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Existence and multiplicity of positive bound states for Schrödinger equations

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Abstract

In this paper, we study the existence and multiplicity of positive bound states of non-autonomous systems of nonlinear Schrödinger equations. The proof is based on the fixed point theorems in a cone.

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1 Introduction

Because of the important background in nonlinear optics and other fields, many authors pay more attention to the study of different types of vector nonlinear Schrödinger equations, we refer the readers to [1–5]. Most of these results have been proven using critical point theory, variational approaches or a fixed point theorem. More recently, Chu has applied another topological approach, a nonlinear alternative principle of Leray-Schauder, to establish some new existence results for the following Schrödinger equations

$$-\ddot{u}(x) + a(x)u(x) = b(x)f(u(x)) + e(x),$$

where $a \in L^\infty(\mathbb{R}, \mathbb{R}^N)$ is nonnegative almost everywhere, $e \in L^1(\mathbb{R}, \mathbb{R}^N)$. The author considered two different cases. One is the singular case, that is, $f \in C(\mathbb{R}^N \setminus \{0\}, \mathbb{R}^N)$ and

$$\lim_{u \rightarrow 0^+} f_i(u) = +\infty, \quad i = 1, 2, \dots, n.$$

The other is the regular case, that is, $f \in C(\mathbb{R}^N, \mathbb{R}^N)$. However, the references [4–6] are not concerned with multiplicity of positive solutions for the scalar Schrödinger equation or system.

Motivated by the study of solitary wave solutions, in this paper we mainly aim to study the existence and multiplicity of positive bound states of the more general system of non-autonomous Schrödinger equations

$$\begin{cases} -\ddot{u}(x) + A(x)u(x) = C(x)F(u(x)), \\ \lim_{|x| \rightarrow \infty} u(x) = \lim_{|x| \rightarrow \infty} \dot{u}(x) = 0, \\ \int_{-\infty}^{+\infty} u^2(x) dx + \int_{-\infty}^{+\infty} \dot{u}^2(x) dx < +\infty, \end{cases} \quad (1)$$

where

$$u(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \\ \vdots \\ u_n(x) \end{pmatrix}, \quad A(x) = \begin{pmatrix} a_{11}(x) & a_{12}(x) & \cdots & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) & \cdots & a_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(x) & a_{n2}(x) & \cdots & a_{nn}(x) \end{pmatrix},$$

$$C(x) = \begin{pmatrix} c_1(x) & 0 & \cdots & 0 \\ 0 & c_2(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_n(x) \end{pmatrix}, \quad F(u(x)) = \begin{pmatrix} f_1(u_1(x), u_2(x), \dots, u_n(x)) \\ f_2(u_1(x), u_2(x), \dots, u_n(x)) \\ \vdots \\ f_n(u_1(x), u_2(x), \dots, u_n(x)) \end{pmatrix}.$$

The methods used here are the Krasnoselskii fixed point theorem and the Leggett-Williams fixed point theorem together with a compactness criterion due to Zima.

We organize the paper as follows. In Section 2, we give some preliminaries; in Section 3, we discuss the existence and multiplicity of positive solutions for (1).

2 Preliminaries

For convenience, we assume that the following conditions hold throughout this paper.

(H1) $a_{ij} \in L^\infty(R, R)$, and $a_{ij}(x)$ satisfies the following property

$$a_{ij}(x) = \begin{cases} a_{ii}(x) \geq 0, & \text{and } \inf a_{ii}(x) > 0 \text{ if } i = j, \\ a_{ij}(x) \leq 0, & \text{and } \text{Supp}(a_{ij}) \text{ is a nonempty compact set if } i \neq j. \end{cases}$$

