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Lower and upper estimates of solutions to systems of delay dynamic equations on time scales

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Abstract

In this paper we study a system of delay dynamic equations on the time scale $\mathbb{T}$ of the form

$$y^{\Delta}(t) = f\left(t, y_{\tau}(t)\right),$$

where $f : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$, $y_\tau(t) = (y_1(\tau_1(t)), \dots, y_n(\tau_n(t)))$ and $\tau_i : \mathbb{T} \to \mathbb{T}$, $i = 1, \dots, n$, are the delay functions. We are interested in the asymptotic behavior of solutions of the mentioned system. More precisely, we formulate conditions on a function f, which guarantee that the graph of at least one solution of the above-mentioned system stays in the prescribed domain. This result generalizes some previous results concerning the asymptotic behavior of solutions of non-delay systems of dynamic equations or of delay dynamic equations. A relevant example is considered.

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Keywords: time scale; dynamic system; delay; asymptotic behavior of solution; retract; retraction

1 Introduction

1.1 Time scale calculus

Time scale calculus, first introduced by Stefan Hilger in his PhD thesis in 1988 (see [1]) is nowadays well-known calculus and often studied in applications. Recall that a time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of reals. Note that $[a,b]_{\mathbb{T}} := [a,b] \cap \mathbb{T}$ (resp. $(a,b)_{\mathbb{T}} := (a,b) \cap \mathbb{T}$ *etc.*, we define any combination of right and left open or closed intervals), $[a,\infty)_{\mathbb{T}} := [a,\infty) \cap \mathbb{T}$, σ, ρ, μ and f^{Δ} stand for the finite time scale interval, infinite time scale interval, forward jump operator, backward jump operator, graininess and Δ-derivative of f. Further, we use the symbols $C_{\mathrm{rd}}(\mathbb{T})$ and $C^1_{\mathrm{rd}}(\mathbb{T})$ to stand for the class of rd-continuous and rd-continuous Δ-differentiable functions defined on the time scale $\mathbb{T}$. Finally, we work with all types of points on the time scale $\mathbb{T}$, *i.e.*, with right-dense points or right-scattered points, respectively with left-dense points or left-scattered points. See [2], which is the initiating paper of the time scale theory, and [3] containing a lot of information on time scale calculus.

Now we remind further aspects of time scales calculus, which will be needed later (see, *e.g.*, [3]). We use the standard symbol $\|\cdot\|$ for an arbitrary vector norm. Note that (in this paper) a type of a norm is not important.

Definition 1 Let $\mathbb{T}$ be a time scale. A function $f : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ is called

(i) *rd-continuous* if g defined by $g(t) := f(t, y(t))$ is rd-continuous for any rd-continuous function $y : \mathbb{T} \to \mathbb{R}^n$;

(ii) *bounded* on a set $S \subset \mathbb{T} \times \mathbb{R}^n$ if there exists a constant $M > 0$ such that

$$\|f(t,y)\| \le M \quad \text{for all } (t,y) \in S;$$

(iii) *Lipschitz continuous* on a set $S \subset \mathbb{T} \times \mathbb{R}^n$ if there exists a constant $L > 0$ such that

$$\|f(t,y_1) - f(t,y_2)\| \le L\|y_1 - y_2\| \quad \text{for all } (t,y_1),(t,y_2) \in S.$$

1.2 System of delay dynamic equations on time scales

Let $\tau_i : \mathbb{T} \to \mathbb{T}$, $i = 1, \dots, n$, $n \in \mathbb{N}$, be increasing rd-continuous functions satisfying $\tau_i(t) \le t$ for all $t \in \mathbb{T}$, and let $y_\tau(t) = (y_1(\tau_1(t)), \dots, y_n(\tau_n(t)))$ be a vector, where its every component y_i is with an own delay τ_i. Let the function $f : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ be rd-continuous. We consider the system of n delay dynamic equations

$$y^\Delta(t) = f(t, y_\tau(t)), \tag{1}$$

i.e.,

$$\begin{aligned}
y_1^\Delta(t) &= f_1\bigl(t, y_1(\tau_1(t)), y_2(\tau_2(t)), \dots, y_n(\tau_n(t))\bigr),\\
y_2^\Delta(t) &= f_2\bigl(t, y_1(\tau_1(t)), y_2(\tau_2(t)), \dots, y_n(\tau_n(t))\bigr),\\
&\vdots\\
y_n^\Delta(t) &= f_n\bigl(t, y_1(\tau_1(t)), y_2(\tau_2(t)), \dots, y_n(\tau_n(t))\bigr)
\end{aligned}$$

on the time scale $\mathbb{T}$.

For given $t_0 \in \mathbb{T}$ and $\alpha_0 := \min\{\tau_i(t_0)\}_{i=1}^n$, a function $y : [\alpha_0, \infty)_\mathbb{T} \to \mathbb{R}^n$ is said to be a solution of (1) on $[\alpha_0, \infty)_\mathbb{T}$ provided $y \in C_{\mathrm{rd}}([\alpha_0, \infty)_\mathbb{T})$, $y \in C^1_{\mathrm{rd}}([t_0, \infty)_\mathbb{T})$ and y satisfies (1) for all $t \in [t_0, \infty)_\mathbb{T}$. If, moreover, we are given an initial function $\varphi : [\alpha_0, t_0]_\mathbb{T} \to \mathbb{R}^n$, $\varphi \in C_{\mathrm{rd}}([\alpha_0, t_0]_\mathbb{T})$ such that

$$y(t) = \varphi(t), \quad t \in [\alpha_0, t_0]_\mathbb{T}, \tag{2}$$

then we say that y is a solution of initial problem (IP) (1), (2).

1.3 Existence and uniqueness of solutions of delay dynamic equations

For the next study, it is important to known whether a solution of IP (1) and (2) exists and if it is uniquely defined. However, the following theorem (in a more general form) can be found in [4, Theorem 2.1].

