Goltser and Domoshnitsky *Advances in Difference Equations* 2013, **2013**:187
http://www.advancesindifferenceequations.com/content/2013/1/187

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

About reducing integro-differential equations with infinite limits of integration to systems of ordinary differential equations

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Abstract

The purpose of this paper is to propose a method for studying integro-differential equations with infinite limits of integration. The main idea of this method is to reduce integro-differential equations to auxiliary systems of ordinary differential equations.

Results: a scheme of the reduction of integro-differential equations with infinite limits of integration to these auxiliary systems is described and a formula for representation of bounded solutions, based on fundamental matrices of these systems, is obtained.

Conclusion: methods proposed in this paper could be a basis for the Floquet theory and studies of stability, bifurcations, parametric resonance and various boundary value problems. As examples, models of tumor-immune system interaction, hematopoiesis and plankton-nutrient interaction are considered.

MSC: 45J05; 45J15; 34A12; 34K05; 34K30; 47G20

Keywords: integro-differential equations; fundamental matrix; Cauchy matrix; hyperbolic systems

1 Introduction

Integro-differential equations appeared very naturally in various applications (see, for example, [1–5]), which explains the interest in the theory of these equations (see, for example, [6, 7]). Various examples, in which the simple enough integro-differential equation

$$x'(t) = X\left(t, x(t), \int_0^t F\bigl(t, s, x(s)\bigr)\, ds\right), \tag{1.1}$$

by elementary operations can be reduced to a system of ordinary differential equations, are known. In this connection, let us refer, for example, to the monograph [8]. Note the idea of the chain trick used in various applications (see, for example, [9, 10]) and its developed form in the paper [11]. Independently, the idea of a reduction to systems of ordinary differential equations in the study of stability, which was, actually, the chain trick, was presented in [12]. Starting with this reduction, approaches to the study of stability and bifurcation of integro-differential equations were proposed in the papers [13–16]. The approach developed in these papers allowed researchers to define a notion of periodic integro-differential systems and to build the Floquet theory for integro-differential equations on this basis in

[17]. The first known results on estimates of distance between two adjacent zeros of oscillating solutions to a linearization of equation (1.1) and results connecting oscillation behavior and the exponential stability were obtained on this basis [17]. A parametric resonance in linear almost periodic systems was studied in [18], and the bifurcation of steady resonance modes for integro-differential systems was investigated in [19]. Stabilization by control in a form of integrals of solutions was studied in [20]. The stability of partial functional differential equations on the basis of this reduction was studied in [21]. Constructive approach to a phase transition model was presented in [22]. A reduction to infinite dimensional systems was considered in [21, 23, 24]. In all these papers the limits of integration in integral terms were 0 and t, and this was very essential.

The main goal of this paper is to present a method reducing integro differential equations with infinite limits of integration

$$x'(t) = f\left(t, x(t), \int_{-\infty}^{+\infty} K(t,s) g\big(s, x(s)\big)\, ds\right), \quad t \in (-\infty, +\infty), \tag{1.2}$$

to systems of ordinary differential equations. In a future we are planning to develop the ideas of noted above papers for equation (1.2). As well as we know, there are no results of this type. Important motivation in the study of integro-differential equation (1.2) can be found also in various applications of such equations in, for example, models of tumor-immune system interaction [9], hematopoiesis [10], stability and persistence in plankton models [25] which will be considered below.

Denote

$$u(t) = \int_{-\infty}^{+\infty} K(t,s) g\big(s, x(s)\big)\, ds, \tag{1.3}$$

$$v(t) = \int_{-\infty}^{t} K(t,s) g\big(s, x(s)\big)\, ds, \tag{1.4}$$

$$w(t) = \int_{t}^{+\infty} K(t,s) g\big(s, x(s)\big)\, ds. \tag{1.5}$$

Using these notations, we can write

$$x'(t) = f\big(t, x(t), u(t)\big), \quad t \in (-\infty, +\infty), \tag{1.6}$$

or

$$x'(t) = f\big(t, x(t), v(t) + w(t)\big), \quad t \in (-\infty, +\infty).$$

It is possible to represent the vector $x \in R^n$ in the form $x = \operatorname{col}\{y, z\}$, where $y \in R^k$, $z \in R^{n-k}$. In many applications, system (1.4) can be represented in the form

$$y'(t) = Y\big(t, x(t), v(t)\big), \qquad z'(t) = Z\big(t, x(t), w(t)\big), \quad t \in (-\infty, +\infty). \tag{1.7}$$

The first equation in (1.7) depends on its integral part v on delay only (see (1.4)) and the second one is dependent on advance only. Note that the cases $k = n$ and $k = 0$ can be also considered. If $k = n$, we get a system with distributed delay, and if $k = 0$, the one with distributed advance. Note that a combination of distributed and concentrated deviations is also possible. Considering such systems, we do not discuss questions of existence of solutions and assume that solutions to these systems exist. Note that even for the Volterra equation, one-point problem (1.1) with the condition $x(t_0) = x_0$, $t_0 > 0$, can have more than one solution or not have solutions at all (see, for example, [26], Chapter 1, Section 9, pp. 70-74).

For system (1.2) our method essentially uses the properties of linear nonhomogeneous systems of ODEs, possessing exponential dichotomy [27] or hyperbolicity [28]. It is known that such systems have (under corresponding conditions) unique bounded on the axis solution. Corresponding bibliography can be found in [28]. The case of autonomous systems was considered in [29, 30]. Below, in the next paragraph, we formulate, in convenient for us form, a result about the existence and structure of the solution for general non-autonomous linear systems of ODEs. This result is based on the theorem about reduction of hyperbolic systems to a block diagonal form [28].

