○ Advances in Difference Equations
a SpringerOpen Journal

RESEARCH   **Open Access**

The properties of solutions of certain type of difference equations

Xiaoguang Qi[1*], Jia Dou[2] and Lianzhong Yang[3]

*Correspondence:
xiaogqi@gmail.com
[1] School of Mathematics, University
of Jinan, Jinan, Shandong 250022,
P.R. China
Full list of author information is
available at the end of the article

Abstract

In this paper, we shall utilize Nevanlinna value distribution theory to study the
solvability of the difference equations of the form: $f(z)^n + p(z)(\Delta_c f)^m = r(z)e^{q(z)}$ and
$f(z)^n + p(z)e^{q(z)}(\Delta_c f)^m = r(z)$, and we shall study the growth of their entire solutions.
Moreover, we will give a number of examples to show that the results in this paper
are the best possible in certain senses. This article extends earlier results by Liu *et al.*
(Czechoslov. Math. J. 61:565-576, 2011; Ann. Pol. Math. 102:129-142, 2011).

MSC: Primary 39A05; secondary 30D35

Keywords: meromorphic functions; difference equation; growth; finite order

1 Introduction and main results

In this paper, we use the basic notions in Nevanlinna theory of meromorphic functions, as
found in [1]. In addition, we use $\delta(f)$, $\lambda(f)$, and $\lambda(\frac{1}{f})$ to denote the order, and the exponents
of the convergence of zeros and poles of a meromorphic function $f(z)$, respectively. We
define difference operators as $\Delta_c f = f(z + c) - f(z)$, where c is a non-zero constant.

In 2001, Yang [2] started to study the existence and uniqueness of finite order entire
solutions of the following type of non-linear differential equation:

$$L(f) - p(z)f(z)^n = h(z), \qquad (1)$$

where $L(f)$ is a linear differential polynomial in f with polynomial coefficients, $p(z)$ is a
non-vanishing polynomial, $h(z)$ is an entire function, and n is an integer such that $n \geq 4$.
Subsequently, several papers have appeared in which the solutions of (1) are studied. The
reader is invited to see [3–5].

Recently, Yang and Laine [6] considered the existence of the non-linear differential-
difference equation of the form

$$f(z)^n + L(z, f) = h(z), \qquad (2)$$

where $L(z, f)$ is a finite sum of product of f, derivatives of f, and their shifts. They obtained
the following result.

Theorem A *Let $n \geq 4$ be an integer, $L(z, f)$ be a linear differential-difference polynomial of
f, with small meromorphic coefficients, and $h(z)$ be a meromorphic function of finite order.*

Springer

Then (2) *possesses at most one admissible transcendental entire solution of finite order, unless $L(z,f)$ vanishes identically. If such a solution $f(z)$ exists, then $f(z)$ is of the same order as $h(z)$.*

In particular, it is shown in [6] that the equation

$$f(z)^2 + P(z)f(z+1) = Q(z)$$

has no transcendental entire solution of finite order, where $P(z)$, $Q(z)$ are polynomials.

Some improvements of Theorem A can be found in [7–10] as well. Replacing $f(z+c)$ with $\Delta_c f$, one of the present authors considered the existence of entire solutions of

$$f(z)^n + P(z)(\Delta_c f)^m = Q(z), \tag{3}$$

we get the following result in [11].

Theorem B *Let $P(z)$, $Q(z)$ be polynomials, n and m be integers satisfying $n > m \geq 0$. Then* (3) *has no transcendental entire solution of finite order.*

In fact, the special case of (3) with $n = m$, and $p(z) = q(z) = 1$, can be viewed as the Fermat type functional equation. It is well known that (3) has no transcendental entire solutions when $n \geq 3$, which can be seen in [12].

It is natural to ask what happens if $P(z)$ or $Q(z)$ is a transcendental entire function in (3). Corresponding to this question, we will investigate the finite order entire (meromorphic) solution of

$$f(z)^n + p(z)(\Delta_c f)^m = r(z)e^{q(z)} \quad \text{and} \quad f(z)^n + p(z)e^{q(z)}(\Delta_c f)^m = r(z).$$

Theorem 1.1 *Consider the non-linear difference equation of the form*

$$f(z)^n + p(z)(\Delta_c f)^m = r(z)e^{q(z)}, \tag{4}$$

where $p(z) \not\equiv 0$, $q(z)$, $r(z)$ are polynomials, n and m are positive integers. Suppose that $f(z)$ is a transcendental entire function of finite order, not of period c. If $n > m$, then $f(z)$ cannot be a solution of (4).

