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Conservation laws for a generalized Ito-type coupled KdV system

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Abstract

In this paper, the conservation laws for a generalized Ito-type coupled Korteweg-de Vries (KdV) system are constructed by increasing the order of the partial differential equations. The generalized Ito-type coupled KdV system is a third-order system of two partial differential equations and does not have a Lagrangian. The transformation $u = U_x$, $v = V_x$ converts the generalized Ito-type coupled KdV system into a system of fourth-order partial differential equations in U and V variables, which has a Lagrangian. Noether's approach is then used to construct the conservation laws. Finally, the conservation laws are expressed in the original variables u and v. Some local and infinitely many nonlocal conserved quantities are found for the generalized Ito-typed coupled KdV system.

1 Introduction

In this paper, we consider the generalized Ito-type coupled Korteweg-de Vries (KdV) system [1]

$$u_t + \alpha u u_x + \beta v v_x + \gamma u_{xxx} = 0, \tag{1a}$$

$$v_t + \beta u v_x + \beta v u_x = 0. \tag{1b}$$

It is well known that coupled nonlinear systems in which a KdV structure is embedded occur naturally in shallow water wave problems [2, 3]. When $\alpha = -6$, $\beta = -2$ and $\gamma = -1$, the system (1a) and (1b) is called Ito's system and it describes the interaction process of two internal long waves [4, 5]. It should be noted that in the absence of the effect of v, the system (1a) and (1b) reduces to the ordinary KdV equation. In [6] it has been shown that this Ito's system can be a member of a bi-Hamiltonian integrable hierarchy. The numerical methods for this system are very limited [1]. However, Xu and Shu [3] developed local discontinuous Galerkin methods for Ito's system and proved the L^2 stability of these methods and, as a result, showed some good numerical results. Recently, in [1], the generalized Ito-type coupled KdV system (1a) and (1b) was constructed as a multi-symplectic Hamiltonian partial differential equation by introducing some new variables, and multi-symplectic numerical methods were applied to investigate this system.

In this paper, we derive conservation laws for the generalized Ito-type coupled KdV system (1a) and (1b). It is well known that the conservation laws play a central role in the solution and reduction of partial differential equations. Conservation laws are mathematical expressions of the physical laws, such as conservation of energy, mass, momentum

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and so on. In the literature, conservation laws have been extensively used in studying the existence, uniqueness and stability of solutions of nonlinear partial differential equations (see, for example, [7–9]). Conservation laws have also been applied in the development and use of numerical methods (see for example, [10, 11]). Recently, conserved vectors associated with Lie point symmetries have been used to find exact solutions (by exploiting a double reduction method) of some classical partial differential equations [12–14]. Thus, it is important to derive all the conservation laws for a given differential equation.

For variational problems, the celebrated Noether theorem [15] provides an elegant way to construct conservation laws. In fact, it gives an explicit formula for determining a conservation law once a Noether symmetry associated with a Lagrangian is known for an Euler-Lagrange equation. Thus, the knowledge of a Lagrangian is essential in this case. However, there are differential equations, such as scalar evolution differential equations, which do not have a Lagrangian. In such cases, several methods [16–27] have been developed by researchers about the construction of conserved quantities. Comparison of several different methods for computing conservation laws can be found in [21, 27].

2 Conservation laws for the Ito-type coupled KdV system

In this section, we construct conservation laws for the Ito-type KdV system (1a) and (1b). We note that this system does not have a Lagrangian. However, we can put the system into a variational form by letting $u = U_x$, $v = V_x$. Then the Ito-type coupled KdV system (1a) and (1b) transforms into a fourth-order system, *viz.*,

$$U_{tx} + \alpha U_x U_{xx} + \beta V_x V_{xx} + \gamma U_{xxxx} = 0, \tag{2a}$$

$$V_{xt} + \beta V_x U_{xx} + \beta U_x V_{xx} = 0. \tag{2b}$$

It can be readily verified that L given by

$$L = \frac{1}{2} \left\{ \gamma U_{xx}^2 - \frac{\alpha}{3} U_x^3 - \beta U_x V_x^2 - U_t U_x - V_t V_x \right\} \tag{3}$$