(H2) The support of $c_i(x) > 0$ denoted by $\text{Supp}(c_i)$ is a nonempty compact set, and

$$0 < \int_M G_i(s, s) c_i(s) ds < +\infty.$$

(H3) $f_i(u(x)) \in C(R^N, R^+)$ is continuous, and the following notations are introduced:

$$f_{i,0} = \lim_{u \rightarrow 0} \frac{f_i(u)}{\sum_{i=1}^n u_i}, \quad f_{i,\infty} = \lim_{u \rightarrow \infty} \frac{f_i(u)}{\sum_{i=1}^n u_i}.$$

Since (H1) holds, then for the homogeneous problem

$$\begin{cases} -\ddot{\phi}(x) + a_{ii}(x)\phi(x) = 0, \\ \phi(-\infty) = 0, \quad \phi(+\infty) = 0, \end{cases}$$

the associated Green's function is expressed by

$$G_i(x, s) = \begin{cases} \phi_i^1(x)\phi_i^2(s), & -\infty < x \leq s < +\infty, \\ \phi_i^1(s)\phi_i^2(x), & -\infty < s \leq x < +\infty, \end{cases}$$

where ϕ_i^1, ϕ_i^2 are solutions such $\phi_i^1(-\infty) = 0, \phi_i^2(+\infty) = 0$. Moreover, ϕ_i^1, ϕ_i^2 can be chosen as positive increasing and positive decreasing functions, respectively. Note that ϕ_i^1, ϕ_i^2

intersect at a unique point x_0 . Therefore, we can define a function $p_i(x) \in BC(R)$ by

$$p_i(x) = \begin{cases} \frac{1}{\phi_i^2(x)}, & x \leq x_0, \\ \frac{1}{\phi_i^1(x)}, & x > x_0, \end{cases}$$

where $BC(R)$ denotes the space of bounded continuous functions.

Lemma 2.1 [4] *For each $i = 1, 2, \dots, n$, Green's function $G_i(x, s)$ satisfies the following properties:*

- (i) $G_i(x, s) > 0$ for every $(x, s) \in R \times R$;
- (ii) $G_i(x, s) \leq G_i(s, s)$ for every $(x, s) \in R \times R$;
- (iii) Given a nonempty compact subset $P \subset R$, we have

$$G_i(x, s) \geq m_i(P)p_i(s)G_i(s, s) \quad \text{for all } (x, s) \in P \times R,$$

where $m_i(P) = \min\{\phi_i^1(\inf P), \phi_i^2(\inf P)\}$.

Lemma 2.2 [4] *Assume that (H1) holds and $e(x) \in L^1(R)$. Then the unique solution of*

$$\begin{cases} -\ddot{u}_i(x) + a_{ii}(x)u_i(x) = e(x), \\ u_i(-\infty) = 0, \quad u_i(+\infty) = 0 \end{cases}$$

belongs to $H^1(R)$, and the solution can be expressed as

$$u_i(x) = \int_R G_i(x, s)e(s) ds.$$

The proof of our main results is based on the following fixed points, which can be found in [7].

Lemma 2.3 *Let E be a Banach space, and let $K \subset E$ be a cone in E . Assume that Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$, and let $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \rightarrow K$ be a completely continuous operator such that either*

- (i) $\|Tu\| \leq \|u\|, u \in K \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|, u \in K \cap \partial\Omega_2$; or
- (ii) $\|Tu\| \geq \|u\|, u \in K \cap \partial\Omega_1$ and $\|Tu\| \leq \|u\|, u \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Let E be a real Banach space and P be a cone in E . A map α is said to be a nonnegative continuous concave functional on P if

$$\alpha : P \rightarrow [0, +\infty)$$

is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in P$ and $t \in [0, 1]$.

