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# Forced oscillation of higher-order nonlinear neutral difference equations with positive and negative coefficients

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**Abstract**

In this paper, we study the forced oscillation of the higher-order nonlinear difference equation of the form

$$\Delta^m\left[x(n)-p(n)x(n-\tau)\right]+q_1(n)\Phi_\alpha(n-\sigma_1)+q_2(n)\Phi_\beta(n-\sigma_2)=f(n),$$

where $m\geq 1$, $\tau$, $\sigma_1$ and $\sigma_2$ are integers, $0<\alpha<1<\beta$ are constants, $\Phi_*(u)=|u|^{*-1}u$, $p(n)$, $q_1(n)$, $q_2(n)$ and $f(n)$ are real sequences with $p(n)>0$. By taking all possible values of $\tau$, $\sigma_1$ and $\sigma_2$ into consideration, we establish some new oscillation criteria for the above equation in two cases: (i) $q_1=q_1(n)\leq 0$, $q_2=q_2(n)>0$; (ii) $q_1\geq 0$, $q_2<0$.

**MSC:** 39A10

**Keywords:** forced oscillation; neutral difference equation; positive and negative coefficients; higher-order

## 1 Introduction

Qualitative theory of difference equations has received much attention in recent years due to its extensive applications in computer, probability theory, queuing problems, statistical problems, stochastic time series, combinatorial analysis, number theory, electrical networks, genetics in biology, economics, psychology, sociology, and so on [1, 2].

In this paper, we consider the oscillation of the following $m$th-order forced nonlinear difference equation of the form

$$\Delta^m\left[x(n)-p(n)x(n-\tau)\right]+q_1(n)\Phi_\alpha(n-\sigma_1)+q_2(n)\Phi_\beta(n-\sigma_2)=f(n), \tag{1}$$

where $m\geq 1$, $\tau$, $\sigma_1$ and $\sigma_2$ are integers, $\Phi_*(u)=|u|^{*-1}u$, $p(n)$, $q_1(n)$, $q_2(n)$ and $f(n)$ are real sequences defined on $N=\{0,1,2,\ldots\}$ with $p(n)>0$, $0<\alpha<1<\beta$ are constants, and

$$\Delta x(n)=x(n+1)-x(n),\qquad \Delta^i x(n)=\Delta\left(\Delta^{i-1}x(n)\right),\quad 2\leq i\leq m.$$

As usual, a solution of Eq. (1) is said to be oscillatory, if for every integer $N\geq 0$, there exists $n\geq N$ such that $x(n)x(n+1)\leq 0$; otherwise, it is called nonoscillatory.

For the continuous version of Eq. (1), many authors have studied its oscillation (see monograph [3] and references therein). To the best of our knowledge, little has been known about the forced oscillation of Eq. (1) with positive and negative coefficients ($q_1 \le 0$, $q_2 > 0$ or $q_1 \ge 0$, $q_2 < 0$) and mixed nonlinearities ($0 < \alpha < 1$, $\beta > 1$). For some particular cases of Eq. (1), there have been many oscillation results in [4–19], to name a few. Motivated by the work in [20–22], we study the forced oscillation of Eq. (1) in this paper.

The main contribution of this paper is that we establish some new oscillation criteria for Eq. (1) with positive and negative coefficients and mixed nonlinearities. Unlike some existing results in the literature, all possible values of delays $\tau$, $\sigma_1$ and $\sigma_2$ are considered.

## 2 Main results

Throughout this paper, we denote

$$\phi_0(n,s) = \phi(n,s) = (n-s)^{(k)} = (n-s)(n-s+1)\cdots(n-s+k-1), \quad k \ge m, \tag{2}$$

$$\phi_i(n,s) = (-1)^i \Delta_s^i \phi(n,s) = C_k^i (n-s)^{(k-i)}, \quad i = 1,2,\ldots,m. \tag{3}$$

By the straightforward computation, it is not difficult to see that

$$\begin{cases} \phi(n,s) = 0, & n \le s \le n+k-1, \\ \phi_i(n+i+1,n+m) = 0, & i = 0,1,2,\ldots,m-1, \\ \phi_m(n,s) \ge 0, & 0 \le s \le n-1, \end{cases} \tag{4}$$

and

$$\lim_{n\to\infty} \frac{\phi_i(n+i+1,n_0)}{\phi(n+1,n_0)} = o(1), \quad i = 1,2,\ldots,m, \tag{5}$$

where $n_0 \ge 0$ is an integer. We also denote $\sum_{s=l}^{k} = 0$ if $k < l$.

