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On a class of q-Bernoulli and q-Euler polynomials

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Abstract
The main purpose of this paper is to introduce and investigate a class of generalized Bernoulli polynomials and Euler polynomials based on the q-integers. The q-analogues of well-known formulas are derived. The q-analogue of the Srivastava-Pintér addition theorem is obtained. We give new identities involving q-Bernstein polynomials.

1 Introduction

Throughout this paper, we always make use of the following notation: $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{N}_0$ denotes the set of nonnegative integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers.

The q-shifted factorial is defined by

$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{j=0}^{n-1}\left(1-q^j a\right), \quad n \in \mathbb{N},$$

$$(a;q)_\infty = \prod_{j=0}^{\infty}\left(1-q^j a\right), \quad |q| < 1, a \in \mathbb{C}.$$

The q-numbers and q-numbers factorial are defined by

$$[a]_q = \frac{1-q^a}{1-q} \quad (q \neq 1); \qquad [0]_q! = 1; \qquad [n]_q! = [1]_q[2]_q \cdots [n]_q, \quad n \in \mathbb{N}, a \in \mathbb{C},$$

respectively. The q-polynomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_{n-k}(q;q)_k}.$$

The q-analogue of the function $(x+y)^n$ is defined by

$$(x+y)_q^n := \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{1}{2}k(k-1)} x^{n-k} y^k, \quad n \in \mathbb{N}_0.$$

The q-binomial formula is known as

$$(1-a)_q^n = (a;q)_n = \prod_{j=0}^{n-1}\left(1-q^j a\right) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{1}{2}k(k-1)}(-1)^k a^k.$$

In the standard approach to the q-calculus, two exponential functions are used

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty} \frac{1}{(1-(1-q)q^k z)}, \quad 0 < |q| < 1, |z| < \frac{1}{|1-q|},$$

$$E_q(z) = \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n-1)} z^n}{[n]_q!} = \prod_{k=0}^{\infty}\left(1+(1-q)q^k z\right), \quad 0 < |q| < 1, z \in \mathbb{C}.$$

From this form, we easily see that $e_q(z)E_q(-z) = 1$. Moreover,

$$D_q e_q(z) = e_q(z), \qquad D_q E_q(z) = E_q(qz),$$

where D_q is defined by

$$D_q f(z) := \frac{f(qz) - f(z)}{qz - z}, \quad 0 < |q| < 1, 0 \neq z \in \mathbb{C}.$$

The above q-standard notation can be found in [1].

Over 70 years ago, Carlitz extended the classical Bernoulli and Euler numbers and polynomials and introduced the q-Bernoulli and the q-Euler numbers and polynomials (see [2, 3] and [4]). There are numerous recent investigations on this subject by, among many other authors, Cenkci *et al.* [5–7], Choi *et al.* [8] and [9], Kim *et al.* [10–13], Ozden and Simsek [14], Ryoo *et al.* [15], Simsek [16, 17] and [18], and Luo and Srivastava [19], Srivastava *et al.* [20], Srivastava [21], Mahmudov [22].

We first give here the definitions of the q-Bernoulli and the q-Euler polynomials of higher order as follows.

Definition 1 Let $q, \alpha \in \mathbb{C}$, $0 < |q| < 1$. The q-Bernoulli numbers $\mathfrak{B}_{n,q}^{(\alpha)}$ and polynomials $\mathfrak{B}_{n,q}^{(\alpha)}(x,y)$ in x, y of order α are defined by means of the generating functions:

$$\left(\frac{t}{e_q(t)-1}\right)^{\alpha} = \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}^{(\alpha)} \frac{t^n}{[n]_q!}, \quad |t| < 2\pi,$$

$$\left(\frac{t}{e_q(t)-1}\right)^{\alpha} e_q(tx)E_q(ty) = \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}^{(\alpha)}(x,y) \frac{t^n}{[n]_q!}, \quad |t| < 2\pi.$$

Definition 2 Let $q, \alpha \in \mathbb{C}$, $0 < |q| < 1$. The q-Euler numbers $\mathfrak{E}_{n,q}^{(\alpha)}$ and polynomials $\mathfrak{E}_{n,q}^{(\alpha)}(x,y)$ in x, y of order α are defined by means of the generating functions:

$$\left(\frac{2}{e_q(t)+1}\right)^{\alpha} = \sum_{n=0}^{\infty} \mathfrak{E}_{n,q}^{(\alpha)} \frac{t^n}{[n]_q!}, \quad |t| < \pi,$$

$$\left(\frac{2}{e_q(t)+1}\right)^{\alpha} e_q(tx)E_q(ty) = \sum_{n=0}^{\infty} \mathfrak{E}_{n,q}^{(\alpha)}(x,y) \frac{t^n}{[n]_q!}, \quad |t| < \pi.$$