For numbers a, b such that $0 < a < b$, letting α be a nonnegative continuous concave functional on P , we define the following convex sets:

$$P_a = \{x \in P : \|x\| < a\}$$

and

$$P(\alpha, a, b) = \{x \in P : a \leq \alpha(x), \|x\| \leq b\}.$$

Lemma 2.4 *Let $T : \bar{P}_c \rightarrow \bar{P}_c$ be completely continuous and α be a nonnegative continuous concave functional on P such that $\alpha(x) \leq \|x\|$ for all $x \in \bar{P}_c$. Suppose that there exist $0 < d < a < b \leq c$ such that*

- (i) $\{x \in P(\alpha, a, b) : \alpha(x) > a\} \neq \emptyset$ and $\alpha(Tx) > a$ for $x \in P(\alpha, a, b)$;
- (ii) $\|Tx\| < d$ for $\|x\| \leq d$;
- (iii) $\alpha(Tx) > a$ for $x \in P(\alpha, a, c)$ with $\|Tx\| > b$.

Then T has at least three fixed points x_1, x_2, x_3 satisfying

$$\begin{aligned} \|x_1\| < d, \quad a < \alpha(x_2), \\ \|x_3\| > d \quad \text{and} \quad \alpha(x_3) < a. \end{aligned}$$

In addition, the following compactness criterion proved by Zima in [8] is also used in our proof.

Lemma 2.5 *Let $\Omega \subset BC(R)$. Let us assume that the functions $u \in \Omega$ are equicontinuous in each compact interval of R and that for all $u \in \Omega$, we have*

$$|u(x)| \leq \xi(x), \quad \forall x \in R,$$

where $\xi \in BC(R)$ verifies

$$\lim_{|x| \rightarrow +\infty} \xi(x) = 0.$$

Then Ω is relatively compact.

3 Main results

From now on, we assume that $M = \bigcup_{i \neq j} \text{Supp}(a_{ij}) \cup \text{Supp}(c_i)$ is a nonempty compact set. Let

E denote the Banach space $\overbrace{BC(R) \times BC(R) \times \cdots \times BC(R)}^n$ with the norm $\|u\| = \sum_{i=1}^n |u_i|_\infty$, $|u_i|_\infty = \max_{x \in M} |u_i(x)|$ for $u = (u_1, u_2, \dots, u_n) \in E$. Define a cone $K \subset E$ as

$$K = \left\{ u = (u_1, u_2, \dots, u_n) \in E : u_i(x) \geq 0 \text{ and } \min_{x \in M} \sum_{i=1}^n u_i(x) \geq \delta \|u\| \right\},$$

where $\delta = \min_{i=1,2,\dots,n} \{m_i p_0^i\} \in (0, 1)$, $p_0^i = \inf_M p_i(x)$ and the constants $m_i \equiv m_i(M)$, $i = 1, 2, \dots, n$, are defined by property (iii) of Lemma 2.1. Since M is compact, then $p_0^i > 0$, $i = 1, 2, \dots, n$. Moreover, from (iii) of Lemma 2.1 it follows that $m_i p_0^i < 1$ for $i = 1, 2, \dots, n$.

Let $T : K \rightarrow E$ be a map with components (T_1, \dots, T_n) defined by

$$T_i(u)(x) = \int_M G_i(x, s) \left[- \sum_{j \neq i} a_{ij} u_j + c_i(s) f_i(u(s)) \right] ds.$$

A fixed point of T is a solution of (1) which belongs to $\overbrace{H^1(R) \times \dots \times H^1(R)}^n$.

Lemma 3.1 *Assume that (H1)-(H3) hold. Then $T(K) \subset K$, and $T : K \rightarrow K$ is completely continuous.*

Proof The continuity is trivial. Since M is compact, there exists a point x_m where $\min_{x \in M} T_i(u)(x)$ is attained. Then, for any $x \in R$, we have

$$\begin{aligned} T_i(u)(x_m) &= \int_M G_i(x, s) \left[- \sum_{j \neq i} a_{ij} u_j + c_i(s) f_i(u(s)) \right] ds \\ &\geq m_i \int_M p_i(s) G_i(s, s) \left[- \sum_{j \neq i} a_{ij} u_j + c_i(s) f_i(u(s)) \right] ds \\ &\geq m_i p_0^i \int_M G_i(s, s) \left[- \sum_{j \neq i} a_{ij} u_j + c_i(s) f_i(u(s)) \right] ds \\ &\geq m_i p_0^i \int_M G_i(x, s) \left[- \sum_{j \neq i} a_{ij} u_j + c_i(s) f_i(u(s)) \right] ds \\ &\geq m_i p_0^i T_i(u)(x), \end{aligned}$$