Remarks (1) In case $n \leq m$, Theorem 1.1 is not true. In the special case that $m = n = 1$, the function $f(z) = e^z$ solves

$$f(z) + z\Delta_{\ln 2} f = (z+1)e^z.$$

Moreover, the entire function $f(z) = e^z + z$ is a solution of the following equation:

$$f(z) + \frac{z}{4\pi}(\Delta_{2\pi i} f)^2 = e^z.$$

(2) From Theorem 1.1, we get the following result: Let $f(z)$ be a transcendental entire function of finite order, then for $n > m$, $f(z)^n + p(z)(\Delta_c f)^m$ assumes zero infinitely often, where $p(z) \not\equiv 0$ is polynomial.

In the following, we will consider the properties of the solutions of the equation

$$f(z)^n + p(z)e^{q(z)}(\Delta_c f)^m = r(z), \tag{5}$$

where $p(z) \not\equiv 0$, $q(z)$, $r(z)$ are polynomials. In fact, (5) may have solutions. For example, the function $f(z) = ze^z$ is a solution of the equation $f(z)^3 - \frac{z^3}{2\pi i}e^{2z}\Delta_{2\pi i}f = 0$. One solution of the equation $f(z) - e^z \frac{1}{2\pi i}\Delta_{2\pi i}f = z$ is the function $f(z) = e^z + z$. In addition, the function $f(z) = e^z + z$ can solve $f(z) + e^z \frac{1}{4\pi}(\Delta_{2\pi i}f)^2 = z$ as well. Hence, here we just study the order of the growth of the solutions. We state our findings as follows.

Theorem 1.2 *Let $p(z) \not\equiv 0$, $q(z)$, $r(z)$ be polynomials, n and m be positive integers satisfying $n > m$. Let $f(z)$ be finite order entire solutions of* (5), *then $\delta(f) = \deg q(z)$.*

Remark We will give the following examples to show that the assumption that $n > m$ in Theorem 1.2 is sharp. Clearly, $f(z) = e^z$ is a solution of the equation $f(z) - \frac{1}{2}\Delta_{\ln 2}f = 0$. The function $f(z) = 4e^{8\pi iz} - e^{4\pi iz} + z$ solves $f(z) - (\Delta_{-\frac{1}{4}}f)^2 = z - \frac{1}{16}$. However, $\delta(f) \neq \deg q(z)$ in the above two examples.

Corollary 1.3 *Let $p(z) \not\equiv 0$, $q(z)$, $r(z)$ be polynomials, $n > 1$. Let $f(z)$ be finite order entire solutions of the equation*

$$f(z)^n + p(z)e^{q(z)}\Delta_c f = r(z), \tag{6}$$

then $\delta(f) = \deg q(z)$.

According to (6), we will give a further discussion on the existence of meromorphic solutions. We get:

Theorem 1.4 *Let $p(z) \not\equiv 0$, $q(z)$, $r(z)$ be polynomials. Assume one of the following assertions holds*:

(i) $n > 1$, $f(z)$ *is a finite order meromorphic function* (*not entire*),

(ii) $n > 1$, $r(z) \equiv 0$, *and $f(z)$ is a finite order entire function with infinitely many zeros*,

(iii) $n \geq 1$, $r(z) \not\equiv 0$, *and $f(z)$ is a finite order entire function satisfying $\lambda(f) < \delta(f)$.*

Then $f(z)$ cannot be a solution of (6).

Remarks (1) From Theorem 1.4, we know that the solution of the equation $f(z)^n + p(z)e^{q(z)}\Delta_c f = 0$ must be expressed as $f(z) = \alpha(z)e^{\beta(z)}$, where $\alpha(z)$ and $\beta(z)$ are polynomials.

(2) The condition that $n > 1$ in Theorem 1.4 is sharp. In fact, if $n = 1$, then we know the function $f(z) = \frac{e^z}{z}$ is a solution of the equation $f(z) + (\frac{z}{2\pi i} + 1)\Delta_{2\pi i}f = 0$. Moreover, the function $f(z) = \cos z$ can solve $f(z) + \frac{1}{2}\Delta_\pi f = 0$.

(3) Some ideas of this paper are from [13].