is a Lagrangian for the system (2a) and (2b). This is because L satisfies

$$\frac{\delta L}{\delta U} = 0 \quad \text{and} \quad \frac{\delta L}{\delta V} = 0, \tag{4}$$

where $\delta/\delta U$ and $\delta/\delta V$ are the Euler-Lagrange operators defined by

$$\frac{\delta}{\delta U} = \frac{\partial}{\partial U} - D_t \frac{\partial}{\partial U_t} - D_x \frac{\partial}{\partial U_x} + D_t^2 \frac{\partial}{\partial U_{tt}} + D_x^2 \frac{\partial}{\partial U_{xx}} + D_x D_t \frac{\partial}{\partial U_{tx}} - \cdots \tag{5}$$

and

$$\frac{\delta}{\delta V} = \frac{\partial}{\partial V} - D_t \frac{\partial}{\partial V_t} - D_x \frac{\partial}{\partial V_x} + D_t^2 \frac{\partial}{\partial V_{tt}} + D_x^2 \frac{\partial}{\partial V_{xx}} + D_x D_t \frac{\partial}{\partial V_{tx}} - \cdots. \tag{6}$$

Consider the vector field

$$X = \xi^1(t, x, U, V) \frac{\partial}{\partial t} + \xi^2(t, x, U, V) \frac{\partial}{\partial x} + \eta^1(t, x, U, V) \frac{\partial}{\partial U} + \eta^2(t, x, U, V) \frac{\partial}{\partial V}, \tag{7}$$

which has the second extension

$$X^{[2]} = \xi^1(t,x,U,V)\frac{\partial}{\partial t} + \xi^2(t,x,U,V)\frac{\partial}{\partial x} + \eta^1(t,x,U,V)\frac{\partial}{\partial U}$$

$$+ \eta^2(t,x,U,V)\frac{\partial}{\partial V} + \zeta_t^1\frac{\partial}{\partial U_t} + \zeta_t^2\frac{\partial}{\partial V_t} + \zeta_x^1\frac{\partial}{\partial U_x} + \zeta_x^2\frac{\partial}{\partial V_x} + \cdots, \tag{8}$$

where

$$\zeta_t^1 = D_t(\eta^1) - U_t D_t(\xi^1) - U_x D_t(\xi^2), \qquad \zeta_x^1 = D_x(\eta^1) - U_t D_x(\xi^1) - U_x D_x(\xi^2),$$

$$\zeta_t^2 = D_t(\eta^2) - V_t D_t(\xi^1) - V_x D_t(\xi^2), \qquad \zeta_x^2 = D_x(\eta^2) - V_t D_x(\xi^1) - V_x D_x(\xi^2)$$

and

$$D_t = \frac{\partial}{\partial t} + U_t\frac{\partial}{\partial U} + V_t\frac{\partial}{\partial V} + U_{tt}\frac{\partial}{\partial U_t} + V_{tt}\frac{\partial}{\partial V_t} + U_{tx}\frac{\partial}{\partial U_x} + V_{tx}\frac{\partial}{\partial V_x} + \cdots,$$

$$D_x = \frac{\partial}{\partial x} + U_x\frac{\partial}{\partial U} + V_x\frac{\partial}{\partial V} + U_{xx}\frac{\partial}{\partial U_x} + V_{xx}\frac{\partial}{\partial V_x} + U_{tx}\frac{\partial}{\partial U_t} + V_{tx}\frac{\partial}{\partial V_t} + \cdots.$$