namely,

$$\min_{x \in M} T_i(u)(x) \geq m_i p_0^i T_i(u)(x) \geq \delta T_i(u)(x).$$

Therefore, it is clear that $T(K) \subset K$.

Finally, we prove that each component of T is compact. Let $\Omega \subset K$ be a bounded set, then there exists a constant $C > 0$ which is uniformly bounded for its element. Since the derivative is bounded in compacts, the functions of $T_i(\Omega)$ are equicontinuous on each compact interval. On the other hand, for any $u \in \Omega$,

$$|T_i(u)(x)| \leq C \int_R G_i(x, s) \left(- \sum_{j \neq i} a_{ij}(s) \right) ds + \max_{\|u\| \leq C} f_i(u) \int_R G_i(x, s) c_i(s) ds = \xi(x). \quad \square$$

Theorem 3.2 *Assume that (H1)-(H3) hold. In addition, $a_{ij}(x)$ ($i \neq j$) satisfies*

$$\int_M G_i(s, s) \left(- \sum_{j \neq i} a_{ij} \right) ds < \frac{1}{2n}.$$

- (a) *If $f_{i,0} = 0, f_{i,\infty} = \infty$ for some $i_0 \in \{1, 2, \dots, n\}$, then (1) has at least one positive solution.*
- (b) *If $f_{i,\infty} = 0, f_{i,0} = \infty$ for some $i_0 \in \{1, 2, \dots, n\}$, then (1) has at least one positive solution.*

Proof (a) On the one hand, since $f_{i,0} = 0$, then there exists $r > 0$ such that

$$f_i(u) \leq \epsilon \sum_{i=1}^n u_i \quad \text{for } 0 < u_1 + \dots + u_n \leq r,$$

where $\epsilon > 0$ is sufficiently small such that

$$\epsilon \int_M G_i(s,s)c_i(s) ds < \frac{1}{2n}.$$

Set $\Omega_r = \{u \in E : \|u\| < r\}$. Then, for any $u \in \partial\Omega_r \cap K$, we have

$$\begin{aligned} T_i(u)(x) &= \int_M G_i(x,s) \left[-\sum_{j \neq i} a_{ij}u_j + c_i(s)f_i(u(s)) \right] ds \\ &\leq \int_M G_i(s,s) \left[-\sum_{j \neq i} a_{ij}(s)u_j(s) + c_i(s)\epsilon \sum_{i=1}^n u_i(s) \right] ds \\ &\leq \left[\int_M G_i(s,s) \left(-\sum_{j \neq i} a_{ij}(s) \right) ds + \epsilon \int_M G_i(s,s)c_i(s) ds \right] \|u\| \\ &\leq \frac{1}{n} \|u\|. \end{aligned}$$

Furthermore, for any $u \in \partial\Omega_r \cap K$, we have

$$\|T(u)\| = \sum_{i=1}^n |T_i(u)|_\infty \leq \|u\|.$$

On the other hand, since $f_{i_0,\infty} = \infty$ for some $i_0 \in \{1, 2, \dots, n\}$, then there exists \bar{R} such that

$$f_{i_0}(u) \geq \eta \sum_{i=1}^n u_i \quad \text{for } \sum_{i=1}^n u_i \geq \bar{R},$$

where $\eta > 0$ is sufficiently large such that

$$\eta \delta^2 \int_M G_{i_0}(s,s)c_{i_0}(s) ds > 1.$$

Let $R = \max\{2r, \frac{\bar{R}}{\delta}\}$ and set $\Omega_R = \{u \in E : \|u\| < R\}$. Then, for any $u \in K \cap \partial\Omega_R$, $\min_{x \in M} \sum_{i=1}^n u_i(x) \geq \delta \|u\| \geq \bar{R}$, and we have

