$$\begin{cases}\|R\|_{H\to H}\leq 1, & \|\widetilde{R}\|_{H\to H}\leq 1,\\ \|J_1\|_{H\to H}\leq 1, & \|\widetilde{J}_1\|_{H\to H}\leq 1,\\ \|\tau A^{1/2}J_2\|_{H\to H}\leq 1, & \|\tau A^{1/2}\widetilde{J}_2\|_{H\to H}\leq 1,\\ \|A^{-1/2}J_3\|_{H\to H}\leq \tau, & \|A^{-1/2}\widetilde{J}_3\|_{H\to H}\leq \tau,\\ \|\tau A^{1/2}J_4\|_{H\to H}\leq 1, & \|\tau A^{1/2}\widetilde{J}_4\|_{H\to H}\leq 1.\end{cases} \tag{76}$$

Now, we will give the first main theorem of the present paper on the stability of difference scheme (3).

Theorem 2 *Let* $\varphi \in D(A)$, $\psi \in D(A^{\frac{1}{2}})$, $f_{2,2} \in D(A^{\frac{1}{2}})$. *Then, for the solution of difference scheme* (3), *the following stability estimates hold*:

$$\max_{1\le k\le N}\left\|\frac{u_k+u_{k-1}}{2}\right\|_H \le M\left\{\sum_{s=1}^{N-1}\left\|A^{-1/2}f_s\right\|_H\tau+\|\varphi\|_H + \left\|A^{-1/2}\psi\right\|_H+\tau\left\|A^{-1/2}f_{2,2}\right\|_H\right\}, \tag{77}$$

$$\max_{1\le k\le N-1}\left\|\frac{u_{k+1}-u_{k-1}}{2\tau}\right\|_H+\max_{1\le k\le N}\left\|A^{1/2}\frac{u_k+u_{k-1}}{2}\right\|_H \le M\left\{\sum_{s=1}^{N-1}\|f_s\|_H\tau+\left\|A^{1/2}\varphi\right\|_H+\|\psi\|_H+\tau\|f_{2,2}\|_H\right\}, \tag{78}$$

$$\max_{1\le k\le N-1}\left\|A^{1/2}\frac{u_{k+1}-u_{k-1}}{2\tau}\right\|_H+\max_{1\le k\le N}\left\|A\frac{u_k+u_{k-1}}{2}\right\|_H \le M\left\{\sum_{s=2}^{N-1}\|f_s-f_{s-1}\|_H+\|f_1\|_H+\|A\varphi\|_H+\left\|A^{1/2}\psi\right\|_H+\tau\left\|A^{1/2}f_{2,2}\right\|_H\right\}, \tag{79}$$

where M does not depend on τ, φ, ψ, $f_{2,2}$, *and* f_s, $1\le s\le N-1$.

Proof First, we will obtain the formula for the solution of problem (1). In exactly the same manner as in Theorem 1, one establishes formula (48) for the solution of initial value problem (47).

We can rewrite (47) into the following difference problem:

$$\begin{aligned}
&\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)u_{k-1}-\left(2-\frac{5\tau^2A}{6}+\frac{\tau^4A^2}{72}\right)u_k+\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)u_{k+1}\\
&\quad=\tau^2f_k,\quad 1\le k\le N-1,\\
&u_0=\varphi,\\
&u_1=\varphi+\tau\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\left(-\frac{\tau}{2}A\varphi+\left(I-\frac{\tau^2A}{12}\right)\psi+\tau f_{2,2}\right).
\end{aligned} \tag{80}$$

Replacing a with $(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144})$, b with $(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144})$, c with

$$\left(2-\frac{5\tau^2A}{6}+\frac{\tau^4A^2}{72}\right),$$

and ψ with

$$\varphi+\tau\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\left(-\frac{\tau}{2}A\varphi+\left(I-\frac{\tau^2A}{12}\right)\psi+\tau f_{2,2}\right),$$

φ_k with $\tau^2 f_k$ and applying formula (48), we obtain the following formula for u_k, $2 \le k \le N$:

$$\begin{aligned} u_k &= \frac{1}{2}\left[R^k + \widetilde{R}^k\right]\varphi + \frac{1}{2}\left[\widetilde{R}^k - R^k\right] iA^{-1/2}\psi \\ &\quad + \left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1} (\widetilde{R} - R)^{-1}\left[\widetilde{R}^k - R^k\right]\tau^2 f_{2,2} \\ &\quad + \left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1} (\widetilde{R} - R)^{-1} \sum_{s=1}^{k-1}\left[\widetilde{R}^{k-s} - R^{k-s}\right]\tau^2 f_s. \end{aligned} \tag{81}$$

Now, we will prove estimates (77), (78), and (78). First, using the formula of difference scheme (3), we obtain the formula

$$\begin{aligned} \frac{u_1 + u_0}{2} &= \left(I - \frac{\tau^2 A}{6} + \frac{\tau^4 A^2}{144}\right)\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1} \varphi \\ &\quad + \frac{1}{2}\left(I - \frac{\tau^2 A}{12}\right)\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1} \tau\psi \\ &\quad + \frac{1}{2}\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1} \tau^2 f_{2,2}. \end{aligned} \tag{82}$$

Using estimates (76), the formula for $\frac{u_1+u_0}{2}$, and the following simple estimates:

$$\left\| \left(I - \frac{\tau^2 A}{6} + \frac{\tau^4 A^2}{144}\right)\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1} \right\|_{H \to H} \le 1, \tag{83}$$

$$\left\| \tau A^{1/2}\left(I - \frac{\tau^2 A}{12}\right)\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1} \right\|_{H \to H} \le 12, \tag{84}$$

$$\left\| \tau A^{1/2}\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1} \right\|_{H \to H} \le \frac{12\sqrt{11}}{12 + \sqrt{11}}, \tag{85}$$

we get

$$\begin{aligned} &\left\| \frac{u_1 + u_0}{2} \right\|_H \\ &\quad \le \left\| \left(I - \frac{\tau^2 A}{6} + \frac{\tau^4 A^2}{144}\right)\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1} \right\|_{H \to H} \|\varphi\|_H \\ &\qquad + \frac{1}{2}\left\| \tau A^{1/2}\left(I - \frac{\tau^2 A}{12}\right)\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1} \right\|_{H \to H} \left\|A^{-1/2}\psi\right\|_H \\ &\qquad + \frac{1}{2}\left\| \tau A^{1/2}\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1} \right\|_{H \to H} \tau\left\|A^{-1/2} f_{2,2}\right\|_H \\ &\quad \le \|\varphi\|_H + 6\left\|A^{-1/2}\psi\right\|_H + \frac{6\sqrt{11}}{12 + \sqrt{11}}\tau\left\|A^{-1/2} f_{2,2}\right\|_H \\ &\quad \le M\left\{ \sum_{s=1}^{N-1}\left\|A^{-1/2} f_s\right\|_H \tau + \|\varphi\|_H + \left\|A^{-1/2}\psi\right\|_H + \tau\left\|A^{-1/2} f_{2,2}\right\|_H \right\}. \end{aligned}$$