It is obvious that

$$\mathfrak{B}_{n,q}^{(\alpha)} = \mathfrak{B}_{n,q}^{(\alpha)}(0,0), \qquad \lim_{q\to 1^-} \mathfrak{B}_{n,q}^{(\alpha)}(x,y) = B_n^{(\alpha)}(x+y), \qquad \lim_{q\to 1^-} \mathfrak{B}_{n,q}^{(\alpha)} = B_n^{(\alpha)},$$
$$\mathfrak{E}_{n,q}^{(\alpha)} = \mathfrak{E}_{n,q}^{(\alpha)}(0,0), \qquad \lim_{q\to 1^-} \mathfrak{E}_{n,q}^{(\alpha)}(x,y) = E_n^{(\alpha)}(x+y), \qquad \lim_{q\to 1^-} \mathfrak{E}_{n,q}^{(\alpha)} = E_n^{(\alpha)},$$
$$\lim_{q\to 1^-} \mathfrak{B}_{n,q}^{(\alpha)}(x,0) = B_n^{(\alpha)}(x), \qquad \lim_{q\to 1^-} \mathfrak{B}_{n,q}^{(\alpha)}(0,y) = B_n^{(\alpha)}(y),$$
$$\lim_{q\to 1^-} \mathfrak{E}_{n,q}^{(\alpha)}(x,0) = E_n^{(\alpha)}(x), \qquad \lim_{q\to 1^-} \mathfrak{E}_{n,q}^{(\alpha)}(0,y) = E_n^{(\alpha)}(y).$$

Here $B_n^{(\alpha)}(x)$ and $E_n^{(\alpha)}(x)$ denote the classical Bernoulli and Euler polynomials of order α which are defined by

$$\left(\frac{t}{e^t-1}\right)^\alpha e^{tx} = \sum_{n=0}^\infty B_n^{(\alpha)}(x)\frac{t^n}{n!} \quad \text{and} \quad \left(\frac{2}{e^t+1}\right)^\alpha e^{tx} = \sum_{n=0}^\infty E_n^{(\alpha)}(x)\frac{t^n}{n!}.$$

In fact, Definitions 1 and 2 define two different types $\mathfrak{B}_{n,q}^{(\alpha)}(x,0)$ and $\mathfrak{B}_{n,q}^{(\alpha)}(0,y)$ of the q-Bernoulli polynomials and two different types $\mathfrak{E}_{n,q}^{(\alpha)}(x,0)$ and $\mathfrak{E}_{n,q}^{(\alpha)}(0,y)$ of the q-Euler polynomials. Both polynomials $\mathfrak{B}_{n,q}^{(\alpha)}(x,0)$ and $\mathfrak{B}_{n,q}^{(\alpha)}(0,y)$ ($\mathfrak{E}_{n,q}^{(\alpha)}(x,0)$ and $\mathfrak{E}_{n,q}^{(\alpha)}(0,y)$) coincide with the classical higher-order Bernoulli polynomials (Euler polynomials) in the limiting case $q\to 1^-$.

For the q-Bernoulli numbers $\mathfrak{B}_{n,q}$ and the q-Euler numbers $\mathfrak{E}_{n,q}$ of order n, we have

$$\mathfrak{B}_{n,q} = \mathfrak{B}_{n,q}(0,0) = \mathfrak{B}_{n,q}^{(1)}(0,0), \qquad \mathfrak{E}_{n,q} = \mathfrak{E}_{n,q}(0,0) = \mathfrak{E}_{n,q}^{(1)}(0,0),$$

respectively. Note that the q-Bernoulli numbers $\mathfrak{B}_{n,q}$ are defined and studied in [23].

The aim of the present paper is to obtain some results for the above newly defined q-Bernoulli and q-Euler polynomials. It should be mentioned that q-Bernoulli and q-Euler polynomials in our definitions are polynomials of x and y, and when $y=0$, they are polynomials of x, but in other definitions they are functions of q^x. First advantage of this approach is that for $q\to 1^-$, $\mathfrak{B}_{n,q}^{(\alpha)}(x,y)$ ($\mathfrak{E}_{n,q}^{(\alpha)}(x,y)$) becomes the classical Bernoulli $\mathfrak{B}_n^{(\alpha)}(x+y)$ (Euler $\mathfrak{E}_n^{(\alpha)}(x+y)$) polynomial, and we may obtain the q-analogues of well-known results, for example, those of Srivastava and Pintér [24], Cheon [25], *etc.* The second advantage is that we find the relation between q-Bernstein polynomials and Phillips q-Bernoulli polynomials and derive the formulas involving the q-Stirling numbers of the second kind, q-Bernoulli polynomials and Phillips q-Bernstein polynomials.