$$\begin{aligned} T_{i_0}(u)(x) &= \int_M G_{i_0}(x,s) \left[-\sum_{j \neq i_0} a_{i_0j}u_j + c_{i_0}(s)f_{i_0}(u(s)) \right] ds \\ &\geq m_{i_0} p_0^{i_0} \int_M G_{i_0}(s,s)c_{i_0}(s) \eta \sum_{i=1}^n u_i(s) ds \\ &\geq \eta \delta^2 \int_M G_{i_0}(s,s)c_{i_0}(s) ds \|u\| \\ &> \|u\|. \end{aligned}$$

Furthermore, we have

$$\|T(u)\| > \|u\| \quad \text{for } u \in K \cap \partial\Omega_R.$$

Now by Lemma 2.3, T has a fixed point $u = (u_1, u_2, \dots, u_n) \in K \cap (\overline{\Omega}_R \setminus \Omega_r)$, namely, (1) has a positive solution.

(b) On the one hand, since $f_{i_0,0} = \infty$ for some $i_0 \in \{1, 2, \dots, n\}$, there exists $r > 0$ such that

$$f_{i_0}(u) \geq \eta \sum_{i=1}^n u_i \quad \text{for } 0 \leq \sum_{i=0}^n u_i \leq r,$$

where $\eta > 0$ is sufficiently large such that

$$\eta \delta^2 \int_M G_{i_0}(s, s) c_{i_0}(s) ds > 1.$$

Set $\Omega_r = \{u \in E : \|u\| < r\}$. Then, for any $u \in K \cap \partial\Omega_r$, we have

$$\begin{aligned} T_{i_0}(u)(x) &= \int_M G_{i_0}(x, s) \left[- \sum_{j \neq i_0} a_{i_0 j} u_j + c_{i_0}(s) f_{i_0}(u(s)) \right] ds \\ &\geq m_{i_0} p_0^{i_0} \int_M G_{i_0}(s, s) c_{i_0}(s) \eta \sum_{i=1}^n u_i(s) ds \\ &\geq \eta \delta^2 \int_M G_{i_0}(s, s) c_{i_0}(s) ds \|u\| \\ &> \|u\|. \end{aligned}$$

Furthermore, we have

$$\|T(u)\| > \|u\| \quad \text{for } u \in K \cap \partial\Omega_r.$$

On the other hand, since $f_{i,\infty} = 0$, then there exists \bar{R} such that

$$f_i(u) \leq \epsilon \sum_{i=1}^n u_i \quad \text{for } u_1 + \dots + u_n \geq \bar{R},$$

where $\epsilon > 0$ is sufficiently small such that

$$\epsilon \int_M G_i(s, s) c_i(s) ds < \frac{1}{2n}.$$

Let $R = \max\{2r, \frac{\bar{R}}{\delta}\}$ and set $\Omega_R = \{u \in E : \|u\| < R\}$. Then, for any $u \in K \cap \partial\Omega_R$, $\min_{x \in M} \sum_{i=1}^n u_i(x) \geq \delta \|u\| \geq \bar{R}$, and we have

$$\begin{aligned} T_i(u)(x) &= \int_M G_i(x, s) \left[- \sum_{j \neq i} a_{ij} u_j + c_i(s) f_i(u(s)) \right] ds \\ &\leq \int_M G_i(s, s) \left[- \sum_{j \neq i} a_{ij}(s) u_j(s) + c_i(s) \epsilon \sum_{i=1}^n u_i(s) \right] ds \end{aligned}$$

$$\begin{aligned} &\leq \left[\int_M G_i(s,s) \left(- \sum_{j \neq i} a_{ij}(s) \right) ds + \epsilon \int_M G_i(s,s) c_i(s) ds \right] \|u\| \\ &\leq \frac{1}{n} \|u\|. \end{aligned}$$