Applying $A^{1/2}$ to the formula for $\frac{u_1+u_0}{2}$ and using estimates (76), (83), (84), and (85), we obtain

$$\begin{aligned}
&\left\| A^{1/2}\frac{u_1+u_0}{2}\right\|_H \\
&\quad\le \left\| \left(I-\frac{\tau^2A}{6}+\frac{\tau^4A^2}{144}\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H} \\
&\qquad\times \left\|A^{1/2}\varphi\right\|_H+\frac{1}{2}\left\|\tau A^{1/2}\left(I-\frac{\tau^2A}{12}\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\|\psi\|_H \\
&\qquad+\frac{1}{2}\left\|\tau A^{1/2}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\tau\|f_{2,2}\|_H \\
&\quad\le \left\|A^{1/2}\varphi\right\|_H+6\|\psi\|_H+\frac{6\sqrt{11}}{12+\sqrt{11}}\tau\|f_{2,2}\|_H \\
&\quad\le M\left\{\sum_{s=1}^{N-1}\|f_s\|_H\tau+\left\|A^{1/2}\varphi\right\|_H+\|\psi\|_H+\tau\|f_{2,2}\|_H\right\}.
\end{aligned}$$

Applying A to the formula for $\frac{u_1+u_0}{2}$ and using estimates (76), (83), (84), and (85), we have

$$\begin{aligned}
\left\| A\frac{u_1+u_0}{2}\right\|_H &\le \left\| \left(I-\frac{\tau^2A}{6}+\frac{\tau^4A^2}{144}\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H} \\
&\quad\times \|A\varphi\|_H+\frac{1}{2}\left\|\tau A^{1/2}\left(I-\frac{\tau^2A}{12}\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H} \\
&\quad\times \left\|A^{1/2}\psi\right\|_H+\frac{1}{2}\left\|\tau A^{1/2}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\tau\left\|A^{1/2}f_{2,2}\right\|_H \\
&\le \|A\varphi\|_H+6\left\|A^{1/2}\psi\right\|_H+\frac{6\sqrt{11}}{12+\sqrt{11}}\tau\left\|A^{1/2}f_{2,2}\right\|_H \\
&\le M\left\{\sum_{s=2}^{N-1}\|f_s-f_{s-1}\|_H+\|f_1\|_H+\|A\varphi\|_H\right. \\
&\quad\left.+\left\|A^{1/2}\psi\right\|_H+\tau\left\|A^{1/2}f_{2,2}\right\|_H\right\}. \qquad (86)
\end{aligned}$$

Using the formula for the solution of difference scheme (3), we obtain

$$\begin{aligned}
\frac{u_2-u_0}{2\tau} &= -\frac{\tau A}{2}\left(I-\frac{\tau^2A}{6}+\frac{\tau^4A^2}{144}\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-2}\varphi \\
&\quad+\left(I-\frac{5\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)\left(I-\frac{\tau^2}{12}A\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-2}\psi \\
&\quad+\left(I-\frac{5\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-2}\tau f_{2,2} \\
&\quad+\frac{1}{2}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\tau f_1. \qquad (87)
\end{aligned}$$

Using formula (87), estimates (76), (83), (85), and the following simple estimates:

$$\left\| \left(I - \frac{5\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1} \right\|_{H\to H} \le 1, \tag{88}$$

$$\left\| \left(I - \frac{\tau^2}{12} A\right)\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1} \right\|_{H\to H} \le 1, \tag{89}$$

$$\left\| \left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1} \right\|_{H\to H} \le 1, \tag{90}$$

we get

$$\begin{aligned}
&\left\| \frac{u_2 - u_0}{2\tau} \right\|_H \\
&\quad \le \left\| \left(I - \frac{\tau^2 A}{6} + \frac{\tau^4 A^2}{144}\right)\left(I + \frac{\tau^2}{12} A + \frac{\tau^4}{144} A^2\right)^{-1} \right\| \\
&\qquad \times \left\| \tau A^{1/2}\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1} \right\|_{H\to H} \left\| A^{1/2}\varphi \right\|_H \\
&\qquad + \left\| \left(I - \frac{5\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1} \right\|_{H\to H} \\
&\qquad \times \left\| \left(I - \frac{\tau^2}{12} A\right)\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1} \right\|_{H\to H} \|\psi\|_H \\
&\qquad + \left\| \left(I - \frac{5\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1} \right\|_{H\to H} \\
&\qquad \times \left\| \left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1} \right\|_{H\to H} \tau \|f_{2,2}\|_H + \frac{1}{2} \left\| \left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1} \right\|_{H\to H} \tau \|f_1\| \\
&\quad \le \frac{12\sqrt{11}}{12 + \sqrt{11}} \left\| A^{1/2}\varphi \right\|_H + \|\psi\|_H + \tau \|f_{2,2}\|_H + \frac{1}{2}\tau \|f_1\|_H \\
&\quad \le M\left\{ \sum_{s=1}^{N-1} \|f_s\|_H \tau + \|f_1\|\tau + \left\| A^{1/2}\varphi \right\|_H + \|\psi\|_H + \tau \|f_{2,2}\|_H \right\}.
\end{aligned} \tag{91}$$

Applying $A^{1/2}$ to (87) and using estimates (76), (85), (89), (90), we obtain

$$\begin{aligned}
\left\| A^{1/2} \frac{u_2 - u_0}{2\tau} \right\|_H &\le \left\| \left(I - \frac{\tau^2 A}{6} + \frac{\tau^4 A^2}{144}\right)\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1} \right\| \\
&\quad \times \left\| \tau A^{1/2}\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1} \right\|_{H\to H} \|A\varphi\|_H \\
&\quad + \left\| \left(I - \frac{5\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1} \right\|_{H\to H} \\
&\quad \times \left\| \left(I - \frac{\tau^2}{12} A\right)\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1} \right\|_{H\to H} \left\| A^{1/2}\psi \right\|_H \\
&\quad + \left\| \left(I - \frac{5\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)\left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144}\right)^{-1} \right\|_{H\to H}
\end{aligned}$$

$$
\begin{aligned}
&\times \left\| \left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144} \right)^{-1} \right\|_{H\to H} \tau \left\| A^{1/2} f_{2,2} \right\|_H \\
&+ \frac{1}{2} \left\| \tau A^{1/2} \left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144} \right)^{-1} \right\|_{H\to H} \|f_1\|_H \\
\le{}& \frac{12\sqrt{11}}{12+\sqrt{11}} \|A\varphi\|_H + \frac{6\sqrt{11}}{12+\sqrt{11}} \|f_1\|_H + \left\| A^{1/2}\psi \right\|_H + \tau \left\| A^{1/2} f_{2,2} \right\|_H \\
\le{}& M \left\{ \sum_{s=2}^{N-1} \|f_s - f_{s-1}\|_H + \|f_1\|_H + \|A\varphi\|_H \right. \\
&\left. + \left\| A^{1/2}\psi \right\|_H + \tau \left\| A^{1/2} f_{2,2} \right\|_H \right\}. \qquad (92)
\end{aligned}
$$