2 Preliminary lemmas

Lemma 2.1 [14, Theorem 2.1] *Let $f(z)$ be a meromorphic function of finite order, and let $c \in \mathbb{C}$, then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = S(r,f).$$

Remark From Lemma 2.1, we know $T(r,f) = T(r,f(z+c)) + S(r,f)$ when $f(z)$ is an entire function of finite order.

Lemma 2.2 [15, Theorem 2.4.2] *Let $f(z)$ be a transcendental meromorphic solution of*

$$f^n A(z,f) = B(z,f),$$

where $A(z,f)$, $B(z,f)$ are differential polynomials in f and its derivatives with small meromorphic coefficients a_λ, in the sense of $m(r,a_\lambda) = S(r,f)$ for all $\lambda \in I$. If $d(B(z,f)) \leq n$, then $m(r,A(z,f)) = S(r,f)$.

Lemma 2.3 [1, Theorem 1.56] *Let $f_j(z)$ ($j = 1, 2, 3$) be meromorphic functions that satisfy*

$$\sum_{j=1}^{3} f_j(z) \equiv 1.$$

If $f_1(z)$ is not a constant, and

$$\sum_{j=1}^{3} N\left(r, \frac{1}{f_j}\right) + 2\sum_{j=1}^{3} \overline{N}(r,f_j) < \left(\lambda + o(1)\right) T(r),$$

where $\lambda < 1$ and $T(r) = \max_{1 \leq j \leq 3}\{T(r,f_j)\}$, then either $f_2(z) \equiv 1$ or $f_3(z) \equiv 1$.

Lemma 2.4 [1, Theorem 1.51] *Suppose that $f_j(z)$ ($j = 1, \ldots, n$) ($n \geq 2$) are meromorphic functions and $g_j(z)$ ($j = 1, \ldots, n$) are entire functions satisfying the following conditions.*
  (1) *$\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$.*
  (2) *$1 \leq j < k \leq n, g_j(z) - g_k(z)$ are not constants for $1 \leq j < k \leq n$.*
  (3) *For $1 \leq j \leq n, 1 \leq h < k \leq n$,*

$$T(r,f_j) = o\left\{T\left(r, e^{g_h - g_k}\right)\right\}, \quad r \to \infty, r \notin E,$$

  where $E \subset (1,\infty)$ is of finite linear measure.
Then $f_j(z) \equiv 0$.

3 Proof of Theorem 1.1

If $q(z)$ is a constant or $r(z) \equiv 0$, then the conclusion follows from Theorem B. It remains to consider the case $q(z)$ is a non-constant polynomial and $r(z) \not\equiv 0$. Assume $f(z)$ is a transcendental entire solution of (4), which is finite order, not of period c. Differentiating (4) and eliminating $e^{q(z)}$, we have

$$f(z)^{n-1}\left(nf'(z) - \left(q'(z) + \frac{r'(z)}{r(z)}\right)f(z)\right)$$

$$= \left(q'(z) + \frac{r'(z)}{r(z)}\right)p(z)(\Delta_c f)^m - p'(z)(\Delta_c f)^m - mp(z)(\Delta_c f)^{m-1}(\Delta_c f)'. \tag{7}$$

If $nf'(z) - (q'(z) + \frac{r'(z)}{r(z)})f(z) \equiv 0$, then we have $f(z)^n = Ar(z)e^{q(z)}$. Writing $f(z) = h(z)e^{\frac{q(z)}{n}}$, where $h(z)$ satisfies $h(z)^n = Ar(z)$ and A is a non-zero constant. Substituting $f(z)$ into (4),

we get

$$(A - 1)r(z)e^{q(z)} + p(z)(\Delta_c f)^m \equiv 0. \tag{8}$$

Clearly, if $A = 1$, then $f(z)$ is a period function with period c, which contradicts the assumption. Hence, $A \neq 1$. Let $g = e^{\frac{q(z)}{n}}$, then $(\Delta_c f)^m$ can be expressed as $\sum_{i=0}^{m} \binom{m}{i}(-1)^i h(z)^i h(z + c)^{m-i} g(z)^i g(z + c)^{m-i}$. Furthermore, from Lemma 2.1, we have

$$
\begin{aligned}
T\left(r, (\Delta_c f)^m\right) &= T\left(r, \sum_{i=0}^{m} \binom{m}{i}(-1)^i h(z)^i h(z + c)^{m-i} g(z)^i g(z + c)^{m-i}\right) \\
&= m\left(r, \sum_{i=0}^{m} \binom{m}{i}(-1)^i h(z)^i h(z + c)^{m-i} g(z)^i g(z + c)^{m-i}\right) \\
&\leq m\left(r, \frac{\sum_{i=0}^{m} \binom{m}{i}(-1)^i g(z + c)^{m-i} g(z)^i}{g(z)^m}\right) + m\left(r, g(z)^m\right) + S(r, g) \\
&\leq T\left(r, g(z)^m\right) + S(r, g) = m T(r, g) + S(r, g). \tag{9}
\end{aligned}
$$

