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Infinitely many solutions for a fourth-order differential equation on a nonlinear elastic foundation

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Abstract
In this paper, existence results of infinitely many solutions for a fourth-order differential equation with nonlinear boundary conditions are established. The proof is based on variational methods. Some recent results are improved and extended.

1 Introduction

In this paper, we consider a beam equation with nonlinear boundary conditions of the type

$$\begin{cases} u^{(4)} = f(x,u), & 0 < x < 1, \\ u(0) = u'(0) = 0, \\ u''(1) = 0, & u'''(1) = g(u(1)), \end{cases} \tag{1.1}$$

where $f \in C([0,1],\mathbb{R})$ and $g \in C(\mathbb{R})$ are real functions. This kind of problem arises in the study of deflections of elastic beams on nonlinear elastic foundations. The problem has the following physical description: a thin flexible elastic beam of length 1 is clamped at its left end $x = 0$ and resting on an elastic device at its right end $x = 1$, which is given by g. Then, the problem models the static equilibrium of the beam under a load, along its length, characterized by f. The derivation of the model can be found in [1, 2].

Owing to the importance of fourth-order two-point boundary value problems in describing a large class of elastic deflection, there is a wide literature that deals with the existence and multiplicity results for such a problem with different boundary conditions (see, for instance, [3–8] and the references therein).

Motivated by the above works, in the present paper we study the existence of infinitely many solutions for problem (1.1) when the nonlinear term $f(x,u)$ satisfies the superlinear condition and sublinear condition at the infinity on u, respectively. As far as we know, this case has never before been considered.

Now we state our main results.

1.1 The superlinear case

We give the following assumptions.

($\mathcal{H}_1$) g is odd and satisfies

$$2\int_0^s g(t)\,dt - g(s)s \geq 0, \qquad \int_0^s g(t)\,dt \geq 0 \quad \text{for all } s \in \mathbb{R}.$$

($\mathcal{H}_2$) There exist constants $a, b \geq 0$ and $\gamma \in [0,1)$ such that

$$\left|g(s)\right| \leq a + b|s|^\gamma \quad \text{for } s \in \mathbb{R}.$$

($\mathcal{H}_3$) $\lim_{|u|\to+\infty} \frac{F(x,u)}{u^2} = +\infty$ uniformly for $x \in [0,1]$, where $F(x,u) = \int_0^u f(x,t)\,dt$.
($\mathcal{H}_4$) $F(x,0) \equiv 0$, $0 \leq F(x,u) = o(|u|^2)$ as $|u| \to 0$ uniformly for $x \in [0,1]$.
($\mathcal{H}_5$) There exist constants $\alpha > 1$, $1 < \beta < 1 + \frac{\alpha-1}{\alpha}$, $c_1, c_2 > 0$ and $L > 0$ such that for every $x \in [0,1]$ and $u \in \mathbb{R}$ with $|u| \geq L$,

$$f(x,u)u - 2F(x,u) \geq c_1|u|^\alpha, \qquad \left|f(x,u)\right| \leq c_2|u|^\beta.$$

Theorem 1.1 *Assume that* ($\mathcal{H}_1$)-($\mathcal{H}_5$) *hold and F is even in u. Then problem* (1.1) *has infinitely many solutions.*

Remark 1.1 There exist some functions satisfying ($\mathcal{H}_3$)-($\mathcal{H}_5$), but not satisfying the well-known (AR)-condition,

$$0 < \theta F(x,u) \leq f(x,u)u, \quad \forall u > 0, x \in [0,1],$$

for some $\theta > 2$.

For example, take $f(x,u) = 2u\ln(1+u^2)^2 + \frac{4u^3}{1+u^2}$. Then $F(x,u) = |u|^2\ln(1+|u|^2)^2$. Obviously, ($\mathcal{H}_3$)-($\mathcal{H}_4$) are satisfied. Note that

$$f(x,u)u - 2F(x,u) = 2|u|^2\bigl(\ln\bigl(1+|u|^2\bigr)\bigr)\frac{2|u|^2}{1+|u|^2} \geq 2|u|^2\ln 2, \quad \forall |u| \geq 1,$$

and

$$\left|f(x,u)\right| \leq 2\bigl(\ln\bigl(1+|u|^2\bigr)\bigr)^2|u| + \frac{2|u|^2}{1+|u|^2}2\bigl(\ln\bigl(1+|u|^2\bigr)\bigr)|u| \leq 2|u|^{\frac{5}{4}}, \quad \forall |u| \geq L,$$

for L being large enough, which implies ($\mathcal{H}_5$). However, it is easy to see that f does not satisfy (AR)-condition.