So, for $k = 1$, the following estimates are proved:

$$
\begin{aligned}
&\left\| \frac{u_k + u_{k-1}}{2} \right\|_H \le M \left\{ \sum_{s=1}^{N-1} \left\| A^{-1/2} f_s \right\|_H \tau + \|\varphi\|_H + \left\| A^{-1/2}\psi \right\|_H + \tau \left\| A^{-1/2} f_{2,2} \right\|_H \right\}, \\
&\left\| A^{1/2} \frac{u_k + u_{k-1}}{2} \right\|_H \le M \left\{ \sum_{s=1}^{N-1} \|f_s\|_H \tau + \left\| A^{1/2}\varphi \right\|_H + \|\psi\|_H + \tau \|f_{2,2}\|_H \right\}, \\
&\left\| A \frac{u_k + u_{k-1}}{2} \right\|_H \le M \left\{ \sum_{s=2}^{N-1} \|f_s - f_{s-1}\|_H + \|f_1\|_H + \|A\varphi\|_H \right. \\
&\qquad \left. + \left\| A^{1/2}\psi \right\|_H + \tau \left\| A^{1/2} f_{2,2} \right\|_H \right\}, \\
&\left\| \frac{u_{k+1} - u_{k-1}}{2\tau} \right\|_H \le M \left\{ \sum_{s=1}^{N-1} \|f_s\|_H \tau + \left\| A^{1/2}\varphi \right\|_H + \|\psi\|_H + \tau \|f_{2,2}\|_H \right\}, \\
&\left\| A^{1/2} \frac{u_{k+1} - u_{k-1}}{2\tau} \right\|_H \le M \left\{ \sum_{s=2}^{N-1} \|f_s - f_{s-1}\|_H + \|f_1\|_H + \|A\varphi\|_H \right. \\
&\qquad \left. + \left\| A^{1/2}\psi \right\|_H + \tau \left\| A^{1/2} f_{2,2} \right\|_H \right\}.
\end{aligned}
$$

Now, we will establish these estimates for any $k \ge 2$. Using formula (81), we obtain

$$
\begin{aligned}
\frac{u_k + u_{k-1}}{2} ={}& \frac{1}{2} \left[J_1 R^{k-1} + \widetilde{J}_1 \widetilde{R}^{k-1} \right] \varphi + \frac{1}{2} \left[\widetilde{J}_1 \widetilde{R}^{k-1} - J_1 R^{k-1} \right] i A^{-1/2} \psi \\
&+ \frac{1}{2} \left[\widetilde{J}_2 \widetilde{R}^{k-1} - J_2 R^{k-1} \right] \tau^2 f_{2,2} + \frac{1}{2} \left(I + \frac{\tau^2 A}{12} + \frac{\tau^4 A^2}{144} \right)^{-1} \tau^2 f_{k-1} \\
&+ \frac{1}{2} \sum_{s=1}^{k-2} \left[\widetilde{J}_2 \widetilde{R}^{k-1-s} - J_2 R^{k-1-s} \right] \tau^2 f_s. \qquad (93)
\end{aligned}
$$

Using formula (93), estimate (76), and the triangle inequality, we get

$$
\begin{aligned}
&\left\| \frac{u_k + u_{k-1}}{2} \right\|_H \\
&\le \frac{1}{2} \left(\|J_1\|_{H\to H} \left\| R^{k-1} \right\|_{H\to H} + \|\widetilde{J}_1\|_{H\to H} \left\| \widetilde{R}^{k-1} \right\|_{H\to H} \right) \|\varphi\|_H
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}|i|\left(\|\widetilde{J}_1\|_{H\to H}\left\|\widetilde{R}^{k-1}\right\|_{H\to H}+\|J_1\|_{H\to H}\left\|R^{k-1}\right\|_{H\to H}\right)\left\|A^{-1/2}\psi\right\|_H \\
&+\frac{1}{2}\left(\left\|\tau A^{1/2}\widetilde{J}_2\right\|_{H\to H}\left\|\widetilde{R}^{k-1}\right\|_{H\to H}+\left\|\tau A^{1/2}J_2\right\|_{H\to H}\left\|R^{k-1}\right\|_{H\to H}\right) \\
&\times\tau\left\|A^{-1/2}f_{2,2}\right\|_H+\frac{1}{2}\left\|\tau A^{1/2}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\tau\left\|A^{-1/2}f_{k-1}\right\|_H \\
&+\frac{1}{2}\sum_{s=1}^{k-2}\left[\left\|\tau A^{1/2}\widetilde{J}_2\right\|_{H\to H}\left\|\widetilde{R}^{k-1-s}\right\|_{H\to H}\right. \\
&\left.+\left\|\tau A^{1/2}J_2\right\|_{H\to H}\left\|R^{k-1-s}\right\|_{H\to H}\right]\left\|A^{-1/2}f_s\right\|_H\tau \\
\le{}& M\left\{\sum_{s=1}^{N-1}\left\|A^{-1/2}f_s\right\|_H\tau+\|\varphi\|_H+\left\|A^{-1/2}\psi\right\|_H+\tau\left\|A^{-1/2}f_{2,2}\right\|_H\right\}
\end{aligned}
$$

for any $k \ge 2$. Combining the estimates $\|\frac{u_k+u_{k-1}}{2}\|_H$ for any k, we obtain (77).