From (8) and (9), we get

$$nT(r, g) \leq m T(r, g) + S(r, g),$$

which contradicts the condition that $n > m$. Therefore, we conclude that $nf'(z) - (q'(z) + \frac{r'(z)}{r(z)})f(z) \not\equiv 0$. We discuss the following two cases.

Case 1. $n > m + 1$. Rewrite (7) in the following forms:

$$
\begin{aligned}
&f(z)^{n-m-1}\left(nf'(z) - \left(q'(z) + \frac{r'(z)}{r(z)}\right)f(z)\right) \\
&= \left(q'(z) + \frac{r'(z)}{r(z)}\right)p(z)\frac{(\Delta_c f)^m}{f^m} - p'(z)\frac{(\Delta_c f)^m}{f^m} - mp(z)\frac{(\Delta_c f)^{m-1}(\Delta_c f)'}{f^m}
\end{aligned}
$$

and

$$
\begin{aligned}
&f(z)^{n-m-2}\left(f\left(nf'(z) - \left(q'(z) + \frac{r'(z)}{r(z)}\right)f(z)\right)\right) \\
&= \left(q'(z) + \frac{r'(z)}{r(z)}\right)p(z)\frac{(\Delta_c f)^m}{f^m} - p'(z)\frac{(\Delta_c f)^m}{f^m} - mp(z)\frac{(\Delta_c f)^{m-1}(\Delta_c f)'}{f^m}.
\end{aligned}
$$

Applying Lemma 2.1, Lemma 2.2, and the lemma on the logarithmic derivative to the above two equations, we obtain

$$T\left(r, nf' - \left(q' + \frac{r'}{r}\right)f\right) = m\left(r, nf' - \left(q' + \frac{r'}{r}\right)f\right) + S(r, f) = S(r, f)$$

and

$$T\left(r, f\left(nf' - \left(q' + \frac{r'}{r}\right)f\right)\right) = m\left(r, f\left(nf' - \left(q' + \frac{r'}{r}\right)f\right)\right) + S(r, f) = S(r, f).$$

Hence

$$T(r,f) = S(r,f),$$

which is a contradiction.

Case 2. $n = m + 1$. For this case, (4) and (7) now take the following forms:

$$f^{m+1} + p(z)(\Delta_c f)^m = r(z)e^{q(z)} \tag{10}$$

and

$$f(z)^m \left((m+1)f'(z) - \left(q'(z) + \frac{r'(z)}{r(z)} \right) f(z) \right)$$
$$= \left(q'(z) + \frac{r'(z)}{r(z)} \right) p(z)(\Delta_c f)^m - p'(z)(\Delta_c f)^m - mp(z)(\Delta_c f)^{m-1}(\Delta_c f)'. \tag{11}$$

Set $H(z) = (m+1)f'(z) - (q'(z) + \frac{r'(z)}{r(z)})f(z)$. Dividing (11) by $f(z)^m$, we have

$$H(z) = \left(q'(z) + \frac{r'(z)}{r(z)} \right) p(z) \frac{(\Delta_c f)^m}{f^m} - p'(z) \frac{(\Delta_c f)^m}{f^m} - mp(z) \frac{(\Delta_c f)^{m-1}(\Delta_c f)'}{f^m}.$$

Since $f(z)$ is entire, applying Lemma 2.1 and the lemma on the logarithmic derivative to the above equation, we conclude that

$$T(r,H) = m(r,H) + S(r,f) = S(r,f). \tag{12}$$

Differentiating $H(z)$, we have

$$(m+1)f'' - \left(q' + \frac{r'}{r} \right)f' - \left(q' + \frac{r'}{r} \right)' f$$
$$= \frac{H'}{H} \cdot H = \frac{H'}{H} \left((m+1)f' - \left(q' + \frac{r'}{r} \right)f \right),$$

that is

$$(m+1)f'' - \left(q' + \frac{r'}{r} + (m+1)\frac{H'}{H} \right)f' - \left(q'' - q'\frac{H'}{H} + \left(\frac{r'}{r} \right)' - \frac{H'}{H} \cdot \frac{r'}{r} \right)f = 0.$$