Theorem 1 (Picard-Lindelöf theorem) *Let* $t_1 \in \mathbb{T}$, $t_1 > t_0$, $m > 0$. *Let*

$$Y_m := \bigl\{y \in \mathbb{R}^n : \|y - \varphi(t)\| \le m \textit{ for some } t \in [\alpha_0, t_0]_\mathbb{T}\bigr\},$$

where the properties of φ and the definition of α_0 are described in previous Section 1.2. *Assume that $f \in C_{\mathrm{rd}}([t_0, t_1]_{\mathbb{T}} \times Y_m)$ is on $[t_0, t_1]_{\mathbb{T}} \times Y_m$ bounded with bound $M > 0$ and Lipschitz continuous. Then initial problem* (1) *and* (2) *has a unique solution y on the interval $[\alpha_0, \sigma(\xi)]_{\mathbb{T}} \subset [\alpha_0, t_1]_{\mathbb{T}}$, where*

$$\xi := \max[t_0, t_0 + \delta]_{\mathbb{T}}$$

and

$$\delta := \min\{t_1 - t_0, m/M\}.$$

Carefully tracing the proof of Theorem 1 in [4], it is easy to verify that if Theorem 1 holds, then the solution of IP (1), (2) depends continuously on the initial data.

2 Problem under consideration

Throughout this paper, we assume that the time scale $\mathbb{T}$ is unbounded above with $t_0 \in \mathbb{T}$. Furthermore (throughout this paper), $\alpha_0 \in \mathbb{T}$ has the same meaning as in the previous section. Let

$$b_i, c_i : \mathbb{T} \to \mathbb{R}, \quad i = 1, \dots, n,$$

be Δ-differentiable functions such that $b_i(t) < c_i(t)$ for each $t \in [\alpha_0, \infty)_{\mathbb{T}}$, $i = 1, \dots, n$, and

$$b_i(t) < \varphi_i(t) < c_i(t) \quad \text{for all } t \in [\alpha_0, t_0]_{\mathbb{T}}, i = 1, \dots, n, \tag{3}$$

where $\varphi_i(t)$ are coordinates of an initial function $\varphi(t)$ used in (2). We define a set $\Omega \subset \mathbb{T} \times \mathbb{R}^n$ as

$$\Omega := \{(t, y) : t \in [\alpha_0, \infty)_{\mathbb{T}}, y \in \omega(t)\},$$

where

$$\omega(t) := \{y \in \mathbb{R}^n : b_i(t) < y_i < c_i(t), i = 1, \dots, n\}.$$

Then the closure $\overline{\Omega}$ equals

$$\overline{\Omega} := \{(t, y) : t \in [\alpha_0, \infty)_{\mathbb{T}}, y \in \overline{\omega}(t)\}$$

with

$$\overline{\omega}(t) = \{y \in \mathbb{R}^n : b_i(t) \le y_i \le c_i(t), i = 1, \dots, n\}.$$

Moreover, we define the y-boundary $\partial_y \Omega$ of Ω as

$$\partial_y \Omega := \{(t, y) : t \in [\alpha_0, \infty)_{\mathbb{T}}, y \in \partial\omega(t)\}$$

with

$$\partial\omega(t) := \overline{\omega}(t) \setminus \omega(t) = \left\{ y \in \mathbb{R}^n : y \in \overline{\omega}(t) \text{ and } \prod_{i=1}^{n} (y_i - b_i(t))(y_i - c_i(t)) = 0 \right\}.$$

Consider delay system (1) and initial problem (2). Let $t_1 \in \mathbb{T}$, where $t_1 > t_0$. Let a function f be bounded and Lipschitz continuous on an open set $S = S(t, y) \subset \mathbb{T} \times \mathbb{R}^n$ and

$$\{(t, y) : t \in [\alpha_0, t_1]_{\mathbb{T}}, y \in \mathbb{R}^n\} \cap \overline{\Omega} \subset S.$$

This condition says that, by Theorem 1, every initial problem (1) and (2) with φ satisfying (3) has exactly one solution on an interval $[\alpha_0, \sigma(\xi)]_{\mathbb{T}}$ where $\sigma(\xi) > t_0$. It is also easy to show that this solution depends continuously on the initial function φ.

Throughout the paper, we assume that the function f is bounded and Lipschitz continuous on an open set S and $\overline{\Omega} \subset S$, which implies that every initial problem (1) and (2) has exactly one solution on an interval $[\alpha_0, \infty)_{\mathbb{T}}$.

The aim of this paper is to establish sufficient conditions on the function f of equation (1) such that there exists at least one solution $y(t)$ of (1) defined on $[\alpha_0, \infty)_{\mathbb{T}}$ such that $(t, y(t)) \in \Omega$ for each $t \in [\alpha_0, \infty)_{\mathbb{T}}$. The main result generalizes some previous results of the first author (and his co-authors) concerning the asymptotic behavior of solutions of discrete and dynamic equations (see, *e.g.*, [5–9]).

In papers [5, 6], to our best knowledge, the retract principle is for the first time extended to discrete equations. In [10] delayed discrete equations are considered by retract technique, and in [8] the retract principle is given for discrete time scales. Paper [7] is devoted to extension of the retract principle to dynamic equations. In [11] the retract principle is extended (under different conditions) to a system of dynamic equations in the plane. In [9] we extended the retract principle to scalar delayed dynamic equations. In the present paper we give an attempt to enlarge the retract principle to systems of delayed dynamic equations.

2.1 Points of strict egress

We define auxiliary sets which are subsets of y-boundary $\partial_y \Omega$:

$$\Omega_B^i := \{(t, y) \in \overline{\Omega} : y_i(t) = b_i(t)\},$$
$$\Omega_C^i := \{(t, y) \in \overline{\Omega} : y_i(t) = c_i(t)\},$$

where $i = 1, 2, \dots, n$. Obviously, $\partial_y \Omega = \bigcup_{i=1}^n (\Omega_B^i \cup \Omega_C^i)$.