2 Preliminaries and lemmas

In this section we provide some basic formulas for the q-Bernoulli and q-Euler polynomials in order to obtain the main results of this paper in the next section. The following result is a q-analogue of the addition theorem for the classical Bernoulli and Euler polynomials.

Lemma 3 (Addition theorems) *For all $x,y\in\mathbb{C}$, we have*

$$\mathfrak{B}_{n,q}^{(\alpha)}(x,y) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{B}_{k,q}^{(\alpha)}(x+y)_q^{n-k}, \qquad \mathfrak{E}_{n,q}^{(\alpha)}(x,y) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{E}_{k,q}^{(\alpha)}(x+y)_q^{n-k},$$

$$\mathfrak{B}_{n,q}^{(\alpha)}(x,y)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q q^{(n-k)(n-k-1)/2}\mathfrak{B}_{k,q}^{(\alpha)}(x,0)y^{n-k}=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q \mathfrak{B}_{k,q}^{(\alpha)}(0,y)x^{n-k}, \tag{1}$$

$$\mathfrak{E}_{n,q}^{(\alpha)}(x,y)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q q^{(n-k)(n-k-1)/2}\mathfrak{E}_{k,q}^{(\alpha)}(x,0)y^{n-k}=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q \mathfrak{E}_{k,q}^{(\alpha)}(0,y)x^{n-k}. \tag{2}$$

In particular, setting $x=0$ and $y=0$ in (1) and (2), we get the following formulas for q-Bernoulli and q-Euler polynomials, respectively

$$\mathfrak{B}_{n,q}^{(\alpha)}(x,0)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q \mathfrak{B}_{k,q}^{(\alpha)}x^{n-k},\qquad \mathfrak{B}_{n,q}^{(\alpha)}(0,y)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q q^{(n-k)(n-k-1)/2}\mathfrak{B}_{k,q}^{(\alpha)}y^{n-k}, \tag{3}$$

$$\mathfrak{E}_{n,q}^{(\alpha)}(x,0)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q \mathfrak{E}_{k,q}^{(\alpha)}x^{n-k},\qquad \mathfrak{E}_{n,q}^{(\alpha)}(0,y)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q q^{(n-k)(n-k-1)/2}\mathfrak{E}_{k,q}^{(\alpha)}y^{n-k}. \tag{4}$$

Setting $y=1$ and $x=1$ in (1) and (2), we get

$$\mathfrak{B}_{n,q}^{(\alpha)}(x,1)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q q^{(n-k)(n-k-1)/2}\mathfrak{B}_{k,q}^{(\alpha)}(x,0),\qquad \mathfrak{B}_{n,q}^{(\alpha)}(1,y)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q \mathfrak{B}_{k,q}^{(\alpha)}(0,y), \tag{5}$$

$$\mathfrak{E}_{n,q}^{(\alpha)}(x,1)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q q^{(n-k)(n-k-1)/2}\mathfrak{E}_{k,q}^{(\alpha)}(x,0),\qquad \mathfrak{E}_{n,q}^{(\alpha)}(1,y)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q \mathfrak{E}_{k,q}^{(\alpha)}(0,y). \tag{6}$$

Clearly, (5) and (6) are q-analogues of

$$B_n^{(\alpha)}(x+1)=\sum_{k=0}^{n}\binom{n}{k}B_k^{(\alpha)}(x),\qquad E_n^{(\alpha)}(x+1)=\sum_{k=0}^{n}\binom{n}{k}E_k^{(\alpha)}(x),$$

respectively.

Lemma 4 *We have*

$$D_{q,x}\mathfrak{B}_{n,q}^{(\alpha)}(x,y)=[n]_q\mathfrak{B}_{n-1,q}^{(\alpha)}(x,y),\qquad D_{q,y}\mathfrak{B}_{n,q}^{(\alpha)}(x,y)=[n]_q\mathfrak{B}_{n-1,q}^{(\alpha)}(x,qy),$$
$$D_{q,x}\mathfrak{E}_{n,q}^{(\alpha)}(x,y)=[n]_q\mathfrak{E}_{n-1,q}^{(\alpha)}(x,y),\qquad D_{q,y}\mathfrak{E}_{n,q}^{(\alpha)}(x,y)=[n]_q\mathfrak{E}_{n-1,q}^{(\alpha)}(x,qy).$$