1.2 The sublinear case

We make the following assumptions.

($\mathcal{S}_1$) g is odd and satisfies $g(s)s \geq 0$ for any $s \in \mathbb{R}$.
($\mathcal{S}_2$) There exist constants $b > 0$ and $\gamma \in [0,1)$ such that

$$\left|g(s)\right| \leq b|s|^\gamma \quad \text{for } s \in \mathbb{R}.$$

($\mathcal{S}_3$) $F(x,0) \equiv 0$ for any $x \in [0,1]$.

($\mathcal{S}_4$) There are constants $k_1 > 0$ and $\zeta_1 \in [1,2)$ with $\zeta_1 < \gamma + 1$ such that

$$F(x,u) \geq k_1|u|^{\zeta_1} \quad \text{for any } (x,u) \in [0,1] \times \mathbb{R}.$$

($\mathcal{S}_5$) There exist constants $k_2 > 0$ and $\zeta_2 \in [1,2)$ such that

$$\left|f(x,u)\right| \leq k_2|u|^{\zeta_2-1} \quad \text{for any } (x,u) \in [0,1] \times \mathbb{R}.$$

Theorem 1.2 *Assume that* ($\mathcal{S}_1$)-($\mathcal{S}_5$) *hold and F is even in u. Then problem* (1.1) *has infinitely many solutions.*

Remark 1.2 The condition ($\mathcal{S}_1$) implies that $\int_0^s g(t)\,dt \geq 0$.

The remainder of this paper is organized as follows. In Section 2, some preliminary results are presented. In Section 3, we give the proofs of our main results.

2 Variational setting and preliminaries

In this section, the following two theorems will be needed in our argument. Let E be a Banach space with the norm $\|\cdot\|$ and $E = \overline{\bigoplus_{j\in\mathbb{N}} X_j}$ with $\dim X_j < \infty$ for any $j \in \mathbb{N}$. Set $Y_k = \bigoplus_{j=0}^k X_j$, $Z_k = \overline{\bigoplus_{j=k}^\infty X_j}$ and $B_k = \{u \in Y_k : \|u\| \leq \rho_k\}$, $N_k = \{u \in Z_k : \|u\| = r_k\}$ for $\rho_k > r_k > 0$. Consider the C^1-functional $\Phi_\lambda : E \to \mathbb{R}$ defined by

$$\Phi_\lambda(u) = A(u) - \lambda B(u), \quad \lambda \in [1,2].$$

Assume that:

(C_1) Φ_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1,2]$. Furthermore, $\Phi_\lambda(-u) = \Phi_\lambda(u)$ for all $(\lambda,u) \in [1,2] \times E$.

(C_2) $B(u) \geq 0$ for all $u \in E$; $A(u) \to \infty$ or $B(u) \to \infty$ as $\|u\| \to \infty$; or

(C_2)′ $B(u) \leq 0$ for all $u \in E$; $B(u) \to -\infty$ as $\|u\| \to \infty$.

For $k \geq 2$, define $\Gamma_k := \{\gamma \in C(B_k,E) : \gamma \text{ is odd}; \gamma|_{\partial B_k} = id\}$,
$c_k(\lambda) := \inf_{\gamma\in\Gamma_k} \max_{u\in B_k} \Phi_\lambda(\gamma(u))$,
$b_k(\lambda) := \inf_{u\in Z_k,\|u\|=r_k} \Phi_\lambda(u)$,
$a_k(\lambda) := \max_{u\in Y_k,\|u\|=\rho_k} \Phi_\lambda(u)$.

Theorem 2.1 ([9, Theorem 2.1]) *Assume that* (C_1) *and* (C_2) (*or* (C_2)′) *hold. If* $b_k(\lambda) > a_k(\lambda)$ *for all* $\lambda \in [1,2]$, *then* $c_k(\lambda) \geq b_k(\lambda)$ *for all* $\lambda \in [1,2]$. *Moreover, for a.e.* $\lambda \in [1,2]$, *there exists a sequence* $\{u_n^k(\lambda)\}_{n=1}^\infty$ *such that* $\sup_n \|u_n^k(\lambda)\| < \infty$, $\Phi'_\lambda(u_n^k(\lambda)) \to 0$ *and* $\Phi_\lambda(u_n^k(\lambda)) \to c_k(\lambda)$ *as* $n \to \infty$.