Definition 2 Let $\alpha_t := \min\{\tau_i(t)\}_{i=1}^n$. A point

$$M_{iB} = (t, y_1, \dots, y_{i-1}, b_i(t), y_{i+1}, \dots, y_n) \in \Omega_B^i, \quad i \in \{1, 2, \dots, n\}, t \geq t_0$$

is called the point of strict egress for the set Ω with respect to system (1) if

$$f_i(t, u_1(\tau_1(t)), u_2(\tau_2(t)), \dots, u_n(\tau_n(t))) < b_i^{\Delta}(t), \tag{4}$$

where $u = (u_1, \dots, u_n) : [\alpha_t, t]_{\mathbb{T}} \to \mathbb{R}^n$ is an arbitrary rd-continuous function such that for every $j = 1, \dots, n$,

$$b_j(s) < u_j(s) < c_j(s), \quad s \in [\alpha_t, t)_{\mathbb{T}},$$
$$b_j(t) \leq u_j(t) \leq c_j(t) \quad \text{for } j \neq i$$

and $u_i(t) = b_i(t)$.

A point

$$M_{iC} = \big(t, y_1, \dots, y_{i-1}, c_i(t), y_{i+1}, \dots, y_n\big) \in \Omega_C^i, \quad i \in \{1, 2, \dots, n\}, t \geq t_0$$

is called the point of strict egress for the set Ω with respect to system (1) if

$$f_i\big(t, u_1\big(\tau_1(t)\big), u_2\big(\tau_2(t)\big), \dots, u_n\big(\tau_n(t)\big)\big) > c_i^{\Delta}(t), \tag{5}$$

where $u = (u_1, \dots, u_n) : [\alpha_t, t]_{\mathbb{T}} \to \mathbb{R}^n$ is an arbitrary rd-continuous function such that for every $j = 1, \dots, n$,

$$b_j(s) < u_j(s) < c_j(s), \quad s \in [\alpha_t, t)_{\mathbb{T}},$$
$$b_j(t) \leq u_j(t) \leq c_j(t) \quad \text{for } j \neq i$$

and $u_i(t) = c_i(t)$.

Remark 1 We will explain the geometrical meaning of the point of strict egress. If a point

$$M_{iB}^* = \big(t^*, y_1, \dots, y_{i-1}, b_i\big(t^*\big), y_{i+1}, \dots, y_n\big) \in \Omega_B^i, \quad i \in \{1, 2, \dots, n\}$$

is a point of strict egress for the set Ω with respect to (1) and $y(t) = (y_1(t), \dots, y_n(t))$ is a (unique) solution of (1) satisfying $(t^*, y(t^*)) = M_{iB}^*$, then, due to (4),

$$\big(y_i\big(t^*\big) - b_i\big(t^*\big)\big)^{\Delta} = f_i\big(t^*, u_1\big(\tau_1\big(t^*\big)\big), u_2\big(\tau_2\big(t^*\big)\big), \dots, u_n\big(\tau_n\big(t^*\big)\big)\big) - b_i^{\Delta}\big(t^*\big) < 0.$$

From the definition of Δ-derivative and the property $y_i(t^*) - b_i(t^*) = 0$, we get $y_i(t) - b_i(t) < 0$ (or $(t, y(t)) \notin \overline{\Omega}$) for $t \in (t^*, t^* + \delta)_{\mathbb{T}}$ with a small positive δ if t^* is a right-dense point and for $t = \sigma(t^*)$ if t^* is right-scattered.

By analogy, if

$$M_{iC}^* = \big(t^*, y_1, \dots, y_{i-1}, c_i\big(t^*\big), y_{i+1}, \dots, y_n\big) \in \Omega_C^i, \quad i \in \{1, 2, \dots, n\}$$

is a point of strict egress for the set Ω with respect to (1) and $y(t) = (y_1(t), \dots, y_n(t))$ is a (unique) solution of (1) satisfying $(t^*, y(t^*)) = M_{iC}^*$, then, due to (5), $y_i(t) - c_i(t) > 0$ (or $(t, y(t)) \notin \overline{\Omega}$) for $t \in (t^*, t^* + \delta)_{\mathbb{T}}$ with a small positive δ if t^* is a right-dense point and for $t = \sigma(t^*)$ if t^* is right-scattered.

We see that in all the cases considered, the solution $y = y(t)$ of (1) with the initial condition $y(t) = \varphi(t)$, $t \in [\alpha_{t^*}, t^*]_{\mathbb{T}}$, and $(t^*, y(t^*)) \in \partial_y \Omega$ satisfies $(t, y(t)) \notin \overline{\Omega}$ for $t \in (t^*, t^* + \delta)_{\mathbb{T}}$ with a small positive δ if t^* is a right-dense point and for $t = \sigma(t^*)$ if t^* is right-scattered.

Definition 3 [12] If $A \subset B$ are subsets of a topological space and $\pi : B \to A$ is a continuous mapping from B onto A such that $\pi(p) = p$ for every $p \in A$, then π is said to be a *retraction* of B onto A. When a retraction of B onto A exists, A is called a *retract* of B.

3 Existence theorem

The following theorem is proved by utilizing the idea of a retract method, which is well known for ordinary differential equations and goes back to Ważewski [13]. In the next theorem, we assume that the function f, except for the indicated conditions, satisfies all the assumptions given in Section 2. Namely, we assume that the function f is bounded and Lipschitz continuous on an open set S and $\overline{\Omega} \subset S$.