The following two facts can be easily proved.

Fact 1. Set $F(x) = ax - bx^\lambda$, where $x \ge 0$, $a \ge 0$ and $b > 0$. If $\lambda > 1$, $F(x)$ obtains its maximum $F_{\max} = (\lambda-1)\lambda^{\frac{\lambda}{1-\lambda}} a^{\frac{\lambda}{\lambda-1}} b^{\frac{1}{1-\lambda}}$.

Fact 2. Set $G(x) = cx - dx^\lambda$, where $x \ge 0$, $c > 0$ and $d \ge 0$. If $0 < \lambda < 1$, $G(x)$ obtains its minimum $G_{\min} = (\lambda-1)\lambda^{\frac{\lambda}{1-\lambda}} c^{\frac{\lambda}{\lambda-1}} d^{\frac{1}{1-\lambda}}$.

We now present the main results of this paper as follows.

**Theorem 1** *Assume that $q_1(n) \le 0$, $q_2(n) > 0$, $\sigma_1 \ge -m$ and $\sigma_2 - \tau \le -m$. If*

$$\lim_{n\to\infty} \sup \frac{1}{\phi(n+1,n_0)} \left\{ \sum_{s=n_0}^{n+m-1} \left[\phi(n,s)f(s) - Q_2(n,s)\right] + \sum_{s=n_0}^{n-\sigma_1-1} Q_1(n,s) \right\} = +\infty, \tag{6}$$

$$\lim_{n\to\infty} \inf \frac{1}{\phi(n+1,n_0)} \left\{ \sum_{s=n_0}^{n+m-1} \left[\phi(n,s)f(s) + Q_2(n,s)\right] - \sum_{s=n_0}^{n-\sigma_1-1} Q_1(n,s) \right\} = -\infty, \tag{7}$$

*where*

$$Q_1(n,s) = (\alpha-1)\alpha^{\frac{\alpha}{1-\alpha}} \left[\phi_m(n+m,s)\right]^{\frac{\alpha}{\alpha-1}} \left[\phi(n,s+\sigma_1)\left|q_1(s+\sigma_1)\right|\right]^{\frac{1}{1-\alpha}}, \tag{8}$$

$$Q_2(n,s) = (\beta-1)\beta^{\frac{\beta}{1-\beta}} \left[\phi_m(n+m,s)p(s)\right]^{\frac{\beta}{\beta-1}} \left[\phi(n,s+\sigma_2-\tau)q_2(s+\sigma_2-\tau)\right]^{\frac{1}{1-\beta}}, \tag{9}$$

*all solutions of Eq. (1) are oscillatory.*

*Proof* Assume to the contrary that there exists a nontrivial solution $x(n)$ of Eq. (1) such that $x(n)$ is nonoscillatory. That is, $x(n)$ does not change sign eventually. Without loss of generality, let $x(n-\sigma_1) \geq 0$, $x(n-\sigma_2) \geq 0$, $x(n-\tau) \geq 0$ for $n \geq n_0$, where $n_0 \geq 0$ is sufficiently large. By the straightforward computation, we have

$$F_1(n,s) - F_2(n,s) = \sum_{s=n_0}^{n+m-1} \phi(n,s)f(s), \tag{10}$$

where

$$F_1(n,s) = \sum_{s=n_0}^{n+m-1} \phi(n,s)\Delta^m x(s) - \sum_{s=n_0}^{n+m-1} \phi(n,s)\big|q_1(s)\big|x^{\alpha}(s-\sigma_1),$$

$$F_2(n,s) = \sum_{s=n_0}^{n+m-1} \phi(n,s)\Delta^m\big[p(s)x(n-\tau)\big] - \sum_{s=n_0}^{n+m-1} \phi(n,s)q_2(s)x^{\beta}(s-\sigma_2).$$