Lemma 5 (Difference equations) *We have*

$$\mathfrak{B}_{n,q}^{(\alpha)}(1,y)-\mathfrak{B}_{n,q}^{(\alpha)}(0,y)=[n]_q\mathfrak{B}_{n-1,q}^{(\alpha-1)}(0,y), \tag{7}$$

$$\mathfrak{E}_{n,q}^{(\alpha)}(1,y)+\mathfrak{E}_{n,q}^{(\alpha)}(0,y)=2\mathfrak{E}_{n,q}^{(\alpha-1)}(0,y), \tag{8}$$

$$\mathfrak{B}_{n,q}^{(\alpha)}(x,0)-\mathfrak{B}_{n,q}^{(\alpha)}(x,-1)=[n]_q\mathfrak{B}_{n-1,q}^{(\alpha-1)}(x,-1),$$
$$\mathfrak{E}_{n,q}^{(\alpha)}(x,0)+\mathfrak{E}_{n,q}^{(\alpha)}(x,-1)=2\mathfrak{E}_{n,q}^{(\alpha-1)}(x,-1).$$

From (7) and (5), (8) and (6), we obtain the following formulas.

Lemma 6 *We have*

$$\mathfrak{B}_{n,q}^{(\alpha-1)}(0,y)=\frac{1}{[n+1]_q}\sum_{k=0}^{n}\begin{bmatrix}n+1\\k\end{bmatrix}_q \mathfrak{B}_{k,q}^{(\alpha)}(0,y), \tag{9}$$

$$\mathfrak{E}_{n,q}^{(\alpha-1)}(0,y)=\frac{1}{2}\left[\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q \mathfrak{E}_{k,q}^{(\alpha)}(0,y)+\mathfrak{E}_{n,q}^{(\alpha)}(0,y)\right]. \tag{10}$$

Putting $\alpha=1$ in (9) and (10) and noting that

$$\mathfrak{B}_{n,q}^{(0)}(0,y)=\mathfrak{E}_{n,q}^{(0)}(0,y)=q^{n(n-1)/2}y^n,$$

we arrive at the following expansions:

$$\begin{aligned} y^n &= \frac{1}{q^{n(n-1)/2}[n+1]_q}\sum_{k=0}^{n}\begin{bmatrix} n+1\\ k\end{bmatrix}_q \mathfrak{B}_{k,q}(0,y),\\ y^n &= \frac{1}{2q^{n(n-1)/2}}\left[\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q \mathfrak{E}_{k,q}(0,y)+\mathfrak{E}_{n,q}(0,y)\right], \end{aligned}$$

which are q-analogues of the following familiar expansions:

$$y^n=\frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}B_k(y),\qquad y^n=\frac{1}{2}\left[\sum_{k=0}^{n}\binom{n}{k}E_k(y)+E_n(y)\right], \tag{11}$$

respectively.

Lemma 7 (Recurrence relationships) *The polynomials $\mathfrak{B}_{n,q}^{(\alpha)}(x,0)$ and $\mathfrak{E}_{n,q}^{(\alpha)}(x,0)$ satisfy the following difference relationships*:

$$\begin{aligned} &\sum_{j=0}^{k}\begin{bmatrix} k\\ j\end{bmatrix}_q m^j\mathfrak{B}_{j,q}^{(\alpha)}(x,0)-\sum_{j=0}^{k}\begin{bmatrix} k\\ j\end{bmatrix}_q m^j\mathfrak{B}_{j,q}^{(\alpha)}(x,-1)\\ &\quad=[k]_q\sum_{j=0}^{k-1}\begin{bmatrix} k-1\\ j\end{bmatrix}_q m^{j+1}\mathfrak{B}_{j,q}^{(\alpha-1)}(x,-1), \end{aligned} \tag{12}$$

$$\begin{aligned} &\mathfrak{B}_{k,q}^{(\alpha)}\left(\frac{1}{m},y\right)-\sum_{j=0}^{k}\begin{bmatrix} k\\ j\end{bmatrix}_q \left(\frac{1}{m}-1\right)_q^{k-j}\mathfrak{B}_{j,q}^{(\alpha)}(0,y)\\ &\quad=[k]_q\sum_{j=0}^{k-1}\begin{bmatrix} k-1\\ j\end{bmatrix}_q \left(\frac{1}{m}-1\right)_q^{k-j-1}\mathfrak{B}_{j,q}^{(\alpha-1)}(0,y), \end{aligned} \tag{13}$$