Applying $A^{1/2}$ to (93), using estimate (76) and the triangle inequality, we get

$$
\begin{aligned}
&\left\|A^{1/2}\frac{u_k+u_{k-1}}{2}\right\|_H \\
\le{}&\frac{1}{2}\left(\|J_1\|_{H\to H}\left\|R^{k-1}\right\|_{H\to H}+\|\widetilde{J}_1\|_{H\to H}\left\|\widetilde{R}^{k-1}\right\|_{H\to H}\right) \\
&\times\left\|A^{1/2}\varphi\right\|_H+\frac{1}{2}|i|\left(\|\widetilde{J}_1\|_{H\to H}\left\|\widetilde{R}^{k-1}\right\|_{H\to H}+\|J_1\|_{H\to H}\left\|R^{k-1}\right\|_{H\to H}\right)\|\psi\|_H \\
&+\frac{1}{2}\left(\left\|\tau A^{1/2}\widetilde{J}_2\right\|_{H\to H}\left\|\widetilde{R}^{k-1}\right\|_{H\to H}+\left\|\tau A^{1/2}J_2\right\|_{H\to H}\left\|R^{k-1}\right\|_{H\to H}\right) \\
&\times\tau\|f_{2,2}\|_H+\frac{1}{2}\left\|\tau A^{1/2}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\tau\|f_{k-1}\|_H \\
&+\frac{1}{2}\sum_{s=1}^{k-2}\left[\left\|\tau A^{1/2}\widetilde{J}_2\right\|_{H\to H}\left\|\widetilde{R}^{k-1-s}\right\|_{H\to H}+\left\|\tau A^{1/2}J_2\right\|_{H\to H}\left\|R^{k-1-s}\right\|_{H\to H}\right]\|f_s\|_H\tau \\
\le{}& M\left\{\sum_{s=1}^{N-1}\|f_s\|_H\tau+\left\|A^{1/2}\varphi\right\|_H+\|\psi\|_H+\tau\|f_{2,2}\|_H\right\}
\end{aligned}
$$

for $k \ge 2$. Combining the estimates for $\|A^{1/2}\frac{u_k+u_{k-1}}{2}\|_H$ for any k, we get

$$
\begin{aligned}
&\max_{1\le k\le N}\left\|A^{1/2}\frac{u_k+u_{k-1}}{2}\right\|_H \\
\le{}& M\left\{\sum_{s=1}^{N-1}\|f_s\|_H\tau+\left\|A^{1/2}\varphi\right\|_H+\|\psi\|_H+\tau\|f_{2,2}\|_H\right\}.
\end{aligned}
\tag{94}
$$

Now, applying Abel's formula to (93), we obtain

$$
\begin{aligned}
\frac{u_k+u_{k-1}}{2}={}&\frac{1}{2}\left[J_1R^{k-1}+\widetilde{J}_1\widetilde{R}^{k-1}\right]\varphi+\frac{1}{2}\left[\widetilde{J}_1\widetilde{R}^{k-1}-J_1R^{k-1}\right]iA^{-1/2}\psi \\
&+\frac{1}{2}\left[\widetilde{J}_2\widetilde{R}^{k-1}-J_2R^{k-1}\right]\tau^2f_{2,2}+\frac{1}{2}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\tau^2f_{k-1}
\end{aligned}
$$

$$+\frac{1}{2}A^{-1}\left(-\sum_{s=2}^{k-2}\left[\widetilde{R}^{k-s}+R^{k-s}\right](f_s-f_{s-1})\right.$$
$$\left.+(\widetilde{R}+R)f_{k-2}-\left(\widetilde{R}^{k-1}+R^{k-1}\right)f_1\right). \tag{95}$$

Applying A to formula (95), using estimate (76) and the triangle inequality, we get

$$\begin{aligned}
&\left\|A\frac{u_k+u_{k-1}}{2}\right\|_H\\
&\quad\leq\frac{1}{2}\left(\|J_1\|_{H\to H}\left\|R^{k-1}\right\|_{H\to H}+\|\widetilde{J}_1\|_{H\to H}\left\|\widetilde{R}^{k-1}\right\|_{H\to H}\right)\\
&\qquad\times\|A\varphi\|_H+\frac{1}{2}|i|\left(\|\widetilde{J}_1\|_{H\to H}\left\|\widetilde{R}^{k-1}\right\|_{H\to H}+\|J_1\|_{H\to H}\left\|R^{k-1}\right\|_{H\to H}\right)\left\|A^{1/2}\psi\right\|_H\\
&\qquad+\frac{1}{2}\left(\left\|\tau A^{1/2}\widetilde{J}_2\right\|_{H\to H}\left\|\widetilde{R}^{k-1}\right\|_{H\to H}+\left\|\tau A^{1/2}J_2\right\|_{H\to H}\left\|R^{k-1}\right\|_{H\to H}\right)\\
&\qquad\times\tau\left\|A^{1/2}f_{2,2}\right\|_H+\frac{1}{2}\left\|\tau A^{1/2}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\tau\left\|A^{1/2}f_{k-1}\right\|_H\\
&\qquad+\frac{1}{2}\left(\sum_{s=2}^{k-2}\left(\left\|\widetilde{R}^{k-s}\right\|_{H\to H}+\left\|R^{k-s}\right\|_{H\to H}\right)\|f_s-f_{s-1}\|_H\right.\\
&\qquad\left.+\left(\|\widetilde{R}\|_{H\to H}+\|R\|_{H\to H}\right)\|f_{k-2}\|_H+\left(\left\|\widetilde{R}^{k-1}\right\|_{H\to H}+\left\|R^{k-1}\right\|_{H\to H}\right)\|f_1\|_H\right)\\
&\quad\leq M\left\{\sum_{s=2}^{N-1}\|f_s-f_{s-1}\|_H+\|f_1\|_H+\|A\varphi\|_H+\left\|A^{1/2}\psi\right\|_H+\tau\left\|A^{1/2}f_{2,2}\right\|_H\right\}.
\end{aligned}$$

Combining the estimates for $\|A\frac{u_k+u_{k-1}}{2}\|_H$ for any k, we obtain

$$\begin{aligned}
\max_{1\leq k\leq N-1}\left\|A\frac{u_k+u_{k-1}}{2}\right\|_H&\leq M\left\{\sum_{s=2}^{N-1}\|f_s-f_{s-1}\|_H+\|f_1\|_H+\left\|A^{1/2}\psi\right\|_H\right.\\
&\qquad\left.+\|A\varphi\|_H+\tau\left\|A^{1/2}f_{2,2}\right\|_H\right\}.
\end{aligned}\tag{96}$$

Using formula (81), we get

$$\begin{aligned}
\frac{u_{k+1}-u_{k-1}}{2\tau}&=\frac{1}{2\tau}\left[J_3R^{k-1}+\widetilde{J}_3\widetilde{R}^{k-1}\right]\varphi+\frac{1}{2\tau}\left[\widetilde{J}_3\widetilde{R}^{k-1}-J_3R^{k-1}\right]iA^{-1/2}\psi\\
&\quad+\frac{1}{2\tau}\left[\widetilde{R}^k+R^k\right]\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\tau^2f_{2,2}\\
&\quad+\frac{1}{2\tau}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\tau^2f_k\\
&\quad+\frac{1}{2\tau}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}[\widetilde{R}+R]\tau^2f_{k-1}\\
&\quad+\frac{1}{2\tau}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\sum_{s=1}^{k-2}\left[\widetilde{R}^{k-s}+R^{k-s}\right]f_s\tau^2.
\end{aligned}\tag{97}$$