Theorem 2.2 ([9, Theorem 2.2]) *Suppose that* (C_1) *holds. Furthermore, we assume that the following conditions hold*:

(D_1) $B(u) \geq 0$; $B(u) \to \infty$ *as* $\|u\| \to \infty$ *on any finite dimensional subspace of* E.

(D_2) *There exist* $\rho_k > r_k > 0$ *such that* $a_k(\lambda) := \inf_{u\in Z_k,\|u\|=\rho_k} \Phi_\lambda(u) \geq 0 > b_k(\lambda) := \max_{u\in Y_k,\|u\|=r_k} \Phi_\lambda(u)$ *for all* $\lambda \in [1,2]$ *and* $d_k(\lambda) := \inf_{u\in Z_k,\|u\|\leq\rho_k} \Phi_\lambda(u) \to 0$ *as* $k \to \infty$ *uniformly for* $\lambda \in [1,2]$.

Then there exist $\lambda_n \to 1$, $u(\lambda_n) \in Y_n$ *such that* $\Phi'_{\lambda_n}|_{Y_n}(u(\lambda_n)) = 0$, $\Phi_{\lambda_n}(u(\lambda_n)) \to c_k \in [d_k(2), b_k(1)]$ *as* $n \to \infty$. *In particular, if* $\{u(\lambda_n)\}$ *has a convergent subsequence for every* k, *then* Φ_1 *has infinitely many nontrivial critical points* $\{u_k\} \subset E \setminus \{0\}$ *satisfying* $\Phi_1(u_k) \to 0^-$ *as* $k \to \infty$.

Now we begin describing the variational formulation of problem (1.1), which is based on the function space

$$E = \{u \in H^2(0,1); u(0) = u'(0) = 0\},$$

where $H^2(0,1)$ is the Sobolev space of all functions $u : [0,1] \to \mathbb{R}$ such that u and its distributional derivative u' are absolutely continuous and u'' belongs to $L^2(0,1)$. Then E is a Hilbert space equipped with the inner product and the norm

$$\langle u, v\rangle = \int_0^1 u''(x)v''(x)\,dx, \qquad \|u\| = \|u\|_2, \tag{2.1}$$

where $\|\cdot\|_p$ denotes the standard L^p norm. In addition, E is compactly embedded in the spaces $L^2(0,1)$ and $C[0,1]$, and therefore, there exist immersion constants $S_2, \bar{S} > 0$ such that

$$\|u\|_2 \le S_2\|u\|, \quad \text{and} \quad \|u\|_\infty \le \bar{S}\|u\|. \tag{2.2}$$

Next, we consider the functional $J : E \to \mathbb{R}$ defined by

$$J(u) = \frac{1}{2}\|u\|^2 - \int_0^1 F(x, u(x))\,dx + G(u(1)), \tag{2.3}$$

where F, G are the primitives

$$F(x,u) = \int_0^u f(x,t)\,dt, \quad \text{and} \quad G(u) = \int_0^u g(t)\,dt. \tag{2.4}$$

Since f, g are continuous, we deduce that J is of class C^1 and its derivative is given by

$$\langle J'(u), \varphi\rangle = \int_0^1 u''(x)\varphi''(x)\,dx - \int_0^1 f(x, u(x))\varphi(x)\,dx + g(u(1))\varphi(1) \tag{2.5}$$

for all $u, \varphi \in E$. Then we can infer that $u \in E$ is a critical point of J if and only if it is a (classical) solution of problem (1.1).

Now we define a class of functionals on E by

$$\begin{aligned} J_\lambda(u) &= \frac{1}{2}\|u\|^2 + G(u(1)) - \lambda\int_0^1 F(x, u(x))\,dx \\ &= A(u) - \lambda B(u), \quad \lambda \in [1,2]. \end{aligned} \tag{2.6}$$

It is easy to know that $J_\lambda \in C^1(E;\mathbb{R})$ for all $\lambda \in [1,2]$ and the critical points of $J_1 = J$ correspond to the weak solutions of problem (1.1). We choose a completely orthonormal basis $\{e_j\}$ of E and define $X_j := \mathbb{R}e_j$. Then Z_k, Y_k can be defined as that at the beginning of Section 2.