$$\begin{aligned} &\sum_{j=0}^{k}\begin{bmatrix} k\\ j\end{bmatrix}_q m^j\mathfrak{E}_{j,q}^{(\alpha)}(x,0)+\sum_{j=0}^{k}\begin{bmatrix} k\\ j\end{bmatrix}_q m^j\mathfrak{E}_{j,q}^{(\alpha)}(x,-1)\\ &\quad=2\sum_{j=0}^{k}\begin{bmatrix} k\\ j\end{bmatrix}_q m^j\mathfrak{E}_{j,q}^{(\alpha-1)}(x,-1), \end{aligned} \tag{14}$$

$$\begin{aligned} &\mathfrak{E}_{k,q}^{(\alpha)}\left(\frac{1}{m},y\right)+\sum_{j=0}^{k}\begin{bmatrix} k\\ j\end{bmatrix}_q \left(\frac{1}{m}-1\right)_q^{k-j}\mathfrak{E}_{j,q}^{(\alpha)}(0,y)\\ &\quad=2\sum_{j=0}^{k}\begin{bmatrix} k\\ j\end{bmatrix}_q \left(\frac{1}{m}-1\right)_q^{k-j}\mathfrak{E}_{j,q}^{(\alpha-1)}(0,y). \end{aligned} \tag{15}$$

3 Explicit relationship between the q-Bernoulli and q-Euler polynomials

In this section we investigate some explicit relationships between the q-Bernoulli and q-Euler polynomials. Here some q-analogues of known results are given. We also obtain new formulas and some of their special cases below. These formulas are some extensions of the formulas of Srivastava and Pintér, Cheon and others.

We present natural q-extensions of the main results in the papers [24] and [26], see Theorems 8 and 13.

Theorem 8 *For $n \in \mathbb{N}_0$, the following relationships hold true*:

$$\begin{aligned}
\mathfrak{B}_{n,q}^{(\alpha)}(x,y) &= \frac{1}{2m^n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \Bigg[m^k \mathfrak{B}_{k,q}^{(\alpha)}(x,0) + \sum_{j=0}^{k}\begin{bmatrix} k \\ j \end{bmatrix}_q m^j \mathfrak{B}_{j,q}^{(\alpha)}(x,-1) \\
&\quad + [k]_q \sum_{j=0}^{k-1}\begin{bmatrix} k-1 \\ j \end{bmatrix}_q m^{j+1}\mathfrak{B}_{j,q}^{(\alpha-1)}(x,-1)\Bigg]\mathfrak{E}_{n-k,q}(0,my), \\
\mathfrak{B}_{n,q}^{(\alpha)}(x,y) &= \frac{1}{2m^n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q m^k \Bigg[\mathfrak{B}_{k,q}^{(\alpha)}(0,y) + \sum_{j=0}^{k}\begin{bmatrix} k \\ j \end{bmatrix}_q \left(\frac{1}{m}-1\right)_q^{k-j} \mathfrak{B}_{j,q}^{(\alpha)}(0,y) \\
&\quad + [k]_q \sum_{j=0}^{k-1}\begin{bmatrix} k-1 \\ j \end{bmatrix}_q \left(\frac{1}{m}-1\right)_q^{k-1-j}\mathfrak{B}_{j,q}^{(\alpha-1)}(0,y)\Bigg]\mathfrak{E}_{n-k,q}(mx,0).
\end{aligned}$$

Proof Using the following identity:

$$\begin{aligned}
&\left(\frac{t}{e_q(t)-1}\right)^{\alpha} e_q(tx)E_q(ty) \\
&\quad = \frac{2}{e_q(\frac{t}{m})+1}\cdot E_q\left(\frac{t}{m}my\right)\cdot \frac{e_q(\frac{t}{m})+1}{2}\cdot\left(\frac{t}{e_q(t)-1}\right)^{\alpha} e_q(tx),
\end{aligned}$$

we have

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!} &= \frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{E}_{n,q}(0,my)\frac{t^n}{m^n[n]_q!}\sum_{n=0}^{\infty}\frac{t^n}{m^n[n]_q!}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(\alpha)}(x,0)\frac{t^n}{[n]_q!} \\
&\quad + \frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{E}_{n,q}(0,my)\frac{t^n}{m^n[n]_q!}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(\alpha)}(x,0)\frac{t^n}{[n]_q!} \\
&=: I_1 + I_2.
\end{aligned}$$