Noting that

$$\begin{aligned}\phi(n,s)\Delta x(s) &= \Delta\big[\phi(n,s-1)x(s)\big] + \phi_1(n,s-1)x(s)\\ &= \Delta\big[\phi(n+1,s)x(s)\big] + \phi_1(n+1,s)x(s),\end{aligned}$$

we can get from (2), (3) and (4) that

$$\begin{aligned}\sum_{s=n_0}^{n+m-1} \phi(n,s)\Delta^m x(s) &= \sum_{s=n_0}^{n+m-1} \Delta\big[\phi(n+1,s)\Delta^{m-1}x(s)\big] + \sum_{s=n_0}^{n+m-1} \phi_1(n+1,s)\Delta^{m-1}x(s)\\ &= -\phi(n+1,n_0)\Delta^{m-1}x(n_0) + \sum_{s=n_0}^{n+m-1} \phi_1(n+1,s)\Delta^{m-1}x(s)\\ &= -\phi(n+1,n_0)\Delta^{m-1}x(n_0) + \sum_{s=n_0}^{n+m-1} \Delta\big[\phi_1(n+2,s)\Delta^{m-2}x(s)\big]\\ &\quad + \sum_{s=n_0}^{n+m-1} \phi_2(n+2,s)\Delta^{m-2}x(s)\\ &= -\phi(n+1,n_0)\Delta^{m-1}x(n_0) - \phi_1(n+2,n_0)\Delta^{m-2}x(n_0)\\ &\quad + \sum_{s=n_0}^{n+m-1} \phi_2(n+2,s)\Delta^{m-2}x(s)\\ &= -\sum_{i=0}^{m-1}\phi_i(n+i+1,n_0)\Delta^{m-i-1}x(n_0) + \sum_{s=n_0}^{n+m-1} \phi_m(n+m,s)x(s).\end{aligned} \tag{11}$$

Since $\phi(n,s) = 0$ for $n \leq s \leq n+m-1$ due to (4), we get from (11) that

$$\begin{aligned}F_1(n,s) = {}&\sum_{s=n_0}^{n+m-1} \phi_m(n+m,s)x(s) - \sum_{s=n_0}^{n-1}\phi(n,s)\big|q_1(s)\big|x^{\alpha}(s-\sigma_1)\\ &- \sum_{i=0}^{m-1}\phi_i(n+i+1,n_0)\Delta^{m-i-1}x(n_0)\end{aligned}$$

$$= \sum_{s=n_0}^{n+m-1} \phi_m(n+m,s)x(s) - \sum_{s=n_0-\sigma_1}^{n-\sigma_1-1} \phi(n,s+\sigma_1)\left|q_1(s+\sigma_1)\right|x^{\alpha}(s)$$

$$- \sum_{i=0}^{m-1} \phi_i(n+i+1,n_0)\Delta^{m-i-1}x(n_0). \tag{12}$$

Noting that $\sigma_1 \geq -m$, we have that $n+m-1 \geq n-\sigma_1-1$. Therefore, we get from (12) that

$$F_1(n,s) \geq \sum_{s=n_0}^{n-\sigma_1-1} \left[\phi_m(n+m,s)x(s) - \phi(n,s+\sigma_1)\left|q_1(s+\sigma_1)\right|x^{\alpha}(s)\right]$$

$$- \sum_{s=n_0-\sigma_1}^{n_0-1} \phi(n,s+\sigma_1)\left|q_1(s+\sigma_1)\right|x^{\alpha}(s)$$

$$- \sum_{i=0}^{m-1} \phi_i(n+i+1,n_0)\Delta^{m-i-1}x(n_0). \tag{13}$$

By Fact 2 and (13), it is not difficult to see that

$$F_1(n,s) \geq \sum_{s=n_0}^{n-\sigma_1-1} Q_1(n,s) - \sum_{s=n_0-\sigma_1}^{n_0-1} \phi(n,s+\sigma_1)\left|q_1(s+\sigma_1)\right|x^{\alpha}(s)$$

$$- \sum_{i=0}^{m-1} \phi_i(n+i+1,n_0)\Delta^{m-i-1}x(n_0), \tag{14}$$

where $Q_1(n,s)$ is defined by (8).