3 Proofs of Theorems 1.1 and 1.2

We will prove Theorem 1.1 by using Theorem 2.1. Firstly, we give the following three useful lemmas.

Lemma 3.1 *Under the assumptions of Theorem* 1.1, *there exists* $\rho_k > 0$ *large enough such that* $a_k(\lambda) := \max_{u\in Y_k, \|u\|=\rho_k} J_\lambda(u) \le 0$ *for all* $\lambda \in [1,2]$.

Proof Let $u \in Y_k$, then there exists $\epsilon_1 > 0$ such that

$$\operatorname{meas}\{x \in [0,1] : |u(x)| \ge \epsilon_1 \|u\|\} \ge \epsilon_1, \quad \forall u \in Y_k \setminus \{0\}. \tag{3.1}$$

Otherwise, for any positive integer n, there exists $u_n \in Y_k \setminus \{0\}$ such that

$$\operatorname{meas}\left\{x \in [0,1] : |u_n(x)| \ge \frac{1}{n}\|u_n\|\right\} < \frac{1}{n}$$

for all k. Set $v_n(x) := \frac{u_n(x)}{\|u_n\|} \in Y_k \setminus \{0\}$, then $\|v_n\| = 1$ and

$$\operatorname{meas}\left\{x \in [0,1] : |v_n(x)| \ge \frac{1}{n}\right\} < \frac{1}{n} \tag{3.2}$$

for all k. Since $\dim Y_k < \infty$, it follows from the compactness of the unit sphere of Y_k that there exists a subsequence, say $\{v_n\}$, such that v_n converges to some v_0 in Y_k. Hence, we have $\|v_0\| = 1$. By the equivalence of the norms on the finite-dimensional space Y_k, we have $v_n \to v_0$ in $L^2[0,1]$, *i.e.*,

$$\int_0^1 |v_n - v_0|^2\,dx \to 0 \quad \text{as } n \to \infty. \tag{3.3}$$

Thus there exist $\xi_1, \xi_2 > 0$ such that

$$\operatorname{meas}\{x \in [0,1] : |v_0(x)| \ge \xi_1\} \ge \xi_2. \tag{3.4}$$

In fact, if not, we have

$$\operatorname{meas}\left\{x \in [0,1] : |v_0(x)| \ge \frac{1}{n}\right\} = 0, \quad \textit{i.e.}, \ \operatorname{meas}\left\{x \in [0,1] : |v_0(x)| < \frac{1}{n}\right\} = 1,$$

for all positive integer n. This implies that

$$0 < \int_0^1 |v_0(x)|^2\,dx < \frac{1}{n^2} \to 0$$

as $n \to \infty$, which gives a contradiction. Therefore, (3.4) holds.

Now let

$$\Omega_0 = \{x \in [0,1] : |v_0(x)| \ge \xi_1\}, \qquad \Omega_n = \left\{x \in [0,1] : |v_n(x)| < \frac{1}{n}\right\}$$

and $\Omega_n^\perp = [0,1] \setminus \Omega_n$. By (3.2) and (3.4), we have

$$\begin{aligned}\operatorname{meas}(\Omega_n \cap \Omega_0) &= \operatorname{meas}\bigl(\Omega_0 \setminus \bigl(\Omega_n^\perp \cap \Omega_0\bigr)\bigr) \\ &\geq \operatorname{meas}(\Omega_0) - \operatorname{meas}\bigl(\Omega_n^\perp \cap \Omega_0\bigr) \\ &\geq \xi_2 - \frac{1}{n}\end{aligned}$$

for all positive integer n. Let n be large enough such that $\xi_2 - \frac{1}{n} \geq \frac{1}{2}\xi_2$ and $\xi_1 - \frac{1}{n} \geq \frac{1}{2}\xi_1$. Then we have

$$\bigl|v_n(x) - v_0(x)\bigr|^2 \geq \left(\xi_1 - \frac{1}{n}\right)^2 \geq \frac{1}{4}\xi_1^2, \quad \forall x \in \Omega_n \cap \Omega_0.$$

This implies that

$$\begin{aligned}\int_0^1 |v_n - v_0|^2\,dx &\geq \int_{\Omega_n \cap \Omega_0} |v_n - v_0|^2\,dx \\ &\geq \frac{1}{4}\xi_1^2 \operatorname{meas}(\Omega_n \cap \Omega_0) \\ &\geq \frac{1}{4}\xi_1^2\left(\xi_2 - \frac{1}{n}\right) \geq \frac{1}{8}\xi_1^2\xi_2 > 0\end{aligned}$$

for all large n, which is a contradiction with (3.3). Therefore, (3.1) holds.