Next, using formula (97), estimate (76), and the triangle inequality, we obtain

$$\begin{aligned}\left\|\frac{u_{k+1}-u_{k-1}}{2\tau}\right\|_H &\le \frac{1}{2\tau}\big(\|A^{-1/2}J_3\|_{H\to H}\|R^{k-1}\|_{H\to H}\\
&\quad + \|A^{-1/2}\widetilde{J}_3\|_{H\to H}\|\widetilde{R}^{k-1}\|_{H\to H}\big)\|A^{1/2}\varphi\|_H\\
&\quad + \frac{1}{2}|i|\big(\|A^{-1/2}\widetilde{J}_3\|_{H\to H}\|\widetilde{R}^{k-1}\|_{H\to H}\\
&\quad + \|A^{-1/2}J_3\|_{H\to H}\|R^{k-1}\|_{H\to H}\big)\|\psi\|_H\\
&\quad + \frac{1}{2}\big(\|\widetilde{R}^k\|_{H\to H} + \|R^k\|_{H\to H}\big)\left\|\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\\
&\quad \times \|f_{2,2}\|_H\tau + \frac{1}{2}\left\|\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\|f_k\|_H\tau\\
&\quad + \frac{1}{2}\big(\|\widetilde{R}\|_{H\to H} + \|R\|_{H\to H}\big)\left\|\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\\
&\quad \times \|f_{k-1}\|_H\tau + \frac{1}{2}\left\|\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\\
&\quad \times \sum_{s=1}^{k-2}\big(\|\widetilde{R}^{k-s}\|_{H\to H} + \|R^{k-s}\|_{H\to H}\big)\|f_s\|_H\tau\\
&\le M\left\{\sum_{s=1}^{N-1}\|f_s\|_H\tau + \|A^{1/2}\varphi\|_H + \|\psi\|_H + \tau\|f_{2,2}\|_H\right\}. \qquad (98)\end{aligned}$$

Combining the estimates for $\|\frac{u_{k+1}-u_{k-1}}{2\tau}\|_H$ for any k, we get

$$\max_{1\le k\le N-1}\left\|\frac{u_{k+1}-u_{k-1}}{2\tau}\right\|_H \le M\left\{\sum_{s=1}^{N-1}\|f_s\|_H\tau + \|A^{1/2}\varphi\|_H + \|\psi\|_H + \tau\|f_{2,2}\|_H\right\}. \qquad (99)$$

From estimates (94) and (99), we obtain (78).

Now, applying Abel's formula to (97), we have

$$\begin{aligned}\frac{u_{k+1}-u_{k-1}}{2\tau} &= \frac{1}{2\tau}\big[J_3R^{k-1}+\widetilde{J}_3\widetilde{R}^{k-1}\big]\varphi + \frac{1}{2\tau}\big[\widetilde{J}_3\widetilde{R}^{k-1}-J_3R^{k-1}\big]iA^{-1/2}\psi\\
&\quad + \frac{1}{2\tau}\big[\widetilde{R}^k+R^k\big]\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\tau^2f_{2,2}\\
&\quad + \frac{1}{2\tau}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\tau^2f_k\\
&\quad + \frac{1}{2\tau}\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}[\widetilde{R}+R]\tau^2f_{k-1}\\
&\quad + \frac{1}{2}\tau\sum_{s=2}^{k-2}\big[\widetilde{J}_2\widetilde{R}^{k-s}+J_2R^{k-s}\big](f_s-f_{s-1})\\
&\quad - \frac{1}{2}\tau[\widetilde{J}_2\widetilde{R}+J_2R]f_{k-2} + \frac{1}{2}\tau\big[\widetilde{J}_2\widetilde{R}^{k-1}+J_2R^{k-1}\big]f_1\}, \quad 2\le k\le N. \qquad (100)\end{aligned}$$

Next, applying $A^{1/2}$ to formula (100), using estimate (76) and the triangle inequality, we obtain

$$\begin{aligned}
&\left\|A^{1/2}\frac{u_{k+1}-u_{k-1}}{2\tau}\right\|_H \\
&\quad\leq \frac{1}{2\tau}\left(\left\|A^{-1/2}J_3\right\|_{H\to H}\left\|R^{k-1}\right\|_{H\to H}\right. \\
&\qquad\left. + \left\|A^{-1/2}\widetilde{J}_3\right\|_{H\to H}\left\|\widetilde{R}^{k-1}\right\|_{H\to H}\right)\|A\varphi\|_H \\
&\qquad + \frac{1}{2}|i|\left(\left\|A^{-1/2}\widetilde{J}_3\right\|_{H\to H}\left\|\widetilde{R}^{k-1}\right\|_{H\to H} + \left\|A^{-1/2}J_3\right\|_{H\to H}\left\|R^{k-1}\right\|_{H\to H}\right)\left\|A^{1/2}\psi\right\|_H \\
&\qquad + \frac{1}{2}\left(\left\|\widetilde{R}^k\right\|_{H\to H} + \left\|R^k\right\|_{H\to H}\right)\left\|\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H} \\
&\qquad \times \left\|A^{1/2}f_{2,2}\right\|_H\tau + \frac{1}{2}\left\|\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H}\left\|A^{1/2}f_k\right\|_H\tau \\
&\qquad + \frac{1}{2}\left(\|\widetilde{R}\|_{H\to H} + \|R\|_{H\to H}\right)\left\|\left(I+\frac{\tau^2A}{12}+\frac{\tau^4A^2}{144}\right)^{-1}\right\|_{H\to H} \\
&\qquad \times \left\|A^{1/2}f_{k-1}\right\|_H\tau + \frac{1}{2}\sum_{s=2}^{k-2}\left(\left\|\tau A^{1/2}\widetilde{J}_2\right\|_{H\to H}\left\|\widetilde{R}^{k-s}\right\|_{H\to H}\right. \\
&\qquad \left. + \left\|\tau A^{1/2}J_2\right\|_{H\to H}\left\|R^{k-s}\right\|_{H\to H}\right)\|f_s-f_{s-1}\|_H \\
&\qquad + \frac{1}{2}\left[\left\|\tau A^{1/2}\widetilde{J}_2\right\|_{H\to H}\|\widetilde{R}\|_{H\to H} + \left\|\tau A^{1/2}J_2\right\|_{H\to H}\|R\|_{H\to H}\right]\|f_{k-2}\|_H \\
&\qquad + \frac{1}{2}\left[\left\|\tau A^{1/2}\widetilde{J}_2\right\|_{H\to H}\left\|\widetilde{R}^{k-1}\right\|_{H\to H} + \left\|\tau A^{1/2}J_2\right\|_{H\to H}\left\|R^{k-1}\right\|_{H\to H}\right]\|f_1\|_H \\
&\quad\leq M\left\{\sum_{s=2}^{N-1}\|f_s-f_{s-1}\|_H + \|f_1\|_H + \|A\varphi\|_H + \left\|A^{1/2}\psi\right\|_H + \tau\left\|A^{1/2}f_{2,2}\right\|_H\right\}.
\end{aligned}$$