It is clear that

$$\begin{aligned}
I_2 &= \frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{E}_{n,q}(0,my)\frac{t^n}{m^n[n]_q!}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(\alpha)}(x,0)\frac{t^n}{[n]_q!} \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q m^{k-n}\mathfrak{B}_{k,q}^{(\alpha)}(x,0)\mathfrak{E}_{n-k,q}(0,my)\frac{t^n}{[n]_q!}.
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
I_1 &= \frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(\alpha)}(x,0)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_q m^{-n}\mathfrak{E}_{j,q}(0,my)\frac{t^n}{[n]_q!} \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{B}_{k,q}^{(\alpha)}(x,0)\sum_{j=0}^{n-k}\begin{bmatrix} n-k \\ j \end{bmatrix}_q m^{k-n}\mathfrak{E}_{j,q}(0,my)\frac{t^n}{[n]_q!} \\
&= \frac{1}{2}\sum_{n=0}^{\infty}m^{-n}\sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_q \mathfrak{E}_{j,q}(0,my)\sum_{k=0}^{n-j}\begin{bmatrix} n-j \\ k \end{bmatrix}_q m^{k}\mathfrak{B}_{k,q}^{(\alpha)}(x,0)\frac{t^n}{[n]_q!}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!} \\
&\quad = \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q m^{k-n}\left[\mathfrak{B}_{k,q}^{(\alpha)}(x,0) + m^{-k}\sum_{j=0}^{k}\begin{bmatrix} k \\ j \end{bmatrix}_q m^{j}\mathfrak{B}_{j,q}^{(\alpha)}(x,0)\right]\mathfrak{E}_{n-k,q}(0,my)\frac{t^n}{[n]_q!}.
\end{aligned}
$$

It remains to use the formula (12). □

Next we discuss some special cases of Theorem 8.

Corollary 9 *For $n \in \mathbb{N}_0$, $m \in \mathbb{N}$, the relationship*

$$
\begin{aligned}
\mathfrak{B}_{n,q}(x,y) &= \frac{1}{2m^n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \left[m^k\mathfrak{B}_{k,q}(x,0) + \sum_{j=0}^{k}\begin{bmatrix} k \\ j \end{bmatrix}_q m^j\mathfrak{B}_{j,q}(x,-1)\right. \\
&\qquad \left. + [k]_q\sum_{j=0}^{k-1}\begin{bmatrix} k-1 \\ j \end{bmatrix}_q m^{j+1}(x-1)_q^j\right]\mathfrak{E}_{n-k,q}(0,my), \\
\mathfrak{B}_{n,q}(x,y) &= \frac{1}{2m^n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \left[m^k\mathfrak{B}_{k,q}(0,y) + \sum_{j=0}^{k}\begin{bmatrix} k \\ j \end{bmatrix}_q \left(\frac{1}{m}-1\right)_q^{k-j}\mathfrak{B}_{j,q}(0,y)\right. \\
&\qquad \left. + [k]_q\sum_{j=0}^{k-1}\begin{bmatrix} k-1 \\ j \end{bmatrix}_q q^{\frac{1}{2}j(j-1)}\left(\frac{1}{m}-1\right)_q^{k-1-j}y^j\right]\mathfrak{E}_{n-k,q}(mx,0)
\end{aligned}
$$

holds true between the q-Bernoulli polynomials and q-Euler polynomials.

Corollary 10 ([26]) *For $n \in \mathbb{N}_0$, $m \in \mathbb{N}$, the following relationship holds true*:

$$
\begin{aligned}
B_n(x+y) &= \sum_{k=0}^{n}\binom{n}{k}\left(B_k(y) + \frac{k}{2}y^{k-1}\right)E_{n-k}(x), \\
B_n(x+y) &= \frac{1}{2m^n}\sum_{k=0}^{n}\binom{n}{k}\left[m^kB_k(x) + m^kB_k\left(x-1+\frac{1}{m}\right)\right. \\
&\qquad \left. + km\bigl(1+m(x-1)\bigr)^{k-1}\right]E_{n-k}(my).
\end{aligned}
$$

Corollary 11 *For $n \in \mathbb{N}_0$, the following relationship holds true*:

$$\mathfrak{B}_{n,q}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \left(\mathfrak{B}_{k,q}(0,y) + q^{\frac{1}{2}(k-1)(k-2)} \frac{[k]_q}{2} y^{k-1} \right) \mathfrak{E}_{n-k,q}(x,0). \tag{16}$$

Corollary 12 *For $n \in \mathbb{N}_0$, the following relationship holds true*:

$$\mathfrak{B}_{n,q}(x,0) = \sum_{\substack{k=0 \\ (k \neq 1)}}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \mathfrak{B}_{k,q} \mathfrak{E}_{n-k,q}(x,0) + \left(\mathfrak{B}_{1,q} + \frac{1}{2} \right) \mathfrak{E}_{n-1,q}(x,0), \tag{17}$$

$$\mathfrak{B}_{n,q}(0,y) = \sum_{\substack{k=0 \\ (k \neq 1)}}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{B}_{k,q} \mathfrak{E}_{n-k,q}(0,y) + \left(\mathfrak{B}_{1,q} + \frac{1}{2} \right) \mathfrak{E}_{n-1,q}(0,y). \tag{18}$$

The formulas (16)-(18) are the q-extension of Cheon's main result [25]. Notice that $\mathfrak{B}_{1,q} = -\frac{1}{[2]_q}$, see [23], and the extra term becomes zero for $q \to 1^-$.