For any $u \in Y_k$, let $\Omega_u = \{x \in [0,1] : |u(x)| \geq \epsilon_1\|u\|\}$. By condition $(\mathcal{H}_3)$, for $M = \frac{1}{\lambda\epsilon_1^2} \geq \frac{1}{2\epsilon_1^2} > 0$, there exists $L_1 > 0$ such that

$$F(x,u) \geq M|u|^2, \quad \forall |u| \geq L_1, \forall x \in [0,1].$$

Hence one has

$$F(x,u) \geq M|u|^2 \geq M\epsilon_1^2\|u\|^2, \quad \forall x \in \Omega_u,$$

for all $u \in Y_k$ with $\|u\| \geq \frac{L_1}{\epsilon_1}$. It follows from $(\mathcal{H}_2)$-$(\mathcal{H}_4)$ and (3.1) that

$$\begin{aligned}J_\lambda(u) &= \frac{1}{2}\|u\|^2 + \int_0^{u(1)} g(x)\,dx - \lambda\int_0^1 F(x,u)\,dx \\ &\leq \frac{1}{2}\|u\|^2 + a\bar{S}\|u\| + b\bar{S}^{\gamma+1}\|u\|^{\gamma+1} - \lambda\int_{\Omega_u} F(x,u)\,dx \\ &\leq \frac{1}{2}\|u\|^2 + a\bar{S}\|u\| + b\bar{S}^{\gamma+1}\|u\|^{\gamma+1} - \lambda M\epsilon_1^2\|u\|^2 \\ &= -\frac{1}{2}\|u\|^2 + a\bar{S}\|u\| + b\bar{S}^{\gamma+1}\|u\|^{\gamma+1},\end{aligned}$$

for all $u \in Y_k$ with $\|u\| \geq \frac{L_1}{\epsilon_1}$. Since $\gamma < 1$, for $\|u\| = \rho_k$ large enough, we have $J_\lambda(u) \leq 0$. □

Lemma 3.2 *Under the assumptions of Theorem* 1.1, *there exist* $r_k > 0$, $\widetilde{b_k} \to \infty$ *such that* $b_k(\lambda) := \inf_{u \in Z_k, \|u\| = r_k} J_\lambda(u) \geq \widetilde{b_k}$ *for all* $\lambda \in [1,2]$.

Proof Set $\gamma_k := \sup_{u\in Z_k,\|u\|=1} \|u\|_\infty$. Then $\gamma_k \to 0$ as $k \to \infty$. Indeed, it is clear that $0 < \gamma_{k+1} \leq \gamma_k$, so that $\gamma_k \to \bar{\gamma} \geq 0$, as $k \to \infty$. For every $k \geq 0$, there exists $u_k \in Z_k$ such that $\|u_k\| = 1$ and $\|u_k\|_\infty > \gamma_k/2$. By the definition of Z_k, $u_k \rightharpoonup 0$ in E. Then it implies that $u_k \to 0$ in $C[0,1]$. Thus we have proved that $\bar{\gamma} = 0$. By $(\mathcal{H}_5)$, we have

$$F(x,u) \leq c_3 + c_4|u|^{\beta+1}, \quad (x,u) \in [0,1] \times \mathbb{R}.$$

By $(\mathcal{H}_4)$, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$F(x,u) \leq \epsilon|u|^2, \quad \forall x \in [0,1], |u| \leq \delta.$$

Therefore, there exists $C = C(\epsilon) > 0$ such that

$$F(x,u) \leq \epsilon|u|^2 + C|u|^{\beta+1}, \quad (x,u) \in [0,1] \times \mathbb{R}. \tag{3.5}$$