Theorem 2 *Let $f : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$. Let $b_i, c_i : \mathbb{T} \to \mathbb{R}$, $i \in \{1, \dots, n\}$ be Δ-differentiable functions on $\mathbb{T}$ such that $b_i(t) < c_i(t)$ for each $t \in [\alpha_0, \infty)_{\mathbb{T}}$. If, moreover, every point $M \in \partial_y \Omega$ is the point of strict egress for the set Ω with respect to system* (1), *then there exists an rd-continuous initial function $\varphi^* : [\alpha_0, t_0]_{\mathbb{T}} \to \mathbb{R}^n$ satisfying*

$$b_i(t) < \varphi_i^*(t) < c_i(t) \quad \textit{for all } t \in [\alpha_0, t_0]_{\mathbb{T}}, i = 1, \dots, n \tag{6}$$

such that the initial problem

$$y(t) = \varphi^*(t), \quad t \in [\alpha_0, t_0]_{\mathbb{T}} \tag{7}$$

defines a solution y of (1) *on the interval* $[\alpha_0, \infty)_{\mathbb{T}}$ *satisfying*

$$\bigl(t, y(t)\bigr) \in \Omega \quad \textit{for every } t \in [\alpha_0, \infty)_{\mathbb{T}}. \tag{8}$$

Proof The idea of the proof is the following. By contrary, we assume that a solution y satisfying (8) does not exist. Then we are able to prove (by a construction of a chain of auxiliary mappings) an existence of a retraction of an n-dimensional ball into its boundary. However, it is well known that the boundary of an n-dimensional ball cannot be its retract (see, *e.g.*, [12]) and we get a contradiction.

Without any special comment, throughout the proof, we use the fact that the initial value problem has a unique solution and this solution depends continuously on the initial data (this is guaranteed by Theorem 1). Suppose now that the initial function φ^* satisfying (6) generates the solution $y = y(t)$ which does not satisfy (8) for at least one $t \in (t_0, \infty)_{\mathbb{T}}$. This means, in general, that for any rd-continuous initial function φ_0 satisfying the inequality

$$b_i(t) < \varphi_{0i}(t) < c_i(t) \quad \text{for all } t \in [\alpha_0, t_0]_{\mathbb{T}}, i = 1, \dots, n, \tag{9}$$

there exists $t^0 \in \mathbb{T}$, $t^0 > t_0$ such that, for a corresponding solution $y = y^0(t)$ of the initial problem

$$y^0(t) = \varphi_0(t), \quad t \in [\alpha_0, t_0]_{\mathbb{T}},$$

we have

$$\bigl(t^0, y^0\bigl(t^0\bigr)\bigr) \notin \Omega$$

and

$$\bigl(t, y^0(t)\bigr) \in \Omega \quad \text{for all } t \in [t_0, t^0)_{\mathbb{T}}.$$

Let us define auxiliary mappings P_1, P_2 and P_3. Note that in this part of the proof, without loss of generality, we admit the eventuality that the function $\varphi_0(t)$ instead of (9) satisfies weaker restrictions

$$b_i(t) < \varphi_{0i}(t) < c_i(t) \quad \text{for all } t \in [\alpha_0, t_0)_{\mathbb{T}}, i = 1, \dots, n,$$

and $\varphi_0(t_0) \in \overline{\omega}(t_0)$.

First, define a mapping P_1. For $(t_0, \varphi_0(t_0))$ with $\varphi_0(t_0) \in \overline{\omega}(t_0)$,

$$P_1\big(t_0, \varphi_0(t_0)\big) := \begin{cases} (t^0, y^0(t^0)) & \text{if } \varphi_0(t_0) \in \omega(t_0), \\ (t^0, y^0(t^0)) \equiv (t_0, y^0(t_0)) & \text{if } \varphi_0(t_0) \in \partial\omega(t_0) \end{cases}$$

with t^0 having the value defined above except for the case $\varphi_0(t_0) \in \partial\omega(t_0)$. In the latter case, we put $t^0 = t_0$.

Second, we define a mapping P_2 for every $(t^0, y^0(t^0))$. For this we will need a set

$$\Omega_{\mathbb{R}} := \big\{(t, y) : t \in [\alpha_0, \infty), y \in \omega_{\mathbb{R}}(t)\big\}$$

with $\omega_{\mathbb{R}}(t) \equiv \omega(t)$ if $t \in \mathbb{T}$ and

$$\begin{aligned} \omega_{\mathbb{R}}(t) := \bigg\{ & y \in \mathbb{R}^n : b_i(t_a) + \big[b_i(t_b) - b_i(t_a)\big] \cdot \frac{t - t_a}{t_b - t_a} \\ & < y_i < c_i(t_a) + \big[c_i(t_b) - c_i(t_a)\big] \cdot \frac{t - t_a}{t_b - t_a}, i = 1, \dots, n \bigg\} \end{aligned}$$

if $t \notin \mathbb{T}$ and $t_a \in \mathbb{T}$, $t_b \in \mathbb{T}$ are such that $t_a < t_b$, $t \in (t_a, t_b)$ and $(t_a, t_b) \cap \mathbb{T} = \emptyset$. It is clear that $\Omega \subseteq \Omega_{\mathbb{R}}$. Further,

$$V(t_a, t_b) := \big\{(t, y) : t_a \le t \le t_b, y \in \overline{\omega}_{\mathbb{R}}(t)\big\}$$

is obviously convex. We define, moreover, the y-part of the boundary of $\Omega_{\mathbb{R}}$ as

$$\partial_y \Omega_{\mathbb{R}} := \big\{(t, y) : t \in [\alpha_0, \infty), y \in \partial\omega_{\mathbb{R}}(t)\big\},$$

where $\partial\omega_{\mathbb{R}}(t) \equiv \partial\omega(t)$ if $t \in \mathbb{T}$ and

$$\begin{aligned} \partial\omega_{\mathbb{R}}(t) := \bigg\{ & y \in \overline{\omega}_{\mathbb{R}}(t) : \prod_{i=1}^{n} \bigg(b_i(t_a) + \big[b_i(t_b) - b_i(t_a)\big] \cdot \frac{t - t_a}{t_b - t_a} - y_i \bigg) \\ & \times \bigg(y_i - c_i(t_a) - \big[c_i(t_b) - c_i(t_a)\big] \cdot \frac{t - t_a}{t_b - t_a} \bigg) = 0 \bigg\} \end{aligned}$$

if $t \notin \mathbb{T}$, where $t_a \in \mathbb{T}$ and $t_b \in \mathbb{T}$ are as above.