We recall that the vector field X, of the form (7), is called a Noether point symmetry generator corresponding to the Lagrangian L if there exists gauge functions $B^1(t,x,U,V)$ and $B^2(t,x,U,V)$ such that

$$X^{[2]}(L) + \{D_t(\xi^1) + D_x(\xi^2)\}L = D_t(B^1) + D_x(B^2). \tag{9}$$

The insertion of L from (3) into Eq. (9) yields

$$-\frac{1}{2}U_x\big[\eta_t^1 + U_t\eta_U^1 + V_t\eta_V^1 - U_t\xi_t^1 - U_t^2\xi_U^1 - U_t V_t\xi_V^1 - U_x\xi_t^2$$

$$- U_t U_t\xi_U^2 - U_x V_t\xi_V^2\big] - \frac{1}{2}V_x\big[\eta_t^2 + U_t\eta_U^2 + V_t\eta_V^2 - V_t\xi_t^1 - U_t V_t\xi_U^1$$

$$- V_t^2\xi_V^1 - V_x\xi_t^2 - U_t V_x\xi_U^2 - V_t V_x\xi_V^2\big]$$

$$- \left(\frac{\alpha}{2}U_x^2 + \frac{\beta}{2}V_x^2 + \frac{1}{2}U_t\right)\big[\eta_x^1 + U_x\eta_U^1 + V_x\eta_V^1$$

$$- U_t\xi_x^1 - U_t U_x\xi_U^1 - U_t V_x\xi_V^1 - U_x\xi_x^2 - U_x^2\xi_U^2 - U_x V_x\xi_V^2\big]$$

$$- \left(\frac{1}{2}V_t + \beta U_x V_x\right)\big[\eta_x^2 + U_x\eta_U^2 + V_x\eta_V^2 - V_t\xi_x^1 - U_x V_t\xi_U^1$$

$$- V_t V_x\xi_V^1 - V_x\xi_x^2 - U_x V_x\xi_U^2 - V_x^2\xi_V^2\big] + \gamma U_{xx}\big[D_x^2\eta^1 - U_t D_x^2\xi^1$$

$$- U_x D_x^2\xi^2 - 2U_{tx}(\xi_x^1 + U_x\xi_U^1 + V_x\xi_V^1) - 2U_{xx}(\xi_x^2 + U_x\xi_U^2 + V_x\xi_V^2)\big]$$

$$+ \frac{1}{2}\Big[\gamma U_{xx}^2 - \frac{\alpha}{3}U_x^3 - \beta U_x V_x^2 - U_t U_x - V_t V_x\Big]\big[\xi_t^1 + U_t\xi_U^1$$

$$+ V_t\xi_V^1 + \xi_x^2 + U_x\xi_U^2 + V_x\xi_V^2\big]$$

$$= B_t^1 + U_t B_U^1 + V_t B_V^1 + B_x^2 + U_x B_U^2 + V_x B_V^2. \tag{10}$$

The analysis of Eq. (10) prompts the following two cases.

2.1 Case 1 $\alpha \neq \beta$

In this case, the splitting of (10) with respect to different combinations of the derivatives of U and V gives an overdetermined system of linear partial differential equations for ξ^1, ξ^2, η^1, η^2, B^1 and B^2. Solving this system yields the following Noether symmetries and gauge terms:

$$
\begin{aligned}
&\xi^1 = c_1, \\
&\xi^2 = c_2, \\
&\eta^1 = E(t), \\
&\eta^2 = F(t), \\
&B^1 = a(t,x), \\
&B^2 = -\frac{1}{2}UE'(t) - \frac{1}{2}VF'(t) + b(t,x), \\
&a_t + b_x = 0.
\end{aligned}
\tag{11}
$$

The above results will now be used to find the components of the conserved vectors for the second-order Lagrangian L. Here we can choose $a = 0$, $b = 0$ as they contribute to the trivial part of the conserved vector. We recall that the conserved vectors for the second-order Lagrangian L are given by [15, 27]