Hence, for any $u \in Z_k$, choose $\epsilon = (4\lambda p^2)^{-1}$, by $(\mathcal{H}_1)$ and (3.5), we have

$$\begin{aligned}
J_\lambda(u) &= \frac{1}{2}\|u\|^2 + \int_0^{u(1)} g(x)\,dx - \lambda\int_0^1 F(x,u)\,dx \\
&\geq \frac{1}{2}\|u\|^2 - \lambda\int_0^1 \left(\epsilon|u|^2 + C|u|^{\beta+1}\right)dx \\
&\geq \frac{1}{2}\|u\|^2 - \lambda\epsilon S^2\|u\|^2 - \lambda C\|u\|_\infty^{\beta+1} \\
&\geq \frac{1}{4}\|u\|^2 - \lambda C\gamma_k^{\beta+1}\|u\|^{\beta+1}.
\end{aligned}$$

Let $r_k := (8\lambda C\gamma_k^{\beta+1})^{\frac{1}{1-\beta}}$. Then, for any $u \in Z_k$ with $\|u\| = r_k$, we have

$$J_\lambda(u) \geq \frac{1}{8}\left(8\lambda CT\gamma_k^{\beta+1}\right)^{\frac{2}{1-\beta}} := \widetilde{b_k} \to \infty$$

uniformly for λ as $k \to \infty$. □

Lemma 3.3 *Under the assumptions of Theorem* 1.1, *there exist* $\lambda_n \to 1$ *as* $n \to \infty$, $\{u_n(k)\}_{n=1}^\infty \subset E$ *such that* $J'_{\lambda_n}(u_n(k)) \to 0$, $J_{\lambda_n}(u_n(k)) \in [\widetilde{b_k}, \widetilde{c_k}]$, *where* $\widetilde{c_k} = \sup_{u\in B_k} \Phi_1(u)$.

Proof It is easy to verify that (C_1) and (C_2) of Theorem 2.1 hold. By Lemmas 3.1, 3.2 and Theorem 2.1, we can obtain the result. □

Proof of Theorem 1.1 For the sake of notational simplicity, in what follows we always set $u_n = u_n(k)$ for all $n \in \mathbb{N}$. By Lemma 3.3, it suffices to prove that $\{u_n\}_{n=1}^\infty$ is bounded and possesses a strong convergent subsequence in E. If not, passing to a subsequence if necessary, we assume that $\|u_n\| \to \infty$ as $n \to \infty$. In view of $(\mathcal{H}_5)$, there exists $c_5 > 0$ such that

$$f(x,u)u - 2F(x,u) \geq c_1|u|^\alpha - c_5 \quad \text{for all } (x,u) \in [0,1] \times \mathbb{R},$$

and combining $(\mathcal{H}_1)$, we have

$$\begin{aligned}2J_{\lambda_n}(u_n) - J'_{\lambda_n}(u_n)u_n &= 2\int_0^{u_n(1)} g(x)\,dx - g\big(u_n(1)\big)u_n(1) \\ &\quad + \lambda_n \int_0^1 \big[f(x,u_n)u_n - 2F(x,u_n)\big]\,dx \\ &\geq \lambda_n \int_0^1 \big(c_1|u_n|^\alpha - c_5\big)\,dx \\ &= c_1\lambda_n \int_0^1 |u_n|^\alpha\,dx - \lambda_n c_5.\end{aligned}$$

This implies that

$$\frac{\int_0^1 |u_n|^\alpha\,dx}{\|u_n\|} \to 0 \quad \text{as } n \to \infty. \tag{3.6}$$

Note that from $(\mathcal{H}_5)$, $1 < \beta < 1 + \frac{\alpha-1}{\alpha}$. Let $\eta = \frac{\alpha-1}{\alpha(\beta-1)}$, then

$$\eta > 1, \qquad \eta\beta - 1 = \eta - \frac{1}{\alpha}. \tag{3.7}$$

By $(\mathcal{H}_5)$, there exists $c_6 > 0$ such that

$$\big|f(x,u)\big|^\eta \leq c_2^\eta |u|^{\eta\beta} + c_6, \quad \forall (x,u) \in [0,1] \times \mathbb{R}. \tag{3.8}$$