Now we are ready to consider an auxiliary mapping P_2. Let $(t^0, y^0(t^0)) \notin \Omega$. Then, due to the convexity of $V(t_a, t_b)$, there exists a unique intersection of the segment connecting the points $(\rho(t^0), y^0(\rho(t^0)))$ and $(t^0, y^0(t^0))$ with $\partial_y \Omega_{\mathbb{R}}$. We denote this point as M^0 and define

$$P_2\big(t^0, y^0\big(t^0\big)\big) := M^0.$$

Note that if $(t^0, y^0(t^0)) \in \partial_y\Omega$, we get the particular case $M^0 = (t^0, y^0(t^0))$ and $P_2(t^0, y^0(t^0)) = (t^0, y^0(t^0))$.

Third, we define a mapping P_3. For this we need a subset of the y-boundary $\partial_y\Omega$. For $s \in \mathbb{T}$, let

$$\partial_y\Omega|_{t=s} := \partial_y\Omega \cap \{(s, y) : y \in \mathbb{R}^n\}.$$

Now we are ready to consider an auxiliary continuous mapping P_3,

$$P_3 : \partial_y\Omega_{\mathbb{R}} \to \partial_y\Omega|_{t=t_0},$$

defined for $M^0 = (t, y) \in \partial_y\Omega_{\mathbb{R}}$ as a point $M^* = (t_0, y^*)$, where $y^* \in \partial\omega(t_0)$ with

$$y_i^* = b_i(t_0) + \frac{c_i(t_0) - b_i(t_0)}{c_i(t) - b_i(t)} \cdot \bigl(y_i - b_i(t)\bigr), \quad i = 1, \dots, n. \tag{10}$$

It is easy to see that the $\partial_y\Omega|_{t=t_0}$ is a retract of $\partial_y\Omega_{\mathbb{R}}$ and satisfies all the assumptions from Definition 3 (with $A := \partial_y\Omega|_{t=t_0}$ and $B := \partial_y\Omega_{\mathbb{R}}$).

We show that the composite mapping

$$P_3 \circ P_2 \circ P_1 : \{(t_0, y) : y \in \overline{\omega}(t_0)\} \to \partial_y\Omega|_{t=t_0}$$

is continuous due to the continuous dependence of the solutions on the initial data, the convexity of sets of the type $V(t_a, t_b)$ defined above and the continuity of the mapping P_3. Let $\delta > 0$ be sufficiently small and $\varphi_0, \varphi_{0,\delta} : [\alpha_0, t_0]_{\mathbb{T}} \to \mathbb{R}^n$ be the initial functions such that

$$\|\varphi_0(t) - \varphi_{0,\delta}(t)\| < \delta \quad \text{for all } t \in [\alpha_0, t_0]_{\mathbb{T}}$$

and

$$\lim_{\delta\to 0} \varphi_{0,\delta} = \varphi_0. \tag{11}$$

Further, let

$$P_1\bigl(t_0, \varphi_0(t_0)\bigr) = \bigl(t^0, y^0\bigl(t^0\bigr)\bigr), \tag{12}$$

$$P_1\bigl(t_0, \varphi_{0,\delta}(t_0)\bigr) = \bigl(t^{0,\delta}, y^{0,\delta}\bigl(t^{0,\delta}\bigr)\bigr). \tag{13}$$

Suppose now that $\varphi_0(t_0), \varphi_{0,\delta}(t_0) \in \omega(t_0)$ (which is equivalent to $t^0 > t_0$). We consider all the possible settings of $(t^0, y^0(t^0))$ and characters of t^0.

(I) Point $(t^0, y^0(t^0)) \in \partial_y\Omega$.

First, assume that the point t^0 is dense, *i.e.*, $\rho(t^0) = t^0 = \sigma(t^0)$. Then, due to the solutions depending continuously on initial data, we have $\lim_{\delta\to 0} t^{0,\delta} = t^0$ and, consequently,

$$\lim_{\delta\to 0} P_1\bigl(t_0, \varphi_{0,\delta}(t_0)\bigr) = P_1\bigl(t_0, \varphi_0(t_0)\bigr) = \bigl(t^0, y^0\bigl(t^0\bigr)\bigr).$$

In this case the mapping P_1 is continuous and the composite mapping $P_2 \circ P_1$ is continuous as well because

$$(P_2 \circ P_1)\big(t_0, \varphi_0(t_0)\big) = \big(t^0, y^0\big(t^0\big)\big)$$

and

$$\lim_{\delta \to 0} (P_2 \circ P_1)\big(t_0, \varphi_{0,\delta}(t_0)\big) = \lim_{\delta \to 0} P_2\big(t^{0,\delta}, y^{0,\delta}\big(t^{0,\delta}\big)\big) = \big(t^0, y^0\big(t^0\big)\big).$$

Further, $P_3 \circ P_2 \circ P_1$ is also continuous because of the continuity of mappings P_3 and $P_2 \circ P_1$.

Second, if the point t^0 is left-scattered and right-dense, *i.e.*, $\rho(t^0) < t^0 = \sigma(t^0)$, we can proceed analogously as in the case before. In this case, for fixed δ, either $t^{0,\delta} = t^0$ (if $y^{0,\delta}(t^0) \notin \omega(t^0)$) or $t^{0,\delta} > t^0$ (if $y^{0,\delta}(t^0) \in \omega(t^0)$). In the alternative $t^{0,\delta} = t^0$, it is obvious that

$$\lim_{\delta \to 0} y^{0,\delta}\big(t^{0,\delta}\big) = y^0\big(t^0\big)$$

and thus

$$\lim_{\delta \to 0} (P_2 \circ P_1)\big(t_0, \varphi_{0,\delta}(t_0)\big) = \lim_{\delta \to 0} P_2\big(t^{0,\delta}, y^{0,\delta}\big(t^{0,\delta}\big)\big) = \big(t^0, y^0\big(t^0\big)\big).$$

Hence the composite mapping $P_2 \circ P_1$ is continuous. Further, $P_3 \circ P_2 \circ P_1$ is also continuous because of the continuity of mappings P_3 and $P_2 \circ P_1$. The alternative $t^{0,\delta} > t^0$ with $\lim_{\delta \to 0} t^{0,\delta} = t^0$ can be proved by the same limit process as in the first case, where the point t^0 is dense.