Combining the estimates for $\|A^{1/2}\frac{u_{k+1}-u_{k-1}}{2\tau}\|_H$ for any k, we get

$$\begin{aligned}
&\max_{1\leq k\leq N-1}\left\|A^{1/2}\frac{u_{k+1}-u_{k-1}}{2\tau}\right\|_H \\
&\quad\leq M\left\{\sum_{s=2}^{N-1}\|f_s-f_{s-1}\|_H + \|f_1\|_H + \left\|A^{1/2}\psi\right\|_H + \|A\varphi\|_H + \tau\left\|A^{1/2}f_{2,2}\right\|_H\right\}.
\end{aligned} \tag{101}$$

From estimates (96) and (101), estimate (78) follows. Theorem 2 is proved. □

3 Numerical analysis

In the present section, finite difference method is used and two numerical examples and some numerical results are presented in order to support our theoretical statements.

Generally, we have not been able to determine a sharp estimate for the constant figuring in the stability inequalities. However, the numerical results are presented by considering the Cauchy problem

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial^2 u(t,x)}{\partial x^2} = 2e^{-t}\sin x, \\ \quad 0 < t < 1, 0 < x < \pi, \\ u(0,x) = \sin x, \quad 0 \le x \le \pi, \\ u_t(0,x) = -\sin x, \quad 0 \le x \le \pi, \\ u(t,0) = u(t,\pi) = 0, \quad 0 \le t \le 1, \end{cases} \tag{102}$$

for a one-dimensional hyperbolic equation. The exact solution of this problem is $u(t,x) = e^{-t}\sin x$.

We consider the set $[0,1]_\tau \times [0,\pi]_h$ of a family of grid points depending on the small parameters τ and h: $[0,1]_\tau \times [0,\pi]_h = \{(t_k,x_n) : t_k = k\tau, 0 \le k \le N, N\tau = 1, x_n = nh, 0 \le n \le M, Mh = \pi\}$. First, for the approximate solution of problem (102), we have applied the third and fourth orders of accuracy difference schemes respectively.

First, let us consider the third order of accuracy in time difference scheme for the approximate solution of Cauchy problem (102). Using difference scheme (2), we obtain

$$\begin{cases} \frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2} - \frac{2}{3}(\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2}) \\ \quad - \frac{1}{6}(\frac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2} + \frac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{h^2}) \\ \quad + \frac{\tau^2}{12}(\frac{u_{n+2}^{k+1}-4u_{n+1}^{k+1}+6u_n^{k+1}-4u_{n-1}^{k+1}+u_{n-2}^{k+1}}{h^4}) = \varphi_n^k, \\ \varphi_n^k = (\frac{4}{3}\exp(-t_k) + \frac{1}{3}(\exp(-t_{k+1}) + \exp(-t_{k-1})))\sin(x_n), \\ \quad t_k = k\tau, 1 \le k \le N-1, N\tau = 1, x_n = nh, 2 \le n \le M-2, Mh = \pi, \\ u_n^0 = \varphi_n^0, \qquad \varphi_n^0 = \sin(x_n), \quad 0 \le n \le M, \\ u_n^1 - u_n^0 - \frac{\tau^2}{12}(\frac{(u_{n+1}^1-u_{n+1}^0)-2(u_n^1-u_n^0)+(u_{n-1}^1-u_{n-1}^0)}{h^2}) \\ \quad + \frac{\tau^4}{144}(\frac{u_{n+2}^1-u_{n+2}^0-4(u_{n+1}^1-u_{n+1}^0)+6(u_n^1-u_n^0)}{h^4} \frac{-4(u_{n-1}^1-u_{n-1}^0)+u_{n-2}^1-u_{n-2}^0}{h^4}) = \varphi_n^N, \\ \varphi_n^N = -\frac{\tau^2}{2}A\sin(x_n) + \tau(I - \frac{\tau^2}{12}A)(-\sin(x_n)) + f_{1,1}\tau, \\ \quad 2 \le n \le M-2, f_{1,1} = (f(0) + (-f(0) + \tau f'(0))\frac{1}{2} - 2f'(0)\frac{\tau}{6}), \\ u_0^k = u_M^k = 0, \quad 0 \le k \le N, \\ u_3^k = 4u_2^k - 5u_1^k, \qquad u_{M-3}^k = 4u_{M-2}^k - 5u_{M-1}^k, \quad 0 \le k \le N, \end{cases} \tag{103}$$

for the approximate solution of Cauchy problem (102). We have $(N+1) \times (N+1)$ system of linear equations in (103) which can be written in the matrix form as follows:

$$\begin{aligned} &Au_{n+2} + Bu_{n+1} + Cu_n + Du_{n-1} + Eu_{n-2} = R\varphi_n, \quad 2 \le n \le M-2, \\ &u_0 = u_M = \widetilde{0}, \qquad u_3^k = 4u_2^k - 5u_1^k, \qquad u_{M-3}^k = 4u_{M-2}^k - 5u_{M-1}^k, \quad 0 \le k \le N, \end{aligned} \tag{104}$$

where

$$A = \begin{bmatrix} 0 & 0 & \cdot & 0 & 0 & 0 \\ x & 0 & \cdot & 0 & 0 & 0 \\ 0 & x & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 0 & x & 0 \\ r & s & \cdot & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)}, \tag{105}$$

$$B=\begin{bmatrix} 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ y & z & t & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & y & z & t & \cdot & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & 0 & y & z & t \\ v & w & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)}, \tag{106}$$

$$C=\begin{bmatrix} 1 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ k & m & n & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & k & m & n & \cdot & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & 0 & k & m & n \\ c & d & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)}, \tag{107}$$

$$D=B, \qquad E=A, \tag{108}$$

$$R=\begin{bmatrix} 1 & \cdot & 0 \\ \cdot & \cdot & \cdot \\ 0 & \cdot & 1 \end{bmatrix}_{(N+1)\times(N+1)}, \tag{109}$$

$$\varphi_n^k=\begin{bmatrix} \varphi_n^0 \\ \varphi_n^1 \\ \cdot \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1}, \quad 0\le k\le N, \tag{110}$$

$$\varphi_n^0=\sin(x_n), \qquad \varphi_n^N=\left(\tau-\frac{\tau^2}{3}\right)\sin(x_n), \tag{111}$$

$$\varphi_n^k=\left(\frac{4}{3}\exp(-t_k)+\frac{1}{3}\bigl(\exp(-t_{k+1})+\exp(-t_{k-1})\bigr)\right)\sin(x_n), \quad 1\le k\le N-1, \tag{112}$$

and

$$u_s^k=\begin{bmatrix} u_s^0 \\ u_s^1 \\ \cdot \\ u_s^N \end{bmatrix}_{(N+1)\times 1}, \quad 0\le k\le N, s=n-2,n-1,n,n+1,n+2. \tag{113}$$