By (2.5), $(\mathcal{H}_2)$ and the Hölder inequality, one has

$$\begin{aligned}J'_{\lambda_n}(u_n)u_n &= \|u_n\|^2 + g\big(u_n(1)\big)u_n(1) - \lambda_n \int_0^1 f(x,u_n)u_n\,dx \\ &\geq \|u_n\|^2 - \big(a + b\big|u_n(1)\big|^\gamma\big)\big|u_n(1)\big| - \lambda_n \left(\int_0^1 \big|f(x,u_n)\big|^\eta\,dx\right)^{\frac{1}{\eta}} C_\eta \|u_n\| \\ &\geq \|u_n\|^2 - a\bar{S}\|u_n\| - b\bar{S}^{\gamma+1}\|u_n\|^{\gamma+1} - \lambda_n \left(\int_0^1 \big|f(x,u_n)\big|^\eta\,dx\right)^{\frac{1}{\eta}} C_\eta \|u_n\|, \end{aligned} \tag{3.9}$$

where $C_\eta > 0$ is a constant independent of n. By (3.8) we obtain

$$\begin{aligned}\int_0^1 \big|f(x,u_n)\big|^\eta\,dx &\leq \int_0^1 \big(c_2^\eta |u_n|^{\eta\beta} + c_6\big)\,dx \\ &\leq c_7 \left(\int_0^1 |u_n|^\alpha\,dx\right)^{1/\alpha} \left(\int_0^1 |u_n|^{\frac{\alpha(\eta\beta-1)}{\alpha-1}}\,dx\right)^{1-\frac{1}{\alpha}} + c_6 \\ &\leq c_8 \left(\int_0^1 |u_n|^\alpha\,dx\right)^{1/\alpha} \|u_n\|^{(\eta\beta-1)} + c_6,\end{aligned}$$

combining this inequality with (3.6) and (3.7) yields that

$$\frac{(\int_0^1 |f(x,u_n)|^\eta\,dx)^{\frac{1}{\eta}}}{\|u_n\|} \leq \left[\frac{c_8(\int_0^1 |u_n|^\alpha\,dx)^{1/\alpha}}{\|u_n\|^{1/\alpha}} \frac{\|u_n\|^{(\eta\beta-1)}}{\|u_n\|^{\eta-\frac{1}{\alpha}}} + \frac{c_6}{\|u_n\|^\eta}\right]^{\frac{1}{\eta}} \to 0$$

as $n \to \infty$. Combining this with (3.9), we have

$$1 = \frac{\|u_n\|^2}{\|u_n\|^2} \le \frac{J'_{\lambda_n}(u_n)u_n}{\|u_n\|^2} + \frac{a\bar{S}}{\|u_n\|} + \frac{b\bar{S}^{\gamma+1}}{\|u_n\|^{1-\gamma}} + \frac{\lambda_n(\int_0^1 |f(x,u_n)|^\eta \, dx)^{\frac{1}{\eta}} C_\eta}{\|u_n\|} \to 0 \quad \text{as } n \to \infty,$$

since $0 \le \gamma < 1$. This is a contradiction. Therefore, $\{u_n\}_{n=1}^\infty$ is bounded in E. Without loss of generality, we may assume $u_n \rightharpoonup w_k$ in E. Then $u_n \to w_k$ in $C[0,1]$. Note that

$$\|u_n - w_k\|^2 = \big(J'_{\lambda_n}(u_n) - J'_{\lambda_n}(w_k)\big)(u_n - w_k) - \big(g\big(u_n(1)\big) - g\big(w_k(1)\big)\big)\big(u_n(1) - w_k(1)\big) + \lambda_n \int_0^1 \big(f(x,u_n) - f(x,w_k)\big)(u_n - w_k)\, dx.$$

Taking $n \to \infty$, we have $\lim_{n\to\infty} \|u_n - w_k\| = 0$, which means that $u_n \to w_k$ in E and $J'_1(w_k) = 0$. Hence, J_1 has a critical point w_k with $J_1(w_k) \in [\widetilde{b_k}, \widetilde{c_k}]$. Consequently, we obtain infinitely many solutions since $\widetilde{b_k} \to \infty$. □

Lemma 3.4 *Under the assumptions of Theorem* 1.2, *there exists* ρ_k *small enough such that* $a_k(\lambda) := \inf_{u\in Z_k, \|u\|=\rho_k} J_\lambda(u) \ge 0$ *and* $d_k(\lambda) := \inf_{u\in Z_k, \|u\|\le\rho_k} J_\lambda(u) \to 0$ *as* $k \to \infty$ *uniformly for* $\lambda \in [1,2]$.