Third, let the point t^0 be left-dense and right-scattered, *i.e.*, $\rho(t^0) = t^0 < \sigma(t^0)$. Then the approach used in previous cases can be modified as follows. Let, as before, (11), (12) and (13) hold. Then, for fixed δ, either $t^{0,\delta} \le t^0$ (if $y^{0,\delta}(t^0) \notin \omega(t^0)$) or $t^{0,\delta} = \sigma(t^0)$ (if $y^{0,\delta}(t^0) \in \omega(t^0)$). The alternative $t^{0,\delta} \le t^0$ with $\lim_{\delta \to 0} t^{0,\delta} = t^0$ can be proved by the same limit process as in the first case, where the point t^0 is dense. However, the alternative $t^{0,\delta} = \sigma(t^0)$ takes into account a possibility that the mapping P_1 cannot be continuous. If $t^{0,\delta} = \sigma(t^0)$ is valid for $\delta \to 0$, then, due to the convexity of $V(t^0, \sigma(t^0))$, there exists a unique intersection of the segment connecting the points $(t^0, y^{0,\delta}(t^0))$ and $(\sigma(t^0), y^{0,\delta}(\sigma(t^0)))$ with $\partial_y \Omega_{\mathbb{R}}$. We denote this point as $M^{0,\delta}$ and, in accordance with the above definition, $P_2(\sigma(t^0), y^{0,\delta}(\sigma(t^0))) = M^{0,\delta}$. We wish to show that $M^{0,\delta} \to M^0 = (t^0, y^0(t^0))$ if $\delta \to 0$. However, in view of the definition of the mapping P_2 and its geometric meaning,

$$\lim_{\delta \to 0} P_2\big(\sigma\big(t^0\big), y^{0,\delta}\big(\sigma\big(t^0\big)\big)\big) = \lim_{\delta \to 0} M^{0,\delta} = \big(t^0, y^0\big(t^0\big)\big)$$

and

$$\lim_{\delta \to 0} (P_2 \circ P_1)\big(t_0, \varphi_{0,\delta}(t_0)\big) = \big(t^0, y^0\big(t^0\big)\big).$$

Hence the continuity of $P_2 \circ P_1$ is proved. Further, $P_3 \circ P_2 \circ P_1$ is also continuous (for the same reason as before).

Fourth, suppose now that t^0 is an isolated point, *i.e.*, $\rho(t^0) < t^0 < \sigma(t^0)$. Let, as before, (11), (12) and (13) hold. Then, for sufficiently small δ, we have either $t^{0,\delta} = t^0$ (if $y^{0,\delta}(t^0) \notin \omega(t^0)$)

or $t^{0,\delta} = \sigma(t^0)$ (if $y^{0,\delta}(t^0) \in \omega(t^0)$). Without any special comment, in the alternative $t^{0,\delta} = t^0$, we proceed in the same way as in the second case, where t^0 is left-scattered and right-dense. Furthermore, in the alternative $t^{0,\delta} = \sigma(t^0)$, we proceed in the same way as in the third case, where t^0 is left-dense and right-scattered.

(II) Point $(t^0, y^0(t^0)) \notin \partial_y\Omega$. Then the point t^0 is left-scattered and only two cases are possible (either t^0 is left-scattered and right-dense or t^0 is isolated). In both mentioned cases, we can proceed in the same way. Let, as before, $\varphi_0(t_0), \varphi_{0,\delta}(t_0) \in \omega(t_0)$ and (11), (12) and (13) hold. Then, for sufficiently small δ, we have only $t^{0,\delta} = t^0$ and, of course, $y^{0,\delta}(t^0) \notin \overline{\omega}(t^0)$. Further,

$$\lim_{\delta\to 0} P_1\big(t_0, \varphi_{0,\delta}(t_0)\big) = \lim_{\delta\to 0}\big(t^{0,\delta}, y^{0,\delta}\big(t^{0,\delta}\big)\big) = \big(t^0, y^0\big(t^0\big)\big) \notin \overline{\Omega}.$$

Due to the convexity of $V(\rho(t^0), t^0)$, there exists a unique intersection of the segment connecting the points $(\rho(t^0), y^{0,\delta}(\rho(t^0)))$ and $(t^0, y^{0,\delta}(t^0))$ with $\partial_y\Omega_{\mathbb{R}}$. We denote this point by $M^{0,\delta}$. In view of the definition of the mapping P_2 and its geometric meaning, we can observe that

$$\lim_{\delta\to 0}(P_2 \circ P_1)\big(t_0, \varphi_{0,\delta}(t_0)\big) = \lim_{\delta\to 0} P_2\big(t^0, y^{0,\delta}\big(t^0\big)\big) = \lim_{\delta\to 0} M^{0,\delta} = M^0.$$

Hence the composite mapping $P_2 \circ P_1$ is continuous. Moreover, $P_3 \circ P_2 \circ P_1$ is also continuous because of the continuity of mappings P_3 and $P_2 \circ P_1$.

We proved that the composite mapping $P_3 \circ P_2 \circ P_1$ is continuous. Note that we omitted the special case $\varphi_0(t_0) \in \partial\omega(t_0)$ (which is equivalent to $t^0 = t_0$). However, this part can be shown in an analogous and simpler way to the one used above.