$$
\begin{aligned}
T^1 =\ & -B^1 + \xi^1 L + W^1 \left[\frac{\partial L}{\partial U_t} - D_t \frac{\partial L}{\partial U_{tt}} - D_x \frac{\partial L}{\partial U_{tx}} \cdots, \right] \\
& + W^2 \left[\frac{\partial L}{\partial V_t} - D_t \frac{\partial L}{\partial V_{xt}} - D_x \frac{\partial L}{\partial V_{tt}} \cdots, \right] \\
& + D_t(W^1)\frac{\partial L}{\partial U_{tt}} + D_t(W^2)\frac{\partial L}{\partial V_{tt}},
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
T^2 =\ & -B^2 + \xi^2 L + W^1 \left[\frac{\partial L}{\partial U_x} - D_t \frac{\partial L}{\partial U_{xt}} - D_x \frac{\partial L}{\partial U_{xx}} \cdots, \right] \\
& + W^2 \left[\frac{\partial L}{\partial V_x} - D_t \frac{\partial L}{\partial V_{xt}} - D_x \frac{\partial L}{\partial V_{xx}} \cdots, \right] \\
& + D_x(W^1)\frac{\partial L}{\partial U_{xx}} + D_x(W^2)\frac{\partial L}{\partial V_{xx}},
\end{aligned}
\tag{13}
$$

where $W^1 = \eta^1 - U_t\xi^1 - U_x\xi^2$ and $W^2 = \eta^2 - V_t\xi^1 - V_x\xi^2$ are the Lie characteristic functions. Thus, using (12) and (13) together with (11) and $u = U_x$, $v = V_x$ yields the following independent conserved vectors for our system (1a) and (1b):

$$
\begin{aligned}
T_1^1 =\ & \frac{\gamma u_x^2}{2} - \frac{\alpha u^3}{6} - \frac{\beta u v^2}{2}, \\
T_1^2 =\ & \frac{\alpha u^2}{2}\int u_t\,dx + \frac{\beta v^2}{2}\int u_t\,dx + \gamma u_{xx}\int u_t\,dx + \beta uv\int v_t\,dx - \gamma u_t u_x \\
& + \frac{1}{2}\left[\int u_t\,dx\right]^2 + \frac{1}{2}\left[\int v_t\,dx\right]^2,
\end{aligned}
\tag{14}
$$

$$T_2^1 = \frac{u^2}{2} + \frac{v^2}{2},$$

$$T_2^2 = -\frac{\gamma u_x^2}{2} + \frac{\alpha u^3}{3} + \beta uv^2 + \gamma uu_{xx},$$

(15)

$$T_{(E,F)}^1 = -\frac{u}{2}E(t) - \frac{v}{2}F(t),$$

$$T_{(E,F)}^2 = -\beta uvF(t) - E(t)\left[\frac{\alpha u^2}{2} + \frac{\beta v^2}{2} + \gamma u_{xx}\right] - \frac{1}{2}E(t)\int u_t\,dx$$

(16)

$$\qquad - \frac{1}{2}F(t)\int v_t\,dx + \frac{1}{2}E'(t)\int u\,dx + \frac{1}{2}F'(t)\int v\,dx.$$

It should be noted that the conserved vectors (14) and (16) are nonlocal conserved vectors and (15) is a local conserved vector for the system (1a) and (1b). We now extract two special cases from the conserved vector (16) by letting $E(t) = 1$ and $F(t) = 0$, which gives a nonlocal conserved vector

$$T_3^1 = -\frac{u}{2},$$

$$T_3^2 = -\frac{\alpha u^2}{2} - \frac{\beta v^2}{2} - \gamma u_{xx} - \frac{1}{2}\int u_t\,dx,$$

(17)

and by choosing $E(t) = 0$ and $F(t) = 1$, which gives a nonlocal conserved vector

$$T_4^1 = -\frac{v}{2},$$

$$T_4^2 = -\beta uv - \frac{1}{2}\int v_t\,dx.$$

We note that since the functions $E(t)$ and $F(t)$ are arbitrary, one obtains infinitely many nonlocal conservation laws for the system (1a) and (1b). It should also be noted that for the special values of α, β and γ, namely, $\alpha = -6$, $\beta = -2$ and $\gamma = -1$, we retrieve the three constants of the motion F_0, F_1 and F_2 obtained in [4].