2 Methods: about bounded solutions of linear nonhomogeneous systems

Consider

$$x'(t) = P(t)x(t) + g(t), \quad t \in (-\infty, +\infty) \tag{2.1}$$

and the corresponding homogeneous system

$$w'(t) = P(t)w(t), \quad t \in (-\infty, +\infty), \tag{2.2}$$

where $x, w \in R^n$, P is an $n \times n$ matrix and g is an n-vector function with continuous bounded elements.

We use the following definition introduced in [28].

Definition 2.1 We say that system (2.2) is hyperbolic if there exist constants $a > 0$ and $\lambda > 0$ and hyperplanes M_+ and M_-: $\dim M_+ = k$, $\dim M_- = n - k$ such that if for $t = t_0$, $w(t_0) = w_0 \in M_+$, then the solution $w(t, t_0, w_0)$ satisfies the inequality

$$\left|w(t, t_0, w_0)\right| \le a|w_0|e^{-\lambda(t-t_0)}, \quad t \ge t_0, \tag{2.3}$$

and if $w_0 \in M_-$, the inequality

$$\left|w(t, t_0, w_0)\right| \le a|w_0|e^{\lambda(t-t_0)}, \quad t \le t_0. \tag{2.4}$$

Theorem 2.1 [28] *Let system* (2.2) *be hyperbolic. Then there exists an* $n \times n$ *matrix* $U(t)$ *with bounded elements such that its inverse matrix* $U^{-1}(t)$ *also possesses bounded elements and the transform* $w = U(t)\eta$ *reduces system* (2.2) *to the form*

$$\xi'(t) = Q^+(t)\xi(t), \qquad \zeta'(t) = Q^-(t)\zeta(t), \tag{2.5}$$

where $\eta = \operatorname{col}\{\xi, \zeta\}$, $\xi \in R^k$, $\zeta \in R^{n-k}$.

If we denote $\Phi_+(t,s) = \phi_+(t)\phi_+^{-1}(s)$, *where* $\phi_+(t)$ *is a fundamental matrix of the first system in* (2.5), $\Phi_-(t,s) = \phi_-(t)\phi_-^{-1}(s)$, *where* $\phi_-(t)$ *is a fundamental matrix of the second system in*

(2.5), *such that* $\Phi_+(s,s) = E_+$, $\Phi_-(s,s) = -E_-$, $\dim E_+ = k$, $\dim E_- = n-k$, *then*

$$\left\| \Phi_+(t,s) \right\| \le a e^{-\lambda(t-s)}, \quad t \ge s, \tag{2.6}$$

$$\left\| \Phi_-(t,s) \right\| \le a e^{\lambda(t-s)}, \quad t \le s. \tag{2.7}$$

We present corresponding constructions, developed in [28] for the proof of this theorem, which will be used below in our paper.

Let

$$\Phi(t) = \big(w_1(t), \dots, w_n(t)\big) \tag{2.8}$$

be a fundamental matrix of system (2.2), where $w_i(t)$ $(i = 1, \dots, n)$ are linearly independent solutions of system (2.2), $M_+ = \operatorname{span}(w_1, \dots, w_k(t))$, $M_- = \operatorname{span}(w_{k+1}, \dots, w_n(t))$. Setting $v_1(t) = w_1(t)$, $u_1(t) = \frac{w_1(t)}{\|w_1(t)\|}$, we define, for $m = 2, 3, \dots, k$, the vectors

$$v_m = w_m - \sum_{i=1}^{m-1} (w_m, u_i) u_i, \qquad u_m = \frac{v_m}{\|v_m\|}. \tag{2.9}$$

For $m = k+1$, we set $v_{k+1}(t) = w_{k+1}(t)$, $u_{k+1}(t) = \frac{w_{k+1}(t)}{\|w_{k+1}(t)\|}$, and for $m = k+2, \dots, n$, we define corresponding vectors according to scheme (2.9). The matrix

$$U(t) = \big(u_1(t), \dots, u_n(t)\big) \tag{2.10}$$

is bounded with its inverse matrix $U^{-1}(t)$ and $\frac{dU}{dt}$. The vectors $u_j(t)$ are pairwise orthogonal and $\|u_j(t)\| = 1$, $j = 1, \dots, n$. Let us set

$$U(t) = \Phi(t) S(t). \tag{2.11}$$

It is clear from the construction of the matrix $U(t)$ that $S(t)$ is a block diagonal

$$S(t) = \operatorname{diag}\big(S_+(t), S_-(t)\big), \qquad \dim S_+ = k, \qquad \dim S_- = n-k. \tag{2.12}$$

Setting in (2.2) $w = U(t)\eta$, we get

$$\frac{d\eta}{dt} = Q(t)\eta, \tag{2.13}$$

where $Q(t) = U^{-1}(PU - \frac{du}{dt}] = -S^{-1}\frac{dS}{dt}$. It follows from (2.12) and (2.13) that $Q(t) = \operatorname{diag}(Q_+(t), Q_-(t))$, where $Q_+(t) = S_+\frac{dS_+}{dt}$, $Q_-(t) = S_-\frac{dS_-}{dt}$. Thus system (2.13) has the form

$$\frac{d\xi}{dt} = Q_+(t)\xi, \qquad \frac{d\zeta}{dt} = Q_-(t)\zeta, \tag{2.14}$$

where $\eta = \operatorname{col}(\xi, \zeta)$.

Define the Cauchy matrices $\Phi_+(t,s)$ and $\Phi_-(t,s)$ such that $\Phi_+(t,t) = E_k$, $\Phi_-(t,t) = -E_{n-k}$, where E_j is a unit $(j \times j)$-matrix.

Let us prove the following assertion about the representation of bounded solutions to system (2.1).