The above equation can be rewritten in the following form:

$$(m+1)\left(\left(\frac{f'}{f} \right)' + \left(\frac{f'}{f} \right)^2 \right) - \left(q' + \frac{r'}{r} + (m+1)\frac{H'}{H} \right)\frac{f'}{f}$$
$$- \left(q'' - q'\frac{H'}{H} + \left(\frac{r'}{r} \right)' - \frac{H'}{H} \cdot \frac{r'}{r} \right) = 0. \tag{13}$$

Suppose z_0 is a zero of $f(z)$ with multiplicity k. If z_0 is a zero of $r(z)$ as well, then the contribution to $N(r, \frac{1}{f})$ is $S(r,f)$. Assuming that z_0 is not a zero of $r(z)$, we will discuss the two subcases:

Subcase 1. Suppose z_0 is a zero of $H(z)$ with multiplicity t. From (13), by simple calculations, we know that $k = 1 + t \le 2t$, which means that z_0 is a contribution of $S(r, f)$ to $N(r, \frac{1}{f})$ by (12).

Subcase 2. Suppose z_0 is not a zero of $H(z)$. By (13), we get $k^2 - k = 0$, then such a zero of $f(z)$ must be simple and we find that $q' + \frac{r'}{r} + (m + 1)\frac{H'}{H}$ must vanish at z_0. That implies that z_0 makes a contribution of $S(r, f)$ to $N(r, \frac{1}{f})$ by (12).

In a word, $N(r, \frac{1}{f}) = S(r, f)$. Therefore, we can assume $f(z)$ to be of the form $f(z) = G(z)e^{s(z)}$, where $s(z)$ is a non-constant polynomial, and $G(z)$ is an entire function satisfying $N(r, \frac{1}{G}) = T(r, G) = S(r, f)$ by the Hadamard factorization theorem. Substituting this expression into (10), we get

$$G(z)^{m+1}e^{(m+1)s(z)} + p(z)\big(G(z + c)e^{s(z+c)} - G(z)e^{s(z)}\big)^m = r(z)e^{q(z)},$$

and so

$$G(z)^{m+1}e^{(m+1)s(z)} + p(z)\left(\sum_{i=0}^{m-1}\binom{m}{i}(-1)^i G(z + c)^{m-i}G(z)^i e^{(m-i)s(z+c)+is(z)}\right)$$
$$+ p(z)(-1)^m G(z)^m e^{ms(z)} = r(z)e^{q(z)},$$

which may be written in the form

$$\frac{G(z)e^{s(z)}}{p(z)(-1)^{m-1}} + \frac{r(z)e^{q(z)}}{p(z)(-1)^m G(z)^m e^{ms(z)}}$$
$$+ \frac{\sum_{i=0}^{m-1}\binom{m}{i}(-1)^i G(z + c)^{m-i}G(z)^i e^{(m-i)s(z+c)+is(z)}}{(-1)^{m-1}G(z)^m e^{ms(z)}} = 1. \tag{14}$$

From Lemma 2.3, we have either

$$f_2(z) = \frac{r(z)e^{q(z)}}{p(z)(-1)^m G(z)^m e^{ms(z)}} \equiv 1$$

or

$$f_3(z) = \frac{\sum_{i=0}^{m-1}\binom{m}{i}(-1)^i G(z + c)^{m-i}G(z)^i e^{(m-i)s(z+c)+is(z)}}{(-1)^{m-1}G(z)^m e^{ms(z)}} \equiv 1.$$

If $f_2(z) \equiv 1$, then we get

$$r(z)e^{q(z)} = p(z)(-1)^m G(z)^m e^{ms(z)} = p(z)(-1)^m f(z)^m.$$

Plugging the above equation into (10), similar to (9), we get $T(r, re^q - p(\Delta_c f)^m) \le mT(r, f) + S(r, f)$. Comparing both sides of (9), we get a contradiction. If $f_3(z) \equiv 1$, then we get $f(z)^{m+1} \equiv r(z)e^{q(z)}$, which means that $\Delta_c f \equiv 0$, and this contradicts the assumption.