2.2 Case 2 $\alpha = \beta$

In this case, we obtain the following Noether symmetries and gauge terms:

$$\xi^1 = c_1,$$

$$\xi^2 = c_2 + \alpha tc_3,$$

$$\eta^1 = xc_3 + E(t),$$

$$\eta^2 = F(t),$$

(18)

$$B^1 = -\frac{c_3}{2}U + a(t,x),$$

$$B^2 = -\frac{1}{2}UE'(t) - \frac{1}{2}VF'(t) + b(t,x),$$

$$a_t + b_x = 0.$$

Again, we can set $a = 0$, $b = 0$ as they contribute to the trivial part of the conserved vector. The independent conserved vectors for the system (1a) and (1b) in this case are as follows:

$$T_1^1 = \frac{\gamma u_x^2}{2} - \frac{\alpha u^3}{6} - \frac{\alpha u v^2}{2},$$

$$T_1^2 = \frac{\alpha u^2}{2} \int u_t\, dx + \frac{\alpha v^2}{2} \int u_t\, dx + \gamma u_{xx} \int u_t\, dx + \alpha uv \int v_t\, dx - \gamma u_t u_x \tag{19}$$

$$+ \frac{1}{2}\left[\int u_t\, dx\right]^2 + \frac{1}{2}\left[\int v_t\, dx\right]^2,$$

$$T_2^1 = \frac{u^2}{2} + \frac{v^2}{2}, \tag{20}$$

$$T_2^2 = -\frac{\gamma u_x^2}{2} + \frac{\alpha u^3}{3} + \alpha u v^2 + \gamma u u_{xx},$$

$$T_3^1 = \frac{\alpha t u^2}{2} + \frac{\alpha t v^2}{2} + \frac{1}{2}\int u\, dx, \tag{21}$$

$$T_3^2 = -\frac{\alpha \gamma t u_x^2}{2} + \frac{\alpha^2 t u^3}{3} + \alpha\alpha t u v^2 + \alpha\gamma t u u_{xx},$$

$$T_{(E,F)}^1 = -\frac{u}{2}E(t) - \frac{v}{2}F(t),$$

$$T_{(E,F)}^2 = -\alpha uv F(t) - E(t)\left[\frac{\alpha u^2}{2} + \frac{\alpha v^2}{2} + \gamma u_{xx}\right] - \frac{1}{2}E(t)\int u_t\, dx \tag{22}$$

$$- \frac{1}{2}F(t)\int v_t\, dx + \frac{1}{2}E'(t)\int u\, dx + \frac{1}{2}F'(t)\int v\, dx.$$

Note that the conserved vectors (19), (21) and (22) are nonlocal conserved vectors whereas the conserved vector (20) is a local conserved vector for the system (1a) and (1b). The conserved vector (22), for $E(t) = 1$ and $F(t) = 0$, gives a nonlocal conserved vector

$$T_4^1 = -\frac{u}{2},$$

$$T_4^2 = -\frac{\alpha u^2}{2} - \frac{\alpha v^2}{2} - \gamma u_{xx} - \frac{1}{2}\int u_t\, dx, \tag{23}$$

and for $E(t) = 0$ and $F(t) = 1$, it gives a nonlocal conserved vector

$$T_5^1 = -\frac{v}{2},$$

$$T_5^2 = -\alpha uv - \frac{1}{2}\int v_t\, dx.$$

We see that for the arbitrary values of $E(t)$ and $F(t)$, infinitely many nonlocal conservation laws exist for the system (1a) and (1b).

3 Concluding remarks

In this paper, we studied the generalized Ito-type coupled Korteweg-de Vries system (1a) and (1b). This system does not have a Lagrangian. In order to apply Noether's theorem, we transformed the system into the fourth-order system (2a) and (2b), which admitted a standard Lagrangian (3). Then Noether's approach was used to derive the conservation

laws in U and V variables. Finally, by reverting back to our original variables u and v, we obtained the conservation laws for the third-order generalized Ito-type coupled KdV system (1a) and (1b). The conservation laws for the generalized Ito-type coupled KdV system consisted of some local and infinite number of nonlocal conserved vectors.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

ETM, BM and CMK worked together in the derivation of the mathematical results. All authors read and approved the final manuscript.

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