Now we are able to finish the proof. We proved that

$$P := P_3 \circ P_2 \circ P_1 : B \to A,$$

where $A := \partial_y\Omega|_{t=t_0}$, $B := \{(t_0, y) : y \in \overline{\omega}(t_0)\}$ is continuous. Moreover,

$$P : A \to A$$

is an identity mapping. In this situation, we have proved that there exists a retraction of the set B onto the set A (see Definition 3). In view of the above-mentioned fact, this is impossible. Our assumption is false and there exists initial problem (7) such that the corresponding solution $y = y^*(t)$ satisfies (8) for every $t \in [\alpha_0, \infty)_{\mathbb{T}}$. The theorem is proved. □

4 Example

Let us consider a dynamic system of type (1)

$$\begin{aligned} y_1^{\Delta}(t) &= f_1\big(t, y_1\big(\tau_1(t)\big), y_2\big(\tau_2(t)\big)\big) \\ &= y_1^4\big(\tau_1(t)\big) + \frac{\cos(ty_2(\tau_2(t)))}{t^3} + \frac{y_2(\tau_2(t))}{t^2+1}, \qquad (14) \\ y_2^{\Delta}(t) &= f_2\big(t, y_1\big(\tau_1(t)\big), y_2\big(\tau_2(t)\big)\big) \\ &= \frac{y_1(\tau_1(t))}{t^2} + y_2^6\big(\tau_2(t)\big) + \frac{\sin(y_2(\tau_2(t)))}{t^3+2} \qquad (15) \end{aligned}$$

defined for each $t \in [1,\infty)_{\mathbb{T}}$ with $\mathbb{T} = \{2^n\}_{n=0}^{\infty}$. Note that in this case $\mu(t) = t$. Let delays be defined as follows:

$$\tau_1(t) = \frac{t}{2}, \qquad \tau_2(t) = \frac{t}{4}.$$

Let, moreover, $t_0 \geq 16$, $t_0 \in \mathbb{T}$ and $\alpha_0 = \min\{\tau_1(t_0), \tau_2(t_0)\} = \tau_2(t_0) = t_0/4$.

With the aid of Theorem 2, we will show that there exists an initial function φ^*,

$$\varphi_i^*(t) \in \left(-t^{-1}, t^{-1}\right), \quad t \in [\alpha_0, t_0]_{\mathbb{T}}, i = 1,2, \tag{16}$$

which defines a solution y for all $t \in [\alpha_0,\infty)_{\mathbb{T}}$ of dynamic system (14), (15) satisfying

$$\left|y_i(t)\right| < t^{-1}, \quad i = 1,2. \tag{17}$$

We define Δ-differentiable functions $b_i, c_i : \mathbb{T} \to \mathbb{R}$, $i = 1,2$, satisfying $b_i(t) < c_i(t)$ for each $t \in [\alpha_0,\infty)_{\mathbb{T}}$ as

$$b_i(t) := -t^{-1}, \qquad c_i(t) := t^{-1}$$

and

$$\Omega := \left\{(t,y) : t \in [\alpha_0,\infty)_{\mathbb{T}}, -t^{-1} < y_i < t^{-1}, i = 1,2\right\}.$$

We will verify that every point $M \in \bigcup_{i=1}^{2}(\Omega_B^i \cup \Omega_C^i)$, where

$$\Omega_B^i := \left\{(t,y) \in \partial_y\Omega : y_i = -t^{-1}, i = 1,2\right\},$$
$$\Omega_C^i := \left\{(t,y) \in \partial_y\Omega : y_i = t^{-1}, i = 1,2\right\},$$

is a point of strict egress for the set Ω with respect to the dynamic system (14), (15).

(1a) Let $(t, b_1(t), y_2) \in \Omega_B^1$. For arbitrary functions $u_1, u_2 : [\alpha_t, t]_{\mathbb{T}} \to \mathbb{R}$, $t \in [t_0,\infty)_{\mathbb{T}}$ such that (for $j = 1,2$) $b_j(s) < u_j(s) < c_j(s)$, $s \in [\alpha_t, t)_{\mathbb{T}}$ and $u_1(t) = b_1(t)$, $b_2(t) \leq u_2(t) \leq c_2(t)$, we need (see (4))

$$u_1^4\big(\tau_1(t)\big) + \frac{\cos(tu_2(\tau_2(t)))}{t^3} + \frac{u_2(\tau_2(t))}{t^2+1} < \left(-\frac{1}{t}\right)^{\Delta}. \tag{18}$$

(1b) Let $(t, c_1(t), y_2) \in \Omega_C^1$. For arbitrary functions $u_1, u_2 : [\alpha_t, t]_{\mathbb{T}} \to \mathbb{R}$, $t \in [t_0,\infty)_{\mathbb{T}}$ such that (for $j = 1,2$) $b_j(s) < u_j(s) < c_j(s)$, $s \in [\alpha_t, t)_{\mathbb{T}}$ and $u_1(t) = c_1(t)$, $b_2(t) \leq u_2(t) \leq c_2(t)$, we need (see (5))

$$u_1^4\big(\tau_1(t)\big) + \frac{\cos(tu_2(\tau_2(t)))}{t^3} + \frac{u_2(\tau_2(t))}{t^2+1} > \left(\frac{1}{t}\right)^{\Delta}. \tag{19}$$

Inequalities (18) and (19) are valid if the inequality

$$\left|u_1^4\big(\tau_1(t)\big) + \frac{\cos(tu_2(\tau_2(t)))}{t^3} + \frac{u_2(\tau_2(t))}{t^2+1}\right| < \left(-\frac{1}{t}\right)^{\Delta}$$

holds for arbitrary u_j such that $-t^{-1} \leq u_j(t) \leq t^{-1}$, $t \in [t_0, \infty)_{\mathbb{T}}$, $j = 1, 2$ ($t_0 \geq 16$). Indeed,

$$\left| u_1^4\bigl(\tau_1(t)\bigr) + \frac{\cos(tu_2(\tau_2(t)))}{t^3} + \frac{u_2(\tau_2(t))}{t^2+1} \right| \leq \frac{1}{(t/2)^4} + \frac{1}{t^3} + \frac{1}{(t/4)(t^2+1)}$$

$$< \frac{1}{2t^2} = \frac{1}{t(t+\mu(t))} = \left(-\frac{1}{t}\right)^{\Delta}.$$