Furthermore, for any $u \in \partial\Omega_R \cap K$, we have

$$\|T(u)\| = \sum_{i=1}^n |T_i(u)|_\infty \leq \|u\|.$$

Now, by Lemma 2.3, T has a fixed point $u = (u_1, u_2, \dots, u_n) \in K \cap (\overline{\Omega}_R \setminus \Omega_r)$, namely, (1) has a positive solution. \square

Corollary 3.3 *Assume that (H1)-(H3) hold. $a_{ij}(x)$ ($i \neq j$) satisfies*

$$\int_M G_i(s,s) \left(- \sum_{j \neq i} a_{ij} \right) ds < \frac{1}{2n}.$$

In addition, the following conditions hold.

(H4) *If there exist constants $\widehat{R}, \rho > 0$ such that for some $i_0 \in \{1, 2, \dots, n\}$,*

$$f_{i_0}(u) \geq \rho \widehat{R} \quad \text{for } \delta \widehat{R} \leq \sum_{i=1}^n u_i \leq \widehat{R},$$

where ρ satisfies

$$\rho \delta \int_M G_{i_0}(s,s) c_{i_0}(s) ds > 1;$$

(H5) $f_{i,0} = 0, f_{i,\infty} = 0$.

Then (1) has at least two positive solutions.

Corollary 3.4 *Assume that (H1)-(H3) hold. $a_{ij}(x)$ ($i \neq j$) satisfies*

$$\int_M G_i(s,s) \left(- \sum_{j \neq i} a_{ij} \right) ds < \frac{1}{2n}.$$

In addition, the following conditions hold.

(H6) *If there exist constants $\widetilde{R}, \vartheta > 0$ such that*

$$f_i(u) \leq \vartheta \widetilde{R} \quad \text{for } \delta \widetilde{R} \leq \sum_{i=1}^n u_i \leq \widetilde{R},$$

where ϑ satisfies

$$\vartheta \int_M G_i(s,s) c_i(s) ds < \frac{1}{2n};$$

(H7) $f_{i_0,0} = \infty, f_{j_0,\infty} = \infty$ for some $i_0, j_0 \in \{1, 2, \dots, n\}$.

Then (1) has at least two positive solutions.

Theorem 3.5 Assume that (H1)-(H3) hold and $a_{ij}(x)$ ($i \neq j$) satisfies

$$\int_M G_i(s, s) \left(- \sum_{j \neq i} a_{ij} \right) ds < \frac{1}{2n}.$$

In addition, there exist numbers a, c and d with $0 < d < a < \frac{c}{4}$ such that the following conditions are satisfied:

(H8) $f_i(u) < \frac{1}{2n \int_M G_i(s, s) c_i(s) ds} d$ for $u_i \geq 0$ and $0 \leq \sum_{i=1}^n u_i < d$;

(H9) there exists $i_0 \in \{1, 2, \dots, n\}$ such that

$$f_{i_0}(u(x)) > \frac{a}{\delta^2 \Gamma} \quad \text{for } x \in M, u_i \geq 0 \text{ and } \sum_{i=1}^n u_i \in \left[a, \frac{a}{\delta} \right],$$

where $\Gamma = \min \{ \int_M G_i(s, s) c_i(s) ds \}$;

(H10) $f_i(u) \leq \frac{1}{2n \int_M G_i(s, s) c_i(s) ds} c$ for $u_i \geq 0$ and $\sum_{i=1}^n u_i < c$.

Then (1) has at least three positive solutions.

Proof For $u = (u_1, u_2, \dots, u_n) \in K$, define

$$\alpha(u) = \min_{x \in M} (u_1(x) + u_2(x) + \dots + u_n(x)),$$

then it is easy to know that α is a nonnegative continuous concave functional on K with $\alpha(u) \leq \|u\|$ for $u \in K$.