On the other hand, similar to the above analysis, we have that

$$F_2(n,s) = \sum_{s=n_0}^{n+m-1} \phi_m(n+m,s)p(s)x(s-\tau) - \sum_{s=n_0}^{n+m-1} \phi(n,s)q_2(s)x^{\beta}(s-\sigma_2)$$

$$- \sum_{i=0}^{m-1} \phi_i(n+i+1,n_0)\Delta^{m-i-1}\left[p(n_0)x(n_0-\tau)\right]$$

$$= \sum_{s=n_0}^{n+m-1} \phi_m(n+m,s)p(s)x(s-\tau) - \sum_{s=n_0}^{n-1} \phi(n,s)q_2(s)x^{\beta}(s-\sigma_2)$$

$$- \sum_{i=0}^{m-1} \phi_i(n+i+1,n_0)\Delta^{m-i-1}\left[x(n_0-\tau)p(n_0)\right]$$

$$= \sum_{s=n_0}^{n+m-1} \phi_m(n+m,s)p(s)x(s-\tau)$$

$$- \sum_{s=n_0-\sigma_2+\tau}^{n-\sigma_2+\tau-1} \phi(n,s+\sigma_2-\tau)q_2(s+\sigma_2-\tau)x^{\beta}(s-\tau)$$

$$- \sum_{i=0}^{m-1} \phi_i(n+i+1,n_0)\Delta^{m-i-1}\left[p(n_0)x(n_0-\tau)\right]. \tag{15}$$

Since $\sigma_2 - \tau \leq -m$, we have that $n - \sigma_2 + \tau - 1 \geq n + m - 1$. By (15), we get

$$\begin{aligned} F_2(n,s) \leq & \sum_{s=n_0}^{n+m-1} \left[\phi_m(n+m,s)p(s)x(s-\tau) - \phi(n,s+\sigma_2-\tau)q_2(s+\sigma_2-\tau)x^\beta(s-\tau)\right] \\ & + \sum_{s=n_0}^{n_0-\sigma_2+\tau-1} \phi(n,s+\sigma_2-\tau)q_2(s+\sigma_2-\tau)x^\beta(s-\tau) \\ & - \sum_{i=0}^{m-1} \phi_i(n+i+1,n_0)\Delta^{m-i-1}\left[p(n_0)x(n_0-\tau)\right]. \end{aligned} \tag{16}$$

By Fact 1 and (16), we have that

$$\begin{aligned} F_2(n,s) \leq & \sum_{s=n_0}^{n+m-1} Q_2(n,s) + \sum_{s=n_0}^{n_0-\sigma_2+\tau-1} \phi(n,s+\sigma_2-\tau)q_2(s+\sigma_2-\tau)x^\beta(s-\tau) \\ & - \sum_{i=0}^{m-1} \phi_i(n+i+1,n_0)\Delta^{m-i-1}\left[p(n_0)x(n_0-\tau)\right], \end{aligned} \tag{17}$$

where $Q_2(n,s)$ is defined by (9).

Multiplying $\frac{1}{\phi(n+1,n_0)}$ on both sides of (10), by (14), (17) and (5), we have that there exists a constant $M_1$ such that

$$\frac{1}{\phi(n+1,n_0)} \left\{ \sum_{s=n_0}^{n+m-1} \left[\phi(n,s)f(s) + Q_2(n,s)\right] - \sum_{s=n_0}^{n-\sigma_1-1} Q_1(n,s) \right\} \geq M_1,$$

which contradicts (7). For the case when $x(n)$ is eventually negative, we can similarly get a contradiction to (6). This completes the proof of Theorem 1. □

**Theorem 2** *Assume that* $q_1(n) \geq 0$, $q_2(n) < 0$, $\sigma_1 - \tau \geq -m$ *and* $\sigma_2 \leq -m$. *If*

$$\lim_{n\to\infty} \sup \frac{1}{\phi(n+1,n_0)} \left\{ \sum_{s=n_0}^{n+m-1} \left[\phi(n,s)f(s) - P_1(n,s)\right] + \sum_{s=n_0}^{n-\sigma_2+\tau-1} P_2(n,s) \right\} = +\infty, \tag{18}$$

$$\lim_{n\to\infty} \inf \frac{1}{\phi(n+1,n_0)} \left\{ \sum_{s=n_0}^{n+m-1} \left[\phi(n,s)f(s) + P_1(n,s)\right] - \sum_{s=n_0}^{n-\sigma_2+\tau-1} P_2(n,s) \right\} = -\infty, \tag{19}$$