Theorem 2.2 *Let all elements of $P(t)$ and $g(t)$ in system* (2.1) *be continuous and bounded for $t \in (-\infty, +\infty)$, and let system* (2.2) *be hyperbolic. Then system* (2.1) *has a unique bounded solution and this solution can be represented in the form*

$$x(t) = U(t)z(t), \quad z(t) = \int_{-\infty}^{+\infty} G(t,s)h(s)\,ds, \tag{2.15}$$

where

$$G(t,s) = \begin{cases} \operatorname{diag}\{\Phi_+(t,s), 0_{n-k}\}, & t > s, \\ \operatorname{diag}\{0_{k,} \Phi_-(t,s)\}, & t < s, \end{cases} \tag{2.16}$$

$$G(s+0,s) - G(s-0,s) = E_n, \tag{2.17}$$

$$h(t) = U^{-1}(t)g(t) = \{h_+(t), h_-(t)\}.$$

Proof Let us substitute

$$y(t) = U(t)z(t) \tag{2.18}$$

into system (2.1), then we get the system

$$\frac{dz}{dt} = Q(t)z + h(t), \tag{2.19}$$

for which the homogeneous system is of the form (2.13), (2.14).

Consider the matrix (2.16). It follows from the properties of the matrices $\Phi_+(t)$, $\Phi_-(t)$ that equality (2.17) is fulfilled. It follows from hyperbolicity of system (2.14) that

$$\begin{aligned} \|\Phi_+(t,s)\| &\le ae^{-\lambda(t-s)}, \quad t > s, \\ \|\Phi_-(t,s)\| &\le ae^{\lambda(t-s)}, \quad t < s. \end{aligned} \tag{2.20}$$

It follows from (2.16) and (2.20) that the integral in (2.15) converges for bounded functions $h(t)$ every t. Computing the derivative of Green's matrix, we get

$$\frac{dG(t,s)}{dt} = Q(t)G(t,s). \tag{2.21}$$

Let us verify now that formula (2.15) defines the solution of equation (2.19). Representing $z(t)$ in the form

$$z(t) = \int_{-\infty}^{t} G(t,s)h(s)\,ds + \int_{t}^{+\infty} G(t,s)h(s)\,ds,$$

differentiating it and taking into account (2.21) and (2.17), we get

$$\begin{aligned} \frac{dz}{dt} &= \int_{-\infty}^{t} Q(t)G(t,s)h(s)\,ds + \int_{t}^{+\infty} Q(t)G(t,s)h(s)\,ds \\ &\quad + \big[G(s+0,s) - G(s-0,s)\big]h(t) = Q(t)z(t) + h(t). \end{aligned}$$

The obtained solution is unique. If we assume the existence of two bounded solutions z_1 and z_2, then $z_1 - z_2$ is a bounded on the axis solution of (2.2). From hyperbolicity, it follows that it is a zero solution. □

Corollary 2.1 *If homogeneous system* (2.2) *is hyperbolic and* $k = n$ ($k = 0$), *then nonhomogeneous system* (2.1) *has a unique bounded for* $t \in (-\infty, +\infty)$ *solution, and this solution can be represented in the following form*:

$$x(t) = U(t)z(t), \quad z(t) = \int_{-\infty}^{t} G(t,s)h(s)\,ds \left(z(t) = -\int_{t}^{+\infty} G(t,s)h(s)\,ds \right), \tag{2.22}$$

where $G(t,s) = \Phi(t)\Phi^{-1}(s)$, $\Phi(t)$ *is a fundamental matrix of system* (2.2).

Remark 2.1 If the matrix $P(t)$ in (2.1) is a constant one, analogous results are obtained in [29, 30]. The existence of a unique bounded solution under the assumption of the exponential dichotomy on $(-\infty, +\infty)$ for system (2.2) with bounded variable coefficients is known (see, [27], p.69, Proposition 2). Similar topics were also studied in [31].

3 Results: about reduction of integro-differential equations to systems of ordinary differential equations

3.1 Reduction to the system of first-order ordinary differential equations

Consider the system

$$x'(t) = X\left(t, x, \int_{-\infty}^{t} K(t,s)g\bigl(s, x(s)\bigr)\,ds\right), \tag{3.1}$$

where the kernel $K(t,s)$ is of the form

$$K(t,s) = \sum_{i,j=1}^{\infty} C_j \Phi_j(t) R_j(s) = \sum_{i,j=1}^{\infty} C_j K_j(t,s). \tag{3.2}$$

Series in (3.2) can be, for example, corresponding orthogonal expansions, series of exponents. One of the interesting cases is a finite sum in (3.2). We assume that all the matrices $\Phi_j(t)$ are differentiable and invertible. We can write

$$K_j(t,s) = \Phi_j(t)\Phi_j^{-1}(s)K_j(s,s). \tag{3.3}$$

Define the so-called multiplicative derivative [32]

$$P_j(t) = \frac{d\Phi_j(t)}{dt}\Phi_j^{-1}(t), \tag{3.4}$$

the matrix

$$G_j(t,s) = \Phi_j(t)\Phi_j^{-1}(s), \tag{3.5}$$

is the Cauchy matrix of the system

$$w_j'(t) = P_j(t)w(t). \tag{3.6}$$

Let us set

$$z_j(t,t_0) = \int_{-\infty}^{t} G_j(t,s)K_j(s,s)g\bigl(s,x(s)\bigr)\,ds = \int_{-\infty}^{t} G_j(t,s)h_j\bigl(s,x(s)\bigr)\,ds, \tag{3.7}$$

where $h_j(s,x(s)) = K_j(s,s)g(s,x(s))$.