Set $b = \frac{a}{\delta}$. First, we show that $T : \bar{K}_c \rightarrow \bar{K}_c$ with $c > b$. For any $u \in \bar{K}_c$, we have

$$\begin{aligned} T_i(u)(x) &= \int_R G_i(x, s) \left[- \sum_{j \neq i} a_{ij} u_j + c_i(s) f_i(u(s)) \right] ds \\ &= \int_M G_i(x, s) \left[- \sum_{j \neq i} a_{ij} u_j + c_i(s) f_i(u(s)) \right] ds \\ &\leq \int_M G_i(s, s) \left[- \sum_{j \neq i} a_{ij} u_j + c_i(s) f_i(u(s)) \right] ds \\ &\leq \int_M G_i(s, s) \left[- \sum_{j \neq i} a_{ij} u_j + c_i(s) \frac{c}{2n \int_M G_i(s, s) c_i(s) ds} \right] ds \\ &\leq \frac{c}{2n} + \frac{c}{2n} \\ &= \frac{c}{n}. \end{aligned}$$

So $\|T(u)\| = \sum_{i=1}^n |T_i(u)|_\infty \leq c$.

In a similar way, we also can prove that $T : \bar{K}_d \rightarrow K_d$. Then (ii) of Lemma 2.4 holds.

Next, we shall show that (i) of Lemma 2.4 is satisfied. It is clearly seen that $u = (\frac{a+b}{2n}, \dots, \frac{a+b}{2n}) \in \{u = (u_1, u_2, \dots, u_n) \in K(\alpha, a, b) : \alpha(u) > a\}$. Then, for any $u \in K(\alpha, a, b)$ and $x \in M$, it is easy to obtain that

$$b \geq \sum_{i=1}^n |u_i|_\infty \geq \sum_{i=1}^n u_i(x) \geq \min_{x \in M} \left(\sum_{i=1}^n u_i(x) \right) = \alpha(u) > a.$$

Then, by (H9), we can have

$$\begin{aligned}
 \alpha(T(u)(x)) &= \min_{x \in M} \left(\sum_{i=1}^n T_i(u)(x) \right) \\
 &\geq \min_{x \in M} T_{i_0}(u)(x) \\
 &\geq \delta \max_{x \in M} T_{i_0}(u)(x) \\
 &\geq \delta \int_M G_{i_0}(x, s) \left[- \sum_{j \neq i_0} a_{i_0 j} u_j + c_{i_0}(s) f_{i_0}(u(s)) \right] ds \\
 &\geq \delta \int_M m_{i_0} p_{i_0}(s) G_{i_0}(s, s) c_{i_0}(s) f_{i_0}(u(s)) ds \\
 &\geq \delta^2 \int_M G_{i_0}(s, s) c_{i_0}(s) \frac{a}{\delta^2 \Gamma} ds \\
 &= a.
 \end{aligned}$$

Finally, we verify that (iii) of Lemma 2.4 is satisfied. Suppose that $u \in K(\alpha, a, c)$ with $\|T(u)\| > b$, then we can have

$$\begin{aligned}
 \alpha(T(u)) &= \min_M \left(\sum_{i=1}^n T_i(u)(x) \right) \\
 &\geq \delta \|T(u)\| \\
 &> b\delta = a.
 \end{aligned}$$

From the above, the hypotheses of Leggett-Williams theorem are satisfied. Hence (1) has at least three positive solutions u^1, u^2 and u^3 such that $\|u^1\| < d, a < \min_{x \in M} (\sum_{i=1}^n u_i^2(x))$, and $\|u^3\| > d$ with

$$\min_{x \in M} \left(\sum_{i=1}^n u_i^3(x) \right) < a.$$

□

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

SS drafted the manuscript. FW and TA gave some suggestions to improve the manuscript. All authors typed, read and approved the final manuscript.

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