Theorem 13 *For $n \in \mathbb{N}_0$, the relationships*

$$\begin{aligned} \mathfrak{E}_{n,q}^{(\alpha)}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{1}{m^{n-1}[k+1]_q} \Bigg[2 \sum_{j=0}^{k+1} \begin{bmatrix} k+1 \\ j \end{bmatrix}_q \left(\frac{1}{m} - 1 \right)_q^{k+1-j} \mathfrak{E}_{j,q}^{(\alpha-1)}(0,y) \\ - \sum_{j=0}^{k+1} \begin{bmatrix} k+1 \\ j \end{bmatrix}_q \left(\frac{1}{m} - 1 \right)_q^{k+1-j} \mathfrak{E}_{j,q}^{(\alpha)}(0,y) - \mathfrak{E}_{k+1,q}^{(\alpha)}(0,y) \Bigg] \mathfrak{B}_{n-k,q}(mx,0), \\ \mathfrak{E}_{n,q}^{(\alpha)}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{1}{m^{n}[k+1]_q} \Bigg[2 \sum_{j=0}^{k+1} \begin{bmatrix} k+1 \\ j \end{bmatrix}_q m^j \mathfrak{E}_{j,q}^{(\alpha-1)}(x,-1) \\ - \sum_{j=0}^{k+1} \begin{bmatrix} k+1 \\ j \end{bmatrix}_q m^j \mathfrak{E}_{j,q}^{(\alpha)}(x,-1) - m^{k+1} \mathfrak{E}_{k+1,q}^{(\alpha)}(x,0) \Bigg] \mathfrak{B}_{n-k,q}(0,my) \end{aligned}$$

hold true between the q-Bernoulli polynomials and q-Euler polynomials.

Proof The proof is based on the following identities:

$$\begin{aligned} \left(\frac{2}{e_q(t)+1} \right)^{\alpha} e_q(tx) E_q(ty) = \left(\frac{2}{e_q(t)+1} \right)^{\alpha} E_q(ty) \cdot \frac{e_q(\frac{t}{m})-1}{t} \cdot \frac{t}{e_q(\frac{t}{m})-1} e_q\left(\frac{t}{m} mx \right), \\ \left(\frac{2}{e_q(t)+1} \right)^{\alpha} e_q(tx) E_q(ty) = \left(\frac{2}{e_q(t)+1} \right)^{\alpha} e_q(tx) \cdot \frac{e_q(\frac{t}{m})-1}{t} \cdot \frac{t}{e_q(\frac{t}{m})-1} E_q\left(\frac{t}{m} my \right) \end{aligned}$$

and is similar to that of Theorem 8. □

Next we discuss some special cases of Theorem 13.

Corollary 14 *For $n \in \mathbb{N}_0$, $m \in \mathbb{N}$, the relationship*

$$\begin{aligned}\mathfrak{E}_{n,q}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{m^{-n}}{[k+1]_q} \Bigg[2 \sum_{j=0}^{k+1} \begin{bmatrix} k+1 \\ j \end{bmatrix}_q m^j (x-1)_q^j \\ - \sum_{j=0}^{k+1} \begin{bmatrix} k+1 \\ j \end{bmatrix}_q m^j \mathfrak{E}_{j,q}(x,-1) - m^{k+1} \mathfrak{E}_{k+1,q}(x,0) \Bigg] \mathfrak{B}_{n-k,q}(0,my)\end{aligned}$$

holds true between the q-Bernoulli polynomials and q-Euler polynomials.

Corollary 15 ([26]) *For $n \in \mathbb{N}_0$, $m \in \mathbb{N}$, the following relationships hold true*:

$$E_n(x+y) = \sum_{k=0}^{n} \frac{2}{k+1} \binom{n}{k} \left(y^{k+1} - E_{k+1}(y) \right) B_{n-k}(x),$$

$$\begin{aligned}E_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} \frac{m^{k-n+1}}{k+1} \Bigg[2 \left(x + \frac{1-m}{m} \right)^{k+1} - E_{k+1} \left(x + \frac{1-m}{m} \right) \\ - E_{k+1}(x) \Bigg] B_{n-k}(my).\end{aligned}$$