If the matrix $G_j(t,s)$, defined by (3.5), satisfies the inequality

$$\bigl|G_j(t,s)\bigr| \le ae^{-\lambda(t-s)}, \quad t \ge s$$

(this is the analog of (2.6)) for $k = n$, then $z_j(t,t_0)$ in formula (3.7), according to Corollary 2.1, can be considered as a solution of the one-point problem

$$\begin{aligned} &z_j'(t) = P_j(t)z(t) + h_j\bigl(t,x(t)\bigr), \\ &z_j(t_0) = z_j^0, \end{aligned} \tag{3.8}$$

where

$$z_j^0 = \int_{-\infty}^{t_0} G_j(t_0,s)h_j\bigl(s,x(s)\bigr)\,ds,$$

if we consider $x(t)$ as a known function bounded on $(-\infty,t_0]$. Adding to equation (3.8) the so-called initial function (continuous and bounded on $(-\infty,t_0]$)

$$x(t) = \varphi(t), \tag{3.9}$$

we can consider representation (3.7) as a substitution, which leads us to the one-point problem

$$z_j'(t) = P_j(t)z(t) + h_j\bigl(t,x(t)\bigr), \quad z_j(t_0) = z_j^0, \tag{3.10}$$

where z_j^0 was defined above.

We have proven the following assertion.

Theorem 3.1 *Let*

(a) *matrices* $\Phi_j(t)$ *in the kernels* (3.2) *be continuously differentiable and invertible for* $t \in (-\infty,+\infty)$, $j = 1,2,\dots,$

(b) *systems* (3.6) *be of dimension* n_j, *where*

$$P_j(t) = \frac{d\Phi_j(t)}{dt}\Phi_j^{-1}(t),$$

be hyperbolic for every j in the sense of Definition 2.1 (*for* $k = n$).

Then the bounded solution $x(t) \in R^n$ *of system* (3.1) *with the kernel of the form* (3.2) *and the initial function* (3.9) *and the first component* $x(t) \in R^n$ *of the solution to the countable* *system*

$$\begin{aligned} &x'(t) = X(t, x, z_1, z_2, \ldots), \\ &z_j'(t) = P_j(t)z(t) + h_j(t, x), \quad t \in [t_0, +\infty), \\ &z_j(t_0) = z_j^0, \qquad x(t_0) = \varphi(t_0), \end{aligned} \tag{3.11}$$

where $x \in R^n$, $z_j \in R^n$, $h_j(t, x) = K_j(t, t)g(t, x)$,

$$z_j^0 = \int_{-\infty}^{t_0} G_j(t_0, s)h_j\bigl(s, x(s)\bigr)\, ds, \quad j = 1, 2, 3, \ldots$$

coincide.

Remark 3.1 If (3.2) is a finite sum, then system (3.11) is finite dimensional.

Remark 3.2 The system of the form

$$x'(t) = X\left(t, x, \int_0^{\infty} K(\xi)g\bigl(\xi, x(t-\xi)\bigr)\, d\xi\right) \tag{3.12}$$

can be found in various applications. It can be reduced by the change of variable $t - \xi = s$ to system (3.1) with the kernel $K(t, s) = K(t - s)$.

Remark 3.3 System (3.11) can be used for studying qualitative properties and for an approximate solution of system (3.1) of integro-differential system (3.1). An important basis is the theory of countable systems [33–37]; see also the papers [24, 38–41].

Remark 3.4 Analogous result could be obtained for the system

$$x'(t) = X\left(t, x, \int_t^{+\infty} K(t, s)g\bigl(s, x(s)\bigr)\, ds\right),$$

in Section 5.

3.2 Reduction to the system of ordinary differential equations of high orders

Consider the nonhomogeneous linear equation of nth order

$$L[y] \equiv y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = f(t) \tag{3.13}$$

and the corresponding homogeneous equation

$$L[z] \equiv z^{(n)} + p_1(t)z^{(n-1)} + \cdots + p_n(t)z = 0, \tag{3.14}$$

where all coefficients p_j ($j = 1, \ldots, n$) and f are essentially bounded on $(-\infty, +\infty)$.

Let

$$z_1(t), \ldots, z_n(t) \tag{3.15}$$

be a fundamental system of solutions of equation (3.14). Using (3.15), we can construct the solution $z = \psi(t, t_0)$ such that

$$z(t_0) = \psi(t_0, t_0) = 0, \qquad \ldots, \qquad z^{(n-2)}(t_0) = 0, \qquad z^{(n-1)}(t_0) = 1. \tag{3.16}$$

The function $\psi(t, s)$ is called the Cauchy function of equation (3.13) [42, 43]. Consider the function

$$y(t) = \int_{-\infty}^{t} \psi(t, s) f(s)\, ds, \tag{3.17}$$

assuming that the integral converges. Let us verify that (3.17) is a solution of (3.13). Actually,

$$y^{(k)}(t) = \int_{-\infty}^{t} \psi^{(k)}(t, s) f(s)\, ds + \psi^{(k-1)}(t, t) f(t) = \int_{-\infty}^{t} \psi^{(k)}(t, s) f(s)\, ds \tag{3.18}$$

for $k = 1, \ldots, n-1$, and

$$y^{(n)}(t) = \int_{-\infty}^{t} \psi^{(k)}(t, s) f(s)\, ds + f(t). \tag{3.19}$$

It follows from (3.18), (3.19) and the equality $L[\psi(t, t_0)] = 0$ that

$$L[y] = f(t). \tag{3.20}$$

The obtained particular solution satisfies the initial conditions

$$\begin{aligned} y(t_0) &= \int_{-\infty}^{t_0} \psi(t_0, s) f(s)\, ds, \\ y^{(k)}(t_0) &= \int_{-\infty}^{t_0} \psi^{(k)}(t_0, s) f(s)\, ds, \quad k = 1, \ldots, n-1. \end{aligned} \tag{3.21}$$

Example 3.1 For the equation

$$y^{(n)}(t) = f(t), \tag{3.22}$$

we get

$$\begin{aligned} \psi(t, s) &= \frac{1}{(n-1)!}(t-s)^{n-1}, \\ y(t) &= \frac{1}{(n-1)!} \int_{-\infty}^{t} (t-s)^{n-1} f(s)\, ds. \end{aligned} \tag{3.23}$$