*where*

$$P_1(n,s) = (\beta-1)\beta^{\frac{\beta}{1-\beta}}\left[\phi_m(n+m,s)\right]^{\frac{\beta}{\beta-1}}\left[\phi(n,s+\sigma_2)\left|q_2(s+\sigma_2)\right|\right]^{\frac{1}{1-\beta}}, \tag{20}$$

$$P_2(n,s) = (\alpha-1)\alpha^{\frac{\alpha}{1-\alpha}}\left[\phi_m(n+m,s)p(s)\right]^{\frac{\alpha}{\alpha-1}}\left[\phi(n,s+\sigma_1-\tau)q_1(s+\sigma_1-\tau)\right]^{\frac{1}{1-\alpha}}, \tag{21}$$

*all solutions of Eq. (1) are oscillatory.*

*Proof* Suppose to the contrary that there exists a nontrivial solution $x(n)$ of Eq. (1) such that $x(n)$ is nonoscillatory. We may let $x(n-\sigma_1) \geq 0$, $x(n-\sigma_2) \geq 0$, $x(n-\tau) \geq 0$ for $n \geq n_0$, where $n_0 \geq 0$ is sufficiently large. By the straightforward computation, we get from Eq. (1) that

$$G_1(n,s) - G_2(n,s) = \sum_{s=n_0}^{n+m-1} \phi(n,s)f(s), \tag{22}$$

where

$$G_1(n,s) = \sum_{s=n_0}^{n+m-1} \phi(n,s)\Delta^m x(s) - \sum_{s=n_0}^{n+m-1} \phi(n,s)\bigl|q_2(s)\bigr|x^\beta(s-\sigma_2),$$

$$G_2(n,s) = \sum_{s=n_0}^{n+m-1} \phi(n,s)\Delta^m\bigl[p(s)x(n-\tau)\bigr] - \sum_{s=n_0}^{n+m-1} \phi(n,s)q_1(s)x^\alpha(s-\sigma_1).$$

Noticing that $\phi(n,s) = 0$ for $n \leq s \leq n+m-1$, we get from (11) that

$$\begin{aligned} G_1(n,s) = {} & \sum_{s=n_0}^{n+m-1} \phi_m(n+m,s)x(s) - \sum_{s=n_0}^{n-1} \phi(n,s)\bigl|q_2(s)\bigr|x^\beta(s-\sigma_2) \\ & - \sum_{i=0}^{m-1} \phi_i(n+i+1,n_0)\Delta^{m-i-1}x(n_0) \\ = {} & \sum_{s=n_0}^{n+m-1} \phi_m(n+m,s)x(s) - \sum_{s=n_0-\sigma_2}^{n-\sigma_2-1} \phi(n,s+\sigma_2)\bigl|q_2(s+\sigma_2)\bigr|x^\beta(s) \\ & - \sum_{i=0}^{m-1} \phi_i(n+i+1,n_0)\Delta^{m-i-1}x(n_0). \end{aligned} \tag{23}$$

Since $\sigma_2 \leq -m$, we have that $n+m-1 \leq n-\sigma_2-1$. Thus, we can get from (23) that

$$\begin{aligned} G_1(n,s) \leq {} & \sum_{s=n_0}^{n+m-1} \bigl[\phi_m(n+m,s)x(s) - \phi(n,s+\sigma_2)\bigl|q_2(s+\sigma_2)\bigr|x^\beta(s)\bigr] \\ & + \sum_{s=n_0}^{n_0-\sigma_2-1} \phi(n,s+\sigma_2)\bigl|q_2(s+\sigma_2)\bigr|x^\beta(s) \\ & - \sum_{i=0}^{m-1} \phi_i(n+i+1,n_0)\Delta^{m-i-1}x(n_0). \end{aligned} \tag{24}$$

By Fact 1 and (24), it is easy to see that

$$\begin{aligned} G_1(n,s) \leq {} & \sum_{s=n_0}^{n+m-1} P_1(n,s) + \sum_{s=n_0}^{n_0-\sigma_2-1} \phi(n,s+\sigma_2)\bigl|q_2(s+\sigma_2)\bigr|x^\beta(s) \\ & - \sum_{i=0}^{m-1} \phi_i(n+i+1,n_0)\Delta^{m-i-1}x(n_0), \end{aligned} \tag{25}$$

where $P_1(n,s)$ is defined by (20).