Corollary 16 *For $n \in \mathbb{N}_0$, the following relationship holds true*:

$$\mathfrak{E}_{n,q}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{2}{[k+1]_q} \left(q^{\frac{1}{2}k(k+1)} y^{k+1} - \mathfrak{E}_{k+1,q}(0,y) \right) \mathfrak{B}_{n-k,q}(x,0).$$

Corollary 17 *For $n \in \mathbb{N}_0$, the following relationships hold true*:

$$\mathfrak{E}_{n,q}(x,0) = -\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{2}{[k+1]_q} \mathfrak{E}_{k+1,q} \mathfrak{B}_{n-k,q}(x,0),$$

$$\mathfrak{E}_{n,q}(0,y) = -\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{2}{[k+1]_q} \mathfrak{E}_{k+1,q} \mathfrak{B}_{n-k,q}(0,y).$$

These formulas are q-analogues of the formula of Srivastava and Pintér [24].

4 q-Stirling numbers and q-Bernoulli polynomials

In this section, we aim to derive several formulas involving the q-Bernoulli polynomials, the q-Euler polynomials of order α, the q-Stirling numbers of the second kind and the q-Bernstein polynomials.

Theorem 18 *Each of the following relationships holds true for the Stirling numbers $S_2(n,k)$ of the second kind*:

$$\mathfrak{B}_{n,q}^{(\alpha)}(x,y) = \sum_{j=0}^{n} \binom{mx}{j} j! \sum_{k=0}^{n-j} \begin{bmatrix} n \\ k \end{bmatrix}_q m^{j-n} \mathfrak{B}_{k,q}^{(\alpha)}(0,y) S_2(n-k,j),$$

$$\mathfrak{E}_{n,q}^{(\alpha)}(x,y) = \sum_{j=0}^{n} \binom{mx}{j} j! \sum_{k=0}^{n-j} \begin{bmatrix} n \\ k \end{bmatrix}_q m^{j-n} \mathfrak{E}_{k,q}^{(\alpha)}(0,y) S_2(n-k,j).$$

The familiar q-Stirling numbers $S(n,k)$ of the second kind are defined by

$$\frac{(e_q(t)-1)^k}{[k]_q!}=\sum_{m=0}^{\infty}S_{2,q}(m,k)\frac{t^m}{[m]_q!},$$

where $k\in\mathbb{N}$. Next we give the relationship between q-Bernstein basis defined by Phillips [27] and q-Bernoulli polynomials

$$b_{n,k}(q;x):=\begin{bmatrix}n\\k\end{bmatrix}_q x^k(1-x)_q^{n-k}.$$

Theorem 19 *We have*

$$b_{n,k}(q;x)=x^k\sum_{m=0}^{n}\begin{bmatrix}n\\m\end{bmatrix}_q S_{2,q}(m,k)\mathfrak{B}_{n-m,q}^{(k)}(1,-x). \tag{19}$$

Proof The proof follows from the following identities:

$$\begin{aligned}\frac{x^kt^k}{[k]_q!}e_q(t)E_q(-xt)&=\frac{x^kt^k}{[k]_q!}\sum_{n=0}^{\infty}\frac{(1-x)_q^nt^n}{[n]_q!}=\sum_{n=k}^{\infty}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{x^k(1-x)_q^{n-k}t^n}{[n]_q!}\\&=\sum_{n=k}^{\infty}b_{n,k}(q;x)\frac{t^n}{[n]_q!}\end{aligned}$$

and

$$\begin{aligned}\frac{x^kt^k}{[k]_q!}e_q(t)E_q(-xt)&=\frac{x^k(e_q(t)-1)^k}{[k]_q!}\frac{t^k}{(e_q(t)-1)^k}e_q(t)E_q(-xt)\\&=x^k\sum_{m=0}^{\infty}S_{2,q}(m,k)\frac{t^m}{[m]_q!}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(k)}(1,-x)\frac{t^n}{[n]_q!}\\&=x^k\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\begin{bmatrix}n\\m\end{bmatrix}_q S_{2,q}(m,k)\mathfrak{B}_{n-m,q}^{(k)}(1,-x)\right)\frac{t^n}{[n]_q!}.\end{aligned}$$

□

Finally, in their limit case when $q\to 1^-$, this last result (19) would reduce to the following formula for the classical Bernoulli polynomials $B_n^{(k)}(x)$ and the Bernstein basis $b_{n,k}(x)=\binom{n}{k}x^k(1-x)^{n-k}$:

$$b_{n,k}(x)=x^k\sum_{m=0}^{n}\binom{n}{m}S_2(m,k)\mathfrak{B}_{n-m}^{(k)}(1-x).$$

Competing interests

The author declares that he has no competing interests.

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