Example 3.2 For the equation

$$\sum_{j=0}^{n} c_n^j \lambda^j y^{(n-j)}(t) = f(t), \tag{3.24}$$

we get

$$\psi(t,s) = \frac{1}{(n-1)!}(t-s)^{n-1}e^{-\lambda(t-s)},$$
$$y(t) = \frac{1}{(n-1)!}\int_{-\infty}^{t}(t-s)^{n-1}e^{-\lambda(t-s)}f(s)\,ds. \tag{3.25}$$

Consider the system

$$x'(t) = X\left(t, x(t), \int_{-\infty}^{t} F\bigl(t,s,y(s)\bigr)\,ds\right), \quad t \in [t_0,+\infty), \tag{3.26}$$

where $x = \operatorname{col}(u,y)$, $u \in R^{n-1}$, $y \in R^1$, and assume that

$$F\bigl(t,s,y(s)\bigr) = \sum_{j=1}^{\infty}\psi_j(t,s)g_j\bigl(s,y(s)\bigr), \tag{3.27}$$

where $\psi_j(t,s)$ $(j = 1,2,\ldots)$ are the Cauchy functions of corresponding linear equations

$$L_j[z_j] \equiv z_j^{(n)} + p_{1j}(t)z_j^{(n-1)} + \cdots + p_{nj}(t)z_j = 0, \quad j = 1,2,\ldots. \tag{3.28}$$

Define

$$v_j(t) = \int_{-\infty}^{t} \psi_j(t,s)g_j\bigl(s,y(s)\bigr)\,ds, \tag{3.29}$$

and set

$$y(t) = \varphi(t), \quad t \in (-\infty, t_0], \tag{3.30}$$

where φ is considered as a known function. Assuming that the Cauchy functions imply convergence of integrals (3.29) for all j and that the function φ is bounded, we obtained that system (3.26) is reduced to the countable system

$$\begin{aligned}
&x'(t) = X(t,x,v_1,v_2,\ldots),\\
&L_j[v_j] = g_j(t,y),\\
&x(t_0) = x_0,\\
&v_j^{(k)}(t_0) = \int_{-\infty}^{t}\psi_j^{(k)}(t,s)g_j\bigl(s,\varphi(s)\bigr)\,ds,\\
&\quad t \in [t_0,+\infty), k = 0,\ldots,n_j, j = 1,2,\ldots
\end{aligned} \tag{3.31}$$

in the sense that the solution of (3.26) coincides with the component x of the solution vector of (3.31).

4 Results: examples of reduction of integro-differential equations to systems of ordinary differential equations

Example 4.1 Model of tumor-immune system [9]

$$\begin{aligned} x'(t) &= x\bigl(f(x) - \Phi(x,y)\bigr), \\ y' &= \beta(z)y - \mu(x)y + \sigma q(x) + \theta(t), \\ z &= \int_{-\infty}^{t} K(t-s)x(s)\,ds. \end{aligned} \tag{4.1}$$

This system is an example of two-dimensional system (3.26) with distributed delay of x. In [9] the following kernel

$$K(t) = Erl_{\lambda,n}t = \frac{\lambda^n}{(n-1)!}t^{n-1}e^{-\lambda t}, \tag{4.2}$$

is used and the case $n = 1$ is studied in detail. It is clear from Example 3.1 (see (3.24) and (3.25)) that the substitution

$$z(t) = \frac{1}{(n-1)!}\int_{-\infty}^{t} (t-s)^{n-1}e^{-\lambda(t-s)}x(s)\,ds \tag{4.3}$$

reduces system (4.1) with the kernel (4.2) to the system of ordinary differential equations

$$\begin{aligned} x'(t) &= x\bigl(f(x) - \Phi(x,y)\bigr), \\ y' &= \beta(z)y - \mu(x)y + \sigma q(x) + \theta(t), \\ \sum_{j=0}^{n} c_n^j \lambda^j z^{(n-j)}(t) &= \lambda^n x(t), \end{aligned} \tag{4.4}$$

which can be written as a system of the order $n + 2$. Note that for $n = 1$, the last equation in system (4.4) is of the form $z' + z = \lambda x$.

Example 4.2 Model of hematopoiesis [10]. This model can be written in the form

$$P'(t) = -\delta(t)P(t) - \frac{\beta(t)P(t)}{1 + P^n(t)} + \alpha(t)\int_0^{\infty} K(\tau)\frac{P(t-\tau)}{1 + P^n(t-\tau)}\,d\tau. \tag{4.5}$$

The coefficients α, β and δ in (4.5) are positive ω-periodic functions, the kernel K satisfies the condition $\int_0^{\infty} K(\tau)\,d\tau = 1$. The change of variable $t - \tau = s$ and then the substitution of the type (4.3) in the case of the kernel (4.2) reduces integro-differential equation (4.5) to the system of ordinary differential equations

$$\begin{aligned} P'(t) &= -\delta(t)P(t) - \frac{\beta(t)P(t)}{1 + P^n(t)} + \alpha(t)z(t), \\ \sum_{j=0}^{n} c_n^j \lambda^j z^{(n-j)}(t) &= \lambda^n \frac{P(t)}{1 + P^n(t)}. \end{aligned} \tag{4.6}$$

Consider now the case of both distributed and concentrated delays in the system

$$\begin{aligned} y'(t) &= Y\left(t, x(t), \int_{-\infty}^{t} K(t,s) q\big(s, y(s)\big)\, ds\right), \\ z'(t) &= Z\big(t, x(t), z(t-\tau)\big). \end{aligned} \tag{4.7}$$

Let us describe the process of reduction, which is similar to the process described in the Section 3.1. The vector x is of the form $x = \operatorname{col}(y, z)$. Denoting

$$v(t) = \int_{-\infty}^{t} K(t,s) q\big(s, y(s)\big)\, ds, \tag{4.8}$$

we make the substitution (compare with (3.7))

$$v(t) = \int_{-\infty}^{t} G_j(t,s) h_j\big(s, y(s)\big)\, ds. \tag{4.9}$$

Introduce the initial functions

$$y(t) = \varphi(t), \quad t \in (-\infty, t_0], \qquad z(t) = \psi(t), \quad t \in (-\tau, 0]. \tag{4.10}$$

Under the assumption of convergence of the integrals, substitution (4.8) reduces (4.7) to the system

$$\begin{aligned} y'(t) &= Y\big(t, x(t), v_1(t), v_2(t), \ldots\big), \\ z'(t) &= Z\big(t, x(t), z(t-\tau)\big), \\ v_j'(t) &= P_j(t) v_j(t) + h_j\big(t, y(t)\big), \quad j = 1, 2, \ldots \end{aligned} \tag{4.11}$$

with the initial conditions defined by (4.10).