Note that

$$x=\frac{\tau^2}{12h^4}, \qquad t=-\frac{1}{6h^2}-\frac{\tau^2}{3h^4}, \qquad z=-\frac{2}{3h^2}, \tag{114}$$

$$y=-\frac{1}{6h^2}, \qquad n=\frac{1}{\tau^2}+\frac{1}{3h^2}+\frac{\tau^2}{2h^4}, \tag{115}$$

$$m=-\frac{2}{\tau^2}+\frac{4}{3h^2}, \qquad k=\frac{1}{\tau^2}+\frac{1}{3h^2}, \qquad r=-\frac{\tau^2}{144h^4}, \qquad s=\frac{\tau^2}{144h^4}, \tag{116}$$

$$v=\frac{\tau^4}{36h^4}+\frac{\tau^2}{12h^2}, \qquad w=-\frac{\tau^4}{36h^4}-\frac{\tau^2}{12h^2}, \tag{117}$$

$$c=-1-\frac{\tau^2}{6h^2}-\frac{\tau^4}{24h^4}, \qquad d=1+\frac{\tau^2}{6h^2}+\frac{\tau^4}{24h^4}. \tag{118}$$

For a solution of difference equation (103), we have applied the procedure of the modified Gauss elimination method with respect to n with matrix coefficients. Therefore, we seek a solution of the matrix equation by using the following iteration formula:

$$u_n = \alpha_{n+1}u_{n+1} + \beta_{n+1}u_{n+2} + \gamma_{n+1}, \quad n = M-2,\dots,2,1,0, \tag{119}$$

where α_j, β_j $(j = 1,\dots,M)$ are $(N+1)\times(N+1)$ square matrices and γ_j are $(N+1)\times 1$ column matrices. Now, we obtain the formulas of α_{n+1}, β_{n+1}, γ_{n+1} as

$$\begin{aligned}
\beta_{n+1} &= -(C + D\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n)^{-1}(A),\\
\alpha_{n+1} &= -(C + D\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n)^{-1}(B + D\beta_n + E\alpha_{n-1}\beta_n),\\
\gamma_{n+1} &= +(C + D\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n)^{-1}(R\varphi_n - D\gamma_n - E\alpha_{n-1}\gamma_n - E\gamma_{n-1})
\end{aligned} \tag{120}$$

for $n = 2 : M-2$ and α_1, β_1, γ_1, α_2, β_2, γ_2

$$\alpha_1 = \begin{bmatrix} 0 & \cdot & 0 \\ \cdot & \cdot & \cdot \\ 0 & \cdot & 0 \end{bmatrix}_{(N+1)\times(N+1)}, \qquad \beta_1 = \begin{bmatrix} 0 & \cdot & 0 \\ \cdot & \cdot & \cdot \\ 0 & \cdot & 0 \end{bmatrix}_{(N+1)\times(N+1)}, \tag{121}$$

$$\gamma_1 = \gamma_2 = \begin{bmatrix} 0 \\ \cdot \\ 0 \end{bmatrix}_{(N+1)\times 1}, \qquad \alpha_2 = \begin{bmatrix} \frac{4}{5} & 0 & \cdot & 0 \\ 0 & \frac{4}{5} & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \frac{4}{5} \end{bmatrix}_{(N+1)\times(N+1)}, \tag{122}$$

$$\beta_2 = \begin{bmatrix} -\frac{1}{5} & 0 & \cdot & 0 \\ 0 & -\frac{1}{5} & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & -\frac{1}{5} \end{bmatrix}_{(N+1)\times(N+1)}, \tag{123}$$

and $u_M = \widetilde{0}$. Applying the formulas

$$\begin{aligned}
u_{M-2} &= \alpha_{M-1}u_{M-1} + \gamma_{M-1},\\
u_{M-3} &= \alpha_{M-2}u_{M-2} + \beta_{M-2}u_{M-1} + \gamma_{M-2},\\
u_{M-3} &= 4u_{M-2} - 5u_{M-1},
\end{aligned} \tag{124}$$

we get

$$(4I - \alpha_{M-2})(\alpha_{M-1}u_{M-1} + \gamma_{M-1}) - (5I + \beta_{M-1})u_{M-1} = \gamma_{M-2}. \tag{125}$$

From that, it follows

$$u_{M-1} = \bigl[(\beta_{M-2} + 5I) - (4I - \alpha_{M-2})\alpha_{M-1}\bigr]^{-1}\bigl[(4I - \alpha_{M-2})\gamma_{M-1} - \gamma_{M-2}\bigr]. \tag{126}$$

Thus, using the formulas and matrices above, the third order of approximation difference scheme with matrix coefficients for the approximate solution of Cauchy problem (102) is obtained.

Second, let us consider the fourth order of accuracy in time difference scheme (3) for the approximate solution of Cauchy problem (102).