Proof For any $u \in Z_k$, by using $\gamma_k := \sup_{u\in Z_k, \|u\|=1} \|u\|_\infty$ defined in Lemma 3.2, together with $(\mathcal{S}_1)$ and $(\mathcal{S}_5)$, we have

$$\begin{aligned} J_\lambda(u) &= \frac{1}{2}\|u\|^2 + \int_0^{u(1)} g(x)\, dx - \lambda \int_0^1 F(x,u)\, dx \\ &\ge \frac{1}{2}\|u\|^2 - \lambda k_2 \int_0^1 |u|^{\zeta_2}\, dx \ge \frac{1}{2}\|u\|^2 - \lambda k_2 \|u\|_\infty^{\zeta_2} \\ &\ge \frac{1}{2}\|u\|^2 - \lambda k_2 \gamma_k^{\zeta_2} \|u\|^{\zeta_2} = \frac{1}{4}\rho_k^2 \ge 0 \end{aligned}$$

for all $u \in Z_k$ with $\|u\| = \rho_k := (4\lambda k_2 \gamma_k^{\zeta_2})^{1/(2-\zeta_2)}$. Obviously, $\rho_k \to 0$ as $k \to \infty$. So $a_k(\lambda) := \inf_{u\in Z_k, \|u\|=\rho_k} J_\lambda(u) \ge 0$ and $d_k(\lambda) := \inf_{u\in Z_k, \|u\|\le\rho_k} J_\lambda(u) \to 0$ as $k \to \infty$ uniformly for $\lambda \in [1,2]$. □

Lemma 3.5 *Under the assumptions of Theorem* 1.2, *there exists* r_k *small enough such that* $b_k(\lambda) := \max_{u\in Y_k, \|u\|=r_k} J_\lambda(u) < 0$ *for all* $\lambda \in [1,2]$.

Proof For any $u \in Y_k$, by $(\mathcal{S}_1)$-$(\mathcal{S}_5)$ and the equivalence of the norms on the finite-dimensional space Y_k, we have

$$\begin{aligned} J_\lambda(u) &= \frac{1}{2}\|u\|^2 + \int_0^{u(1)} g(x)\, dx - \lambda \int_0^1 F(x,u)\, dx \\ &\le \frac{1}{2}\|u\|^2 + b\bar{S}^{\gamma+1}\|u\|^{\gamma+1} - \lambda k_1 \int_0^1 |u|^{\zeta_1}\, dx \\ &\le \frac{1}{2}\|u\|^2 + b\bar{S}^{\gamma+1}\|u\|^{\gamma+1} - \lambda k_1 c_{14} \|u\|^{\zeta_1}. \end{aligned}$$

Since $\zeta_1 < \gamma + 1 < 2$, for $\|u\| = r_k < \rho_k$ small enough, we can get $J_\lambda(u) < 0$ for all $\lambda \in [1,2]$. □

Proof of Theorem 1.2 It is easy to verify that (C_1) and (D_1) hold under the assumptions of Theorem 1.2. By Lemmas 3.4 and 3.5, the condition (D_2) is also satisfied. Therefore, by Theorem 2.2 there exist $\lambda_n \to 1$, $u(\lambda_n) := u_n \in Y_n$ such that $J'_{\lambda_n}|_{Y_n}(u_n) = 0$, $J_{\lambda_n}(u_n) \to c_k \in [d_k(2), b_k(1)]$ as $n \to \infty$. In the following we show that $\{u_n\}_{n=1}^{\infty}$ is bounded. Indeed, note that

$$\begin{aligned} \|u_n\|^2 &= 2J_{\lambda_n}(u_n) - 2\int_0^{u_n(1)} g(x)\,dx + 2\lambda_n \int_0^1 F(x,u_n)\,dx \\ &\leq M_1 + 2b\bar{S}^{\gamma+1}\|u_n\|^{\gamma+1} + 4k_2 \int_0^1 |u_n|^{\zeta_2}\,dx \\ &\leq M_1 + 2b\bar{S}^{\gamma+1}\|u_n\|^{\gamma+1} + 4k_2\bar{S}^{\zeta_2}\|u_n\|^{\zeta_2}, \quad \forall n \in \mathbb{N}, \end{aligned} \tag{3.10}$$

for some $M_1 > 0$. Since $1 < \gamma + 1 < 2$, (3.10) yields that $\{u_n\}$ is bounded in E. By a standard argument, this yields a critical point u^k of J_1 such that $J_1(u^k) \in [d_k(2), c_k(1)]$. Since $d_k(2) \to 0^-$ as $k \to \infty$, we can obtain infinitely many critical points. □

Competing interests
The author declares that she has no competing interests.

Authors' contributions
The author read and approved the final manuscript.

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