Therefore, from the discussions above, we find that a transcendental finite order entire function $f(z)$ which is not of period c, cannot be a solution of (4). The proof of Theorem 1.1 is finished completely.

4 Proof of Theorem 1.2

Applying Lemma 2.1 to (5), we get

$$nm(r,f) = m\left(r, r - pe^q (\Delta_c f)^m\right)$$

$$\leq m\left(r, e^q\right) + m\left(r, \frac{(\Delta_c f)^m}{f^m}\right) + m\left(r, f^m\right) + S(r,f)$$

$$\leq m\left(r, e^q\right) + mm(r,f) + S(r,f),$$

which means that $(n - m)m(r,f) \leq m(r, e^q) + S(r,f)$. From the assumption that $n > m$ and the above inequality, we conclude that $\delta(f) \leq \deg q(z)$.

Now we show that $\delta(f) = \deg q(z)$. Suppose to the contrary that $\delta(f) < \deg q(z)$. Then $\delta(f^n + pe^q (\Delta_c f)^m) = \deg q(z) > 0$ from Remark below Lemma 2.1, and $\delta(r) = 0$. This is a contradiction by (5).

5 Proof of Theorem 1.4

First, we confirm that one period function with period c cannot be a solution of (6). Actually, if $f(z)$ is a period function, then the order of $f(z)$ satisfies $\delta(f) \geq 1$. At the same time, we get $f(z)^n = r(z)$, which is impossible. It remains to consider the function which is not of period c.

(i) Suppose that $f(z)$ is a finite order meromorphic solution of (6), and z_0 is a pole of $f(z)$ with multiplicity t. Then we get $z_0 + c$ is a pole of $f(z)$ with multiplicity $\geq nt$. By calculation, we see that $z_0 + kc$ is a pole of $f(z)$ with multiplicity $n^k t$, where k is a positive integer. Since $n > 1$, we conclude that $\lambda(f) = \infty$, *i.e.*, $\delta(f) = \infty$. This is a contradiction.

(ii) Suppose that a finite order entire solution $f(z)$ has infinitely many zeros and $r(z) \equiv 0$. We assume that all zeros of $p(z)$ are in $D = \{z : |\operatorname{Re} z| \leq M, |\operatorname{Im} z| \leq M\}$, where $M > 0$ is some constant. From the assumption, we know that $f(z)$ has infinitely many zeros which are not in D. If $z_0 \notin D$ satisfies $f(z_0) = 0$, then from (6) we know that $z_0 + kc$ are zeros of $f(z)$. Moreover, we have $z_0 + kc$ are not in D, as $k \to \infty$. In the same way as (i), we get $\lambda(\frac{1}{f}) = \infty$, *i.e.*, $\delta(f) = \infty$, which contradicts the assumption that $\delta(f) < \infty$.

(iii) Suppose $f(z)$ is an entire solution of finite order satisfying $\lambda(f) < \delta(f)$, then we know $\delta(f) \geq 1$ by the conclusion of Corollary 1.3. Hence, from the Hadamard factorization theorem, $f(z)$ can be written as $f(z) = \alpha(z)e^{\beta(z)}$, where $\beta(z)$ is a non-constant polynomial.

Case 1. $n > 1$. From the conclusion of Corollary 1.3, we know $\lambda(f) = \delta(\alpha) < \delta(f) = \deg q(z) = q \geq 1$. So we can rewrite $f(z)$ in the following form:

$$f(z) = \gamma(z)e^{a_q z^q}, \tag{15}$$

where a_q is a non-zero constant, $\delta(\gamma) < q$, and $T(r, \gamma) = S(r,f)$. From the conclusion of Corollary 1.3, we conclude that

$$e^{q(z)} = \mu(z)e^{b_q z^q}, \tag{16}$$

where b_q is a non-zero constant, $\delta(\mu) < q$, and $T(r, \mu) = S(r, f)$. By (6), (15), and (16), we have

$$\gamma(z)^n e^{na_q z^q} + p(z)\mu(z)\big(\gamma(z+c)e^{\nu(z)} - \gamma(z)\big)e^{(a_q+b_q)z^q} = r(z), \tag{17}$$

where $\nu(z)$ is a polynomial such that $\deg \nu(z) \leq q - 1$.