Using difference scheme (3), we obtain

$$\begin{cases} \frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2} - \frac{5}{6}(\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2}) \\ \quad - \frac{1}{12}(\frac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2} + \frac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{h^2}) \\ \quad - \frac{\tau^2}{72}(\frac{u_{n+2}^k-4u_{n+1}^k+6u_n^k-4u_{n-1}^k+u_{n-2}^k}{h^4}) \\ \quad + \frac{\tau^2}{144}(\frac{u_{n+2}^{k+1}-4u_{n+1}^{k+1}+6u_n^{k+1}-4u_{n-1}^{k+1}+u_{n-2}^{k+1}}{h^4} \\ \quad + \frac{u_{n+2}^{k-1}-4u_{n+1}^{k-1}+6u_n^{k-1}-4u_{n-1}^{k-1}+u_{n-2}^{k-1}}{h^4}) = \varphi_n^k, \\ \varphi_n^k = (\frac{5}{3}(\exp(-t_k)) + \frac{1}{6}(\exp(-t_{k+1}) + \exp(-t_{k-1}))) \sin(x_n), \\ \quad t_k = k\tau, 1 \le k \le N-1, N\tau = 1, x_n = nh, 1 \le n \le M-1, Mh = \pi, \\ u_n^0 = \varphi_n^0, \qquad \varphi_n^0 = \sin(x_n), \quad 0 \le n \le M, \\ u_n^1 - u_n^0 - \frac{\tau^2}{12}(\frac{(u_{n+1}^1-u_{n+1}^0)-2(u_n^1-u_n^0)+(u_{n-1}^1-u_{n-1}^0)}{h^2}) \\ \quad + \frac{\tau^4}{144}(\frac{u_{n+2}^1-u_{n+2}^0-4(u_{n+1}^1-u_{n+1}^0)+6(u_n^1-u_n^0)}{h^4} \frac{-4(u_{n-1}^1-u_{n-1}^0)+u_{n-2}^1-u_{n-2}^0}{h^4}) = \varphi_n^N, \\ \varphi_n^N = -\frac{\tau^2}{2}A\sin(x_n) + \tau(I - \frac{\tau^2}{12}A)(-\sin(x_n)) + \tau f_{2,2}, \\ \quad f_{2,2} = (I - \frac{\tau^2 A}{12})f(0)\tau - ((I - \frac{5\tau^2 A}{12}) - \tau f'(0))\frac{\tau}{2} \\ \qquad + (-A\tau f(0) - 2f'(0) + \tau f''(0))\frac{\tau^2}{6} + (Af(0) - 3f''(0))\frac{\tau^3}{24}, \\ \quad 2 \le n \le M-2, u_0^k = u_M^k = 0, 0 \le k \le N, \\ u_3^k = 4u_2^k - 5u_1^k, \qquad u_{M-3}^k = 4u_{M-2}^k - 5u_{M-1}^k, \quad 0 \le k \le N, \end{cases} \tag{127}$$

for an approximate solution of Cauchy problem (102). We have again the same $(N+1) \times (N+1)$ system of linear equations (104), where

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ x & y & x & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & x & y & x & \cdot & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & 0 & x & y & x \\ a & aa & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)}, \tag{128}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ v & w & v & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & v & w & v & \cdot & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & 0 & v & w & v \\ p & q & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)}, \tag{129}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ m & n & m & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & m & n & m & \cdot & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & 0 & m & n & m \\ c & d & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)}, \tag{130}$$

$$D = B, \qquad E = A, \tag{131}$$

$$R = \begin{bmatrix} 1 & \cdot & 0 \\ \cdot & \cdot & \cdot \\ 0 & \cdot & 1 \end{bmatrix}_{(N+1)\times(N+1)}, \tag{132}$$

and

$$u_s^k = \begin{bmatrix} u_s^0 \\ u_s^1 \\ \cdot \\ u_s^N \end{bmatrix}_{(N+1)\times 1}, \quad 0 \le k \le N, s = n-2, n-1, n, n+1, n+2, \tag{133}$$

$$\varphi_n^k = \begin{bmatrix} \varphi_n^0 \\ \varphi_n^1 \\ \cdot \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1}, \quad 0 \le k \le N, \tag{134}$$

$$\begin{aligned} &\varphi_n^0 = \sin(x_n), \\ &\varphi_n^N = \left(-\tau + \frac{\tau^2}{2} - \frac{\tau^3}{4} + \frac{\tau^4}{12} - \frac{\tau^5}{36} + \frac{\tau^6}{16} - \frac{\tau^7}{108} - \frac{\tau^8}{864}\right)\sin(x_n), \\ &\varphi_n^k = \left(\frac{5}{3}\big(\exp(-t_k)\big) + \frac{1}{6}\big(\exp(-t_{k+1}) + \exp(-t_{k-1})\big)\right)\sin(x_n)\sin(x_n), \\ &\quad 1 \le k \le N-1. \end{aligned} \tag{135}$$

Note that

$$x = \frac{\tau^2}{144h^4}, \qquad y = -\frac{\tau^2}{72h^4}v = -\frac{1}{12h^2} - \frac{4\tau^2}{144h^4}, \tag{136}$$

$$w = -\frac{5}{6h^2} + \frac{4\tau^2}{72h^4}, \qquad m = \frac{1}{\tau^2} + \frac{1}{6h^2} + \frac{\tau^2}{24h^4}, \tag{137}$$

$$\begin{aligned} &n = -\frac{2}{\tau^2} + \frac{5}{3h^2} - \frac{\tau^2}{12h^4}, \qquad r = -\frac{\tau^2}{144h^4}, \\ &s = \frac{\tau^2}{144h^4}, \qquad p = \frac{\tau^4}{36h^4} + \frac{\tau^2}{12h^2}, \end{aligned} \tag{138}$$

$$q = -\frac{\tau^4}{36h^4} - \frac{\tau^2}{12h^2}, \qquad c = -1 - \frac{\tau^2}{6h^2} - \frac{\tau^4}{24h^4}, \qquad cc = 1 + \frac{\tau^2}{6h^2} + \frac{\tau^4}{24h^4}. \tag{139}$$

For the solution of difference equation (127), we use exactly the same method that we have used for the solution of difference equation (103). Thus, using the formulas and matrices above, the fourth order of approximation difference scheme with matrix coefficients for the approximate solution of Cauchy problem (102) is obtained. The implementations of the numerical experiments are carried out by Matlab.

The errors are computed by the following formula:

$$E_M^N = \max_{1 \le k \le N-1, 1 \le k \le M-1} \left| u(t_k, x_n) - u_n^k \right|. \tag{140}$$

Table 1 Errors for the approximate solutions of problem (102)

Method	$N = 10, M = 100$	$N = 20, M = 400$	$N = 40, M = 1600$
Difference scheme (2)	0,0001504	0,00001952	0,000002241
Difference scheme (3)	0,00002459	0,000001628	0,0000001047

Here, $u(t_k, x_n)$ represents the exact solution and u_n^k represents the numerical solution at (t_k, x_n). The errors are presented in Table 1.

In the table, the results are presented for different M and N values which are the step numbers for time and space variables respectively.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
OY carried out the studies, participated in the sequence alignment and drafted the manuscript and AA carried out the studies, participated in the sequence alignment.

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Acknowledgements
The authors would like to thank Prof. Dr. P. E. Sobolevskii for his helpful suggestions to the improvement of this paper.

Received: 30 September 2012 Accepted: 29 January 2013 Published: 15 February 2013

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doi:10.1186/1687-2770-2013-29
Cite this article as: Ashyralyev and Yildirim: **On the numerical solution of hyperbolic IBVP with high-order stable finite difference schemes.** *Boundary Value Problems* 2013 **2013**:29.