On the other hand, similar to the computation of (11), we can get

$$\begin{aligned} G_2(n,s) &= \sum_{s=n_0}^{n+m-1} \phi_m(n+m,s)p(s)x(s-\tau) - \sum_{s=n_0}^{n-1} \phi(n,s)q_1(s)x^\alpha(s-\sigma_1) \\ &\quad - \sum_{i=0}^{m-1} \phi_i(n+i+1,n_0)\Delta^{m-i-1}\bigl[p(n_0)x(n_0-\tau)\bigr] \\ &= \sum_{s=n_0}^{n+m-1} \phi_m(n+m,s)p(s)x(s-\tau) \\ &\quad - \sum_{s=n_0-\sigma_1+\tau}^{n-\sigma_1+\tau-1} \phi(n,s+\sigma_1-\tau)q_1(s+\sigma_1-\tau)x^\alpha(s-\tau) \\ &\quad - \sum_{i=0}^{m-1} \phi_i(n+i+1,n_0)\Delta^{m-i-1}\bigl[p(n_0)x(n_0-\tau)\bigr]. \end{aligned} \tag{26}$$

Noting that $\sigma_1 - \tau \geq -m$ implies $n+m-1 \geq n-\sigma_1+\tau-1$, we get from (26) that

$$\begin{aligned} G_2(n,s) \geq{}& \sum_{s=n_0}^{n-\sigma_1+\tau-1} \bigl[\phi_m(n+m,s)p(s)x(s-\tau) - \phi(n,s+\sigma_1-\tau)q_1(s+\sigma_1-\tau)x^\alpha(s-\tau)\bigr] \\ & - \sum_{s=n_0-\sigma_1+\tau}^{n_0-1} \phi(n,s+\sigma_1-\tau)q_1(s+\sigma_1-\tau)x^\alpha(s-\tau) \\ & - \sum_{i=0}^{m-1} \phi_i(n+i+1,n_0)\Delta^{m-i-1}\bigl[p(n_0)x(n_0-\tau)\bigr]. \end{aligned} \tag{27}$$

By Fact 2 and (27), we have that

$$\begin{aligned} G_2(n,s) \geq{}& \sum_{s=n_0}^{n-\sigma_1+\tau-1} P_2(n,s) - \sum_{s=n_0-\sigma_1+\tau}^{n_0-1} \phi(n,s+\sigma_1-\tau)q_1(s+\sigma_1-\tau)x^\alpha(s-\tau) \\ & - \sum_{i=0}^{m-1} \phi_i(n+i+1,n_0)\Delta^{m-i-1}\bigl[p(n_0)x(n_0-\tau)\bigr], \end{aligned} \tag{28}$$

where $P_2(n,s)$ is defined by (21).

Multiplying $\frac{1}{\phi(n+1,n_0)}$ on both sides of (22), from (25), (28) and (5), we have that there exists a constant $M_2$ such that

$$\frac{1}{\phi(n+1,n_0)} \left\{ \sum_{s=n_0}^{n+m-1} \bigl[\phi(n,s)f(s) - P_1(n,s)\bigr] + \sum_{s=n_0}^{n-\sigma_2+\tau-1} P_2(n,s) \right\} \leq M_2.$$

This is a contradiction to (18). For the case when $x(n)$ is eventually negative, we can similarly get a contradiction to (19). This completes the proof of Theorem 2. □

By Theorems 1 and 2, the following two corollaries are immediate.

**Corollary 1** *Assume that* $q_1(n) \geq 0$, $q_2(n) > 0$ *and* $\sigma_2 - \tau \leq -m$. *If*

$$\lim_{n\to\infty} \sup \frac{1}{\phi(n+1,n_0)} \sum_{s=n_0}^{n+m-1} \left[\phi(n,s)f(s) - Q_2(n,s)\right] = +\infty,$$

$$\lim_{n\to\infty} \inf \frac{1}{\phi(n+1,n_0)} \sum_{s=n_0}^{n+m-1} \left[\phi(n,s)f(s) + Q_2(n,s)\right] = -\infty,$$

*where* $Q_2(n,s)$ *is defined by (9), all solutions of Eq. (1) are oscillatory for any constant* $\sigma_1$.