Subcase 1. If $na_q = b_q + a_q$, then (17) can be expressed as

$$\gamma(z)^n + p(z)\mu(z)\big(\gamma(z+c)e^{\nu(z)} - \gamma(z)\big) = \frac{r(z)}{e^{na_q z^q}}.$$

Since $r(z) \not\equiv 0$, we get $\gamma(z)^n + p(z)\mu(z)(\gamma(z+c)e^{\nu(z)} - \gamma(z)) \not\equiv 0$. Comparing the characteristics function of both sides of the above equation, we get a contradiction.

Subcase 2. If $na_q \neq b_q + a_q$, then we rewrite (17) in the form

$$\gamma(z)^n e^{na_q z^q} + p(z)\mu(z)\big(\gamma(z+c)e^{\nu(z)} - \gamma(z)\big)e^{(a_q+b_q)z^q} - r(z) = 0.$$

Applying Lemma 2.4 to the above equation, we get $r(z) \equiv 0$, which contradicts the assumption.

Case 2. $n = 1$. Comparing both sides of (6), we obtain $\delta(f) \geq \deg q(z) = q$ by Remark below Lemma 2.1. If $\delta(f) = q$, then in the same way as Case 1, the conclusion follows. If $k = \delta(f) > q$, then $f(z)$ can be written as $f(z) = \phi(z)e^{a_k z^k}$, where $\delta(\phi) < k$, and $T(r, \phi) = S(r, f)$. Substituting this expression into (6), we get

$$\big(\phi(z) + p(z)e^{q(z)}\big(\phi(z+c)\varphi(z) - \phi(z)\big)\big)e^{a_k z^k} = r(z), \tag{18}$$

where $\delta(\varphi) < k$, and $T(r, \varphi) = S(r, f)$. Since $r(z) \not\equiv 0$, we get $\phi(z) + p(z)e^{q(z)}(\phi(z+c)\varphi(z) - \phi(z)) \not\equiv 0$. Comparing the characteristics function of both sides of the above equation, we get a contradiction.

Author details
[1] School of Mathematics, University of Jinan, Jinan, Shandong 250022, P.R. China. [2] Quancheng Middle School, Jinan, Shandong 250000, P.R. China. [3] School of Mathematics, Shandong University, Jinan, Shandong 250100, P.R. China.

References
1. Yang, CC, Yi, HX: Uniqueness Theory of Meromorphic Functions. Kluwer Academic, Dordrecht (2003)
2. Yang, CC: On entire solutions of a certain type of nonlinear differential equation. Bull. Aust. Math. Soc. **64**, 377-380 (2001)
3. Heittokangas, J, Korhonen, R, Laine, I: On meromorphic solutions of certain nonlinear differential equations. Bull. Aust. Math. Soc. **66**, 331-343 (2002)

4. Li, P, Yang, CC: On the nonexistence of entire solution of certain type of nonlinear differential equations. J. Math. Anal. Appl. **320**, 827-835 (2006)
5. Li, P: Entire solution of certain type of differential equations. J. Math. Anal. Appl. **344**, 253-259 (2008)
6. Yang, CC, Laine, I: On analogies between nonlinear difference and differential equations. Proc. Jpn. Acad., Ser. A, Math. Sci. **86**, 10-14 (2010)
7. Liu, K, Liu, XL, Yang, LZ: Existence of entire solutions of nonlinear difference equations. Czechoslov. Math. J. **61**, 565-576 (2011)
8. Liu, K, Cao, TB: Entire solutions of Fermat type q-difference differential equations. Electron. J. Differ. Equ. **2013**, 59 (2013)
9. Peng, CW, Chen, ZX: On a conjecture concerning some nonlinear difference equations. Bull. Malays. Math. Sci. Soc. **36**, 221-227 (2013)
10. Wen, ZT, Heittokangas, J, Laine, I: Exponential polynomials as solutions of certain nonlinear difference equations. Acta Math. Sin. **28**, 1295-1306 (2012)
11. Qi, XG: Value distribution and uniqueness of difference polynomials and entire solutions of difference equations. Ann. Pol. Math. **102**, 129-142 (2011)
12. Gross, F: On the equation $f^n + g^n = 1$. Bull. Am. Math. Soc. **72**, 86-88 (1966)
13. Laine, I: A note on value distribution of difference polynomials. Bull. Aust. Math. Soc. **81**, 353-360 (2010)
14. Halburd, R, Korhonen, R: Nevanlinna theory for the difference operator. Ann. Acad. Sci. Fenn., Math. **31**, 463-478 (2006)
15. Laine, I: Nevanlinna Theory and Complex Differential Equations. de Gruyter, Berlin (1993)