Example 4.3 The model of the plankton-nutrient interaction [25]

$$\begin{aligned} N'(t) &= D\big(N^0 - N(t)\big) - aP(t)U\big(N(t)\big) + \gamma \int_{-\infty}^{t} F(t-s) P(s)\, ds, \\ P'(t) &= P\big\{a_1 U\big(N(t-\tau)\big) - (\gamma + D_1)\big\}. \end{aligned} \tag{4.12}$$

The initial functions

$$\begin{aligned} P(t) &= \psi(t), \quad t \in (-\infty, t_0], \\ N(t) &= \varphi(t), \quad t \in (-\tau, 0]. \end{aligned} \tag{4.13}$$

The description of all parameters can be found in the paper [25]. $U(N)$ is a known function. Concerning the function $U(N)$, it is assumed that

$$U(0) = 0, \quad \frac{dU}{dN} > 0, \qquad \lim_{N \to \infty} U(N) = 1. \tag{4.14}$$

A particular case of $U(N)$ is

$$U(N) = \frac{N}{k+N}, \quad \text{where } k > 0. \tag{4.15}$$

Concerning the kernel, it is assumed that $F(t)$ is a bounded nonnegative function such that $\int_{-\infty}^{+\infty} F(t)\,dt = 1$.

In [25] the properties of system (4.12) are considered in various particular cases of the kernel $F(t)$. The most general of them is the following:

$$F(t) = \alpha e^{-\alpha t}, \quad \text{where } \alpha > 0. \tag{4.16}$$

It is clear from (4.2) for $n = 1$ that

$$F(t) = Erl_{\alpha,1}(t). \tag{4.17}$$

The substitution (4.3) for $n = 1$ is of the form

$$z(t) = \int_{-\infty}^{t} e^{-\alpha(t-s)} P(s)\,ds \tag{4.18}$$

and it reduces system (4.12) to the system

$$\begin{aligned} &N'(t) = D\bigl(N^0 - N(t)\bigr) - aP(t)U\bigl(N(t)\bigr) + \gamma z(t), \\ &P'(t) = P\bigl\{a_1 U\bigl(N(t-\tau)\bigr) - (y + D_1)\bigr\}, \\ &z'(t) + z(t) = \alpha P(t). \end{aligned} \tag{4.19}$$

5 Results: systems with advanced argument

Using results of Section 2 and approach of Section 3 (Section 3.1), we describe reduction of the integro-differential system

$$x'(t) = X\left(t, x, \int_{t}^{+\infty} K(t,s) g\bigl(s, x(s)\bigr)\,ds\right), \tag{5.1}$$

and the kernel

$$K(t,s) = \sum_{i,j=1}^{\infty} C_j \Phi_j(t) R_j(s) = \sum_{i,j=1}^{\infty} C_j K_j(t,s) \tag{5.2}$$

to a system of ordinary differential equations. Introducing the matrix $G_j(t,s)$ and the equation

$$w_j'(t) = P_j(t) w(t),$$

by the formulas (3.3), (3.4) and (3.5), let us require that $G_j(t,s)$ $(j = 1, 2, \ldots)$ satisfy inequalities (2.6) under the assumption that $k = 0$ in the condition of hyperbolicity.

Introduce the substitution

$$z_j(t, t_0) = -\int_{t}^{+\infty} G_j(t,s) h_j\bigl(s, x(s)\bigr)\,ds. \tag{5.3}$$

According to Corollary 2.1, we can consider (5.3) for $k = 0$ as the solution of the one-point problem for system (3.8), supposing $x(t)$ is a known function

$$x(t) = \psi(t), \quad t \in [t_0, +\infty). \tag{5.4}$$

As a result, we obtain an analog of Theorem 3.1 for equation (5.1) with the kernel (5.2).

Theorem 5.1 *Let*

(a) *matrices* $\Phi_j(t)$ *in the kernels* (5.2) *be continuously differentiable and invertible for* $t \in (-\infty, +\infty)$, $j = 1, 2, \dots$,

(b) *systems* (3.6) *be of dimension* n_j, *where*

$$P_j(t) = \frac{d\Phi_j(t)}{dt} \Phi_j^{-1}(t),$$

be hyperbolic for every j *in the sense of Definition* 2.1 (*for* $k = n$).

Then the bounded solution $x(t) \in R^n$ *of system* (5.1) *with the kernel of the form* (5.2) *and the end function* (5.4) *and the first component* $x(t) \in R^n$ *of the solution to the system*

$$\begin{aligned} &x'(t) = X(t, x, z_1, z_2, \dots), \\ &z_j'(t) = P_j(t) z(t) + h_j(t, x), \quad t \in [t_0, +\infty), \\ &z_j(t_0) = z_j^0, \qquad x(t_0) = \psi(t_0), \end{aligned} \tag{5.5}$$

where $x \in R^n$, $z_j \in R^n$, $h_j(t, x) = K_j(t, t) g(t, x)$,

$$z_j^0 = -\int_{-\infty}^{t_0} G_j(t_0, s) h_j\bigl(s, x(s)\bigr)\, ds, \quad j = 1, 2, 3, \dots$$

coincide.