(2a) Let $(t, y_1(t), b_2(t)) \in \Omega_B^2$. For arbitrary functions $u_1, u_2 : [\alpha_t, t]_{\mathbb{T}} \to \mathbb{R}$, $t \in [t_0, \infty)_{\mathbb{T}}$ such that (for $j = 1, 2$) $b_j(s) < u_j(s) < c_j(s)$, $s \in [\alpha_t, t)_{\mathbb{T}}$ and $u_2(t) = b_2(t)$, $b_1(t) \leq u_1(t) \leq c_1(t)$, we need (see (4))

$$\frac{u_1(\tau_1(t))}{t^2} + u_2^6\bigl(\tau_2(t)\bigr) + \frac{\sin(u_2(\tau_2(t)))}{t^3+2} < \left(-\frac{1}{t}\right)^{\Delta}. \tag{20}$$

(2b) Let $(t, y_1(t), c_2(t)) \in \Omega_C^2$. For arbitrary functions $u_1, u_2 : [\alpha_t, t]_{\mathbb{T}} \to \mathbb{R}$, $t \in [t_0, \infty)_{\mathbb{T}}$ such that (for $j = 1, 2$) $b_j(s) < u_j(s) < c_j(s)$, $s \in [\alpha_t, t)_{\mathbb{T}}$ and $u_2(t) = c_2(t)$, $b_1(t) \leq u_1(t) \leq c_1(t)$, we need (see (5))

$$\frac{u_1(\tau_1(t))}{t^2} + u_2^6\bigl(\tau_2(t)\bigr) + \frac{\sin(u_2(\tau_2(t)))}{t^3+2} > \left(\frac{1}{t}\right)^{\Delta}. \tag{21}$$

Inequalities (20) and (21) are valid if the inequality

$$\left| \frac{u_1(\tau_1(t))}{t^2} + u_2^6\bigl(\tau_2(t)\bigr) + \frac{\sin(u_2(\tau_2(t)))}{t^3+2} \right| < \left(-\frac{1}{t}\right)^{\Delta}$$

holds for arbitrary u_j such that $-t^{-1} \leq u_j(t) \leq t^{-1}$, $t \in [t_0, \infty)_{\mathbb{T}}$, $j = 1, 2$ ($t_0 \geq 16$). Indeed,

$$\left| \frac{u_1(\tau_1(t))}{t^2} + u_2^6\bigl(\tau_2(t)\bigr) + \frac{\sin(u_2(\tau_2(t)))}{t^3+2} \right| \leq \frac{1}{t^2(t/2)} + \frac{1}{(t/4)^6} + \frac{1}{t^3+2}$$

$$< \frac{1}{2t^2} = \frac{1}{t(t+\mu(t))} = \left(-\frac{1}{t}\right)^{\Delta}.$$

In view of Definition 2, every point $M \in \bigcup_{i=1}^{2}(\Omega_B^i \cup \Omega_C^i)$ is a point of strict egress for the set Ω. Therefore, all the assumptions of Theorem 2 hold and there exists an initial value function φ^* with property (16) such that the initial problem $y(t) = \varphi^*(t)$ defines a solution y on the interval $[\alpha_0, \infty)_{\mathbb{T}}$ of dynamic system (14), (15) satisfying inequalities (17) for every $t \in [\alpha_0, \infty)_{\mathbb{T}}$. This solution (due to (17)) tends to zero as $t \to \infty$.

Remark 2 Note that the choice $\mathbb{T} = \{2^n\}_{n=0}^{\infty}$ in the previous example is not important, and system (14), (15) can be considered on an arbitrary time scale $\mathbb{T}$ with $\mu(t) = O(t)$ (it means that there exists $q > 1$ such that $\mu(t) \leq (q-1)t$ for each $t \in \mathbb{T}$). Indeed, let us slightly modify the previous example. Consider system (14), (15) on an arbitrary time scale with $\mu(t) = O(t)$. (For example, $\mathbb{T} = \{q^n\}_{n=0}^{\infty}$ with $q > 1$ satisfies this condition.) Let, moreover, $\tau_1 = t/q$, $\tau_2 = t/q^k$ with $q > 1$, $k \in \mathbb{N}$. Then one can show that all calculations used in the previous example are true for sufficiently large $t_0 \in \mathbb{T}$.

5 Concluding remarks

The difference between delay dynamic equations and non-delay dynamic equations (resp. a system of delay dynamic equations and a system of non-delay dynamic equations) with respect to controlling their solutions is as follows. The conditions on the function f to get a bounded solution in a 'delay' case are a little bit harder than in a 'non-delay' case. More precisely, the bigger the delays τ_i are, the harder it is to construct a set Ω of considered equations to get a bounded solution $y \in \Omega$. The reason is that in a delay case the history of solutions plays an important role and influences conditions for points, which are strict egress with respect to the investigated equation. It corresponds to the form of Definition 2, where functions $u_i(t)$ have to satisfy some conditions before they touch or pass the boundary of the set Ω.

A further possible complication in dynamic equations (resp. a system of dynamic equations) - the graininess of the time scale $\mathbb{T}$ - was discussed in [9]. Moreover, it is obvious that the bigger the graininess is, the bigger the delays are. This fact also implies a problem to control the solution to stay in domain Ω.

Finally, let us consider initial problem (1), (2) with $\tau_i(t) = t$ for every $i = 1, \ldots, n$. In this case, we get a non-delay dynamic system and the initial function φ defined in (2) can be replaced by the initial condition $y(t_0) = y_0$. Moreover, in this case, carefully tracing the proof of Theorem 2, we can observe that it does not need any change. Hence we can say that Theorem 2 generalizes a result given in [7] as well.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors read and approved the final manuscript.

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