6 Results: about systems with both delayed and advanced argument

Let us consider system (1.2) with distributed delay and advance. Denote $x = \operatorname{col}(y, z)$ in such a form that system (1.2) can be written in the form

$$\begin{aligned} y'(t) &= Y\left(t, x(t), \int_{-\infty}^{t} K_1(t, s) g_1\bigl(s, y(s)\bigr)\, ds\right), \\ z'(t) &= Z\left(t, x(t), \int_{t}^{+\infty} K_2(t, s) g_2\bigl(s, z(s)\bigr)\, ds\right). \end{aligned} \tag{6.1}$$

Using the technique of Sections 3 and 5, we introduce

$$u_j(t) = \int_{-\infty}^{t} G_j^1(t, s) h_j^1\bigl(s, x(s)\bigr)\, ds, \tag{6.2}$$

$$v_j(t) = -\int_{t}^{+\infty} G_j^2(t, s) h_j^2\bigl(s, x(s)\bigr)\, ds, \tag{6.3}$$

where $h_j^i(t, x) = K_i(t, t) g_j(t, x)$, $i = 1, 2$, $j = 1, 2, \dots$ and denote $P_j^1(t) = \frac{d\Phi_j}{dt} \Phi_j^{-1}(t)$, $j = 1, 2, \dots$, $P_r^2(t) = \frac{d\Phi_r}{dt} \Phi_r^{-1}(t)$, $r = 1, 2, \dots$.

Requiring $G_j^i(t,s)$ satisfies inequality (2.6), the first for $k = n$ and the second for $k = 0$, we obtain that solution $x = \operatorname{col}(y,z)$ of system (6.1) satisfies also the following problem:

$$\begin{aligned}
&y'(t) = Y\big(t, x(t), u_1(t), u_2(t), \dots\big),\\
&z'(t) = Z\big(t, x(t), v_1(t), v_2(t), \dots\big),\\
&u_j'(t) = P_j^1(t)u_j(t) + h_j^1\big(t, y(t)\big), \quad j = 1, 2, \dots,\\
&v_r'(t) = P_r^2(t)u_r(t) + h_r^1\big(t, z(t)\big), \quad r = 1, 2, \dots
\end{aligned} \tag{6.4}$$

with conditions

$$\begin{aligned}
&y(t_0) = \varphi(t_0), \qquad u_j(t_0) = \int_{-\infty}^{t_0} G_j^1(t_0,s)h_j^1\big(s, x(s)\big)\,ds, \quad j = 1, 2, \dots,\\
&z'(t) = \psi(t_0), \qquad v_r(t_0) = \int_{t_0}^{+\infty} G_r^2(t_0,s)h_r^2\big(s, x(s)\big)\,ds, \quad r = 1, 2, \dots,\\
&y(t) = \varphi(t), \quad t \in (-\infty, t_0),\\
&z(t) = \psi(t), \quad t \in (t_0 < +\infty).
\end{aligned} \tag{6.5}$$

7 Conclusions

The method described above allows us to reduce systems of integro-differential systems with distributed delay and/or advance to systems of ordinary differential equations. For Volterra systems of the type (1.1), it was a basis for studying stability, bifurcation, Floquet theory, parametric resonance, stabilization and oscillation properties for integro-differential equations with ordinary [13–18, 20] and partial [21, 22] derivatives. We could extend the main results of these works to integro-differential equation (1.2).

Generally speaking, after the reduction, we get infinity dimensional systems of ordinary differential equations. For their analysis, the theory of countable differentiable systems could be used [33–37].

In the study of various biological systems, the linear chain trick method was used (see, for example, [9, 10]). It is clear (see Section 4) that our approach includes the linear trick method. Note also the use of W-transform, which also allows researchers to reduce integro-differential equations to systems of ordinary differential equations [11].

The proposed method allows us also to study generalized and impulsive systems. For example, in the case of discontinuous solutions described by Heaviside functions $H_a(t)$, we can use its connection with δ-function: $H_a(t) = \int_{-\infty}^t \delta_a(\xi)\,d\xi$ and to get to a system of integro-differential equations. Introducing the sequence, for example,

$$Erl_{\lambda,n} t = \frac{\lambda^n}{(n-1)!} t^{n-1} e^{-\lambda t},$$

where $\lambda \to \infty$, we can consider the obtained system of integro-differential equations as an approximation of generalized equations. This allows us in corresponding cases to reduce the study of a generalized and impulsive system to the analysis of the sequence of integro-differential equations, and consequently to the analysis of the corresponding sequence of systems of ordinary differential equations.

Competing interests
There are no competing interests.

Authors' contributions
Results are obtained by both authors as a result of their many years of collaboration. All authors read and approved the final manuscript.

Acknowledgements
Authors thank the reviewers for their reports, which have essentially improved the paper.

Received: 21 January 2013 Accepted: 30 May 2013 Published: 26 June 2013

References

1. Drozdov, AD, Kolmanovskii, VB: Stability in Viscoelasticity. North Holland, Amsterdam (1994)
2. Fabrizio, M, Morro, A: Mathematical Problems in Linear Viscoelasticity. SIAM Stud. Appl. Math. SIAM, Philadelphia (1992)
3. Golden, JM, Graham, GAC: Boundary Value Problems in Linear Viscoelasticity. Springer, Berlin (1988)
4. Gurtin, ME, Pipkin, AC: A general theory of heat conduction with finite wave speeds. Arch. Ration. Mech. Anal. **31**, 113-126 (1968)
5. Novick-Cohen, A: Conserved phase-field equations with memory. In: Damlamian, A, Spruck, J, Visintin, A (eds.) Curvature Flows and Related Topics, pp. 179-197. GAKUTO Internat. Ser. Math. Sci. Appl, vol. 5, Gakkotosho, Tokyo (1995)
6. Corduneanu, C: Integral Equations and Stability Feedback Systems. Academic Press, New York (1973)
7. Corduneanu, C: Integral Equations and Applications. Cambridge University Press, Cambridge (1991)
8. Burton, TA: Volterra Integral and Differential Equations. Academic Press, New York (1983)
9. D'Onofrio, A, Gatti, F, Cerrai, P, Freshi, L: Delay-induced oscillatory dynamics of tumor-immune system interaction. Math. Comput. Model. **52**, 572-591 (2010)
10. Weng, P-X: Global attractive periodic solution in a model of hematopoiesis. Comput. Math. Appl. **44**, 1019-1030 (2002)
11. Ponosov, A, Shindiapin, A, Miguel, J: The W-transform links delay and ordinary differential equations. Funct. Differ. Equ. **9**, 437-470 (2002)
12. Domoshnitsky, A: Exponential stability of convolution integro-differential equations. Funct. Differ. Equ. **5**, 445-455 (1998)
13. Domoshnitsky, A, Gotser, Y: Hopf bifurcation of integro-differential equations. Electron. J. Qual. Theory Differ. Equ. **3**, 1-11 (2000)
14. Domoshnitsky, A, Goltser, Y: Approach to study of bifurcations and stability of integro-differential equations. Math. Comput. Model. **36**, 663-678 (2002)
15. Domoshnitsky, A, Goltser, Y: On stability and boundary value problems for integro-differential equations. Nonlinear Anal., Theory Methods Appl. **63**(5-7), e761-e767 (2005)
16. Goltser, Y, Domoshnitsky, A: Bifurcation and stability of integro-differential equations. Nonlinear Anal., Theory Methods Appl. **47**, 953-967 (2001)
17. Agarwal, RP, Bohner, M, Domoshnitsky, A, Goltser, Ya: Floquet theory and stability of nonlinear integro-differential equations. Acta Math. Hung. **109**(4), 305-330 (2005)
18. Goltser, Y, Agronovich, G: Normal form and parametric resonance in linear almost periodic systems. Funct. Differ. Equ. **9**, 117-134 (2002)
19. Goltser, Y: On bifurcation of steady resonance modes for nonlinear integro-differential equations. Differ. Equ. Dyn. Syst. **2**, 239-251 (2006)
20. Domoshnitsky, A, Maghakyan, A, Puzanov, N: About stabilization by feedback control in integral form. Georgian Math. J. (2012). doi:10.1515/gmj-2012-0033
21. Agarwal, RP, Domoshnitsky, A, Goltser, Y: Stability of partial functional integro-differential equations. J. Dyn. Control Syst. **12**, 1-31 (2006)
22. Domoshnitsky, A, Goltser, Y: Constructive approach to phase transition model. Funct. Differ. Equ. **7**, 269-278 (2000)
23. Goltser, Y, Litsyn, L: Volterra integro-differential systems of ordinary differential equations. Math. Comput. Model. **42**, 221-233 (2005)
24. Goltser, Y, Litsyn, L: Non-linear Volterra IDE, infinite systems and normal form of ODE. Nonlinear Anal., Theory Methods Appl. **68**, 1553-1569 (2008)
25. Shingui, R: The effect of delays on stability and persistence in plankton models. Nonlinear Anal., Theory Methods Appl. **24**, 575-585 (1995)
26. Filatov, AN: Averaging in Differential and Integro-Differential Equations. FAN, Tashkent (1971) (in Russian)
27. Coppel, WA: Dichotomies in Stability Theory. Lecture Notes in Mathematics, vol. 629. Springer, Berlin (1978)
28. Pliss, VA: Integral Sets of Periodic Systems of Differential Equations. Nauka, Moscow (1977)
29. Bibikov, YN: Course of Ordinary Differential Equations. Vysshaya Shkola, Moscow (1991)
30. Demidovich, BP: Lectures on the Mathematical Theory of Stability. Nauka, Moscow (1967)
31. Mitropolsky, YA, Samoilenko, AM, Kulik, VL: Dichotomies and Stability in Nonautonomous Linear Systems. Stability and Control: Theory, Methods and Applications, vol. 14. Taylor & Francis, London (2003)
32. Gantmacher, FR: Theory of Matrices. Chelsea Publishing, New York (1959)
33. Persidskii, KP: Selected Works, vol. 2. Nauka, Alma-Ata (1976)
34. Persidskii, KP: On stability of solutions to a countable systems of differential equations. Izv. AS Kaz. SSR, Math. Mech. **2**, 2-35 (1948)
35. Persidskii, KP: Infinite systems of differential equations. Izv. AS Kaz. SSR, Math. Mech. **4**(8), 3-11 (1956)
36. Persidskii, KP: Countable systems of differential equations and stability of their solutions. Izv. AS Kaz. SSR, Math. Mech. **7**(11), 52-71 (1959)

37. Valeev, KG, Zhautykov, OA: Infinite Systems of Differential Equations. Nauka, Alma-Ata (1974)
38. Domoshnitsky, A, Litsyn, E: Positivity of the Green's matrix of infinite system. Panam. Math. J. **16**(2), 27-40 (2006)
39. Domoshnitsky, A, Goltser, Y: Positivity of solutions to boundary value problems for infinite functional differential systems. Math. Comput. Model. **45**(11-12), 1395-1404 (2007)
40. Litsyn, L: On the formula for general solution of infinite system of functional differential equations. Funct. Differ. Equ. **2**, 111-121 (1994)
41. Litsyn, L: On the general theory of liner functional differential equations. Differ. Uravn. (Minsk) **24**, 977-986 (1988)
42. Goursat, E: Cours d'analyse mathematique. Gauthier-Villars, Paris (1925)
43. Stepanov, VV: Course of Differential Equations. Moscow (1958)

doi:10.1186/1687-1847-2013-187
Cite this article as: Goltser and Domoshnitsky: **About reducing integro-differential equations with infinite limits of integration to systems of ordinary differential equations.** *Advances in Difference Equations* 2013 **2013**:187.