*Proof* In fact, we have that $F_1(n,s) \geq 0$ for any constant $\sigma_1$ since $q_1(n,s) \geq 0$. So, we can drop $F_1(n,s)$ in the estimation of (10). The other proof runs as that of Theorem 1, and hence it is omitted. □

**Corollary 2** *Assume that* $q_1(n) \leq 0$, $q_2(n) < 0$ *and* $\sigma_2 \leq -m$. *If*

$$\lim_{n\to\infty} \sup \frac{1}{\phi(n+1,n_0)} \sum_{s=n_0}^{n+m-1} \left[\phi(n,s)f(s) - P_1(n,s)\right] = +\infty,$$

$$\lim_{n\to\infty} \inf \frac{1}{\phi(n+1,n_0)} \sum_{s=n_0}^{n+m-1} \left[\phi(n,s)f(s) + P_1(n,s)\right] = -\infty,$$

*where* $P_1(n,s)$ *is defined by (20), all nontrivial solutions of Eq. (1) are oscillatory.*

*Proof* For this case, we have that $G_2(n,s) \geq 0$ for any constant $\sigma_1$ since $q_1(n,s) \leq 0$. Therefore, we can drop $G_2(n,s)$ in the estimation of (22). The other proof runs as that of Theorem 2. □

For other cases of $\sigma_1$ and $\sigma_2$ that are not covered by Theorem 1 and Theorem 2, the above method usually does not give sufficient conditions for the oscillation of all solutions of Eq. (1). However, when assuming that the solutions of Eq. (1) satisfy appropriate conditions, sufficient conditions for such solutions can also be derived. In the following, we are focused on the oscillation of all solutions of Eq. (1) satisfying $x(n) = O(n^r)$ for some $r > 0$. Here, $x(n) = O(n^r)$ means that there exists a constant $c > 0$ such that $|x(n)| \leq cn^r$ for $n \geq n_0$.

**Theorem 3** *Assume that* $q_1(n) \leq 0$, $q_2(n) > 0$, *and (6) and (7) hold. All solutions satisfying* $x(n) = O(n^r)$ *are oscillatory if one of the following conditions holds:*

*(i)* $\sigma_1 < -m$, $\sigma_2 - \tau \leq -m$, *and*

$$\lim_{n\to\infty} \sup \frac{1}{\phi(n+1,n_0)} \sum_{s=n+m}^{n-\sigma-1} \left[\phi_m(n+m,s)s^r\right] < \infty, \tag{29}$$

*(ii)* $\sigma_1 \geq -m$, $\sigma_2 - \tau > -m$, *and*

$$\lim_{n\to\infty} \sup \frac{1}{\phi(n+1,n_0)} \sum_{s=n-\sigma_2+\tau}^{n+m-1} \left[\phi(n,s+\sigma_2-\tau)q_2(s+\sigma_2-\tau)s^{r\beta}\right] < \infty, \tag{30}$$

*(iii)* $\sigma_1 < -m$, $\sigma_2 - \tau > -m$, *(29) and (30) hold.*

*Proof* Assume that there exists a nontrivial solution $x(n)$ of Eq. (1) such that $x(n)$ is nonoscillatory. Without loss of generality, let $x(n-\sigma_1)\geq 0$, $x(n-\sigma_2)\geq 0$, $x(n-\tau)\geq 0$ for $n\geq n_0$, where $n_0\geq 0$ is sufficiently large.

(i) For the case $\sigma_1 < -m$, we have that $n+m-1 < n-\sigma_1-1$. Therefore, we get from (12) that

$$\begin{aligned} F_1(n,s) = {} & \sum_{s=n_0}^{n-\sigma_1-1}\bigl[\phi_m(n+m,s)x(s)-\phi(n,s+\sigma_1)\bigl|q_1(s+\sigma_1)\bigr|x^{\alpha}(s)\bigr] \\ & - \sum_{s=n+m}^{n-\sigma_1-1}\phi_m(n+m,s)x(s) + \sum_{s=n_0}^{n_0-\sigma_1-1}\phi(n,s+\sigma_1)\bigl|q_1(s+\sigma_1)\bigr|x^{\alpha}(s) \\ & - \sum_{i=0}^{m-1}\phi_i(n+i+1,n_0)\Delta^{m-i-1}x(n_0). \end{aligned} \tag{31}$$

By Fact 2 and (31), and noting that $x(n)\leq cn^r$ for $n\geq n_0$ and some constant $c>0$, we get

$$\begin{aligned} F_1(n,s) \geq {} & \sum_{s=n_0}^{n-\sigma_1-1} Q_1(n,s) - c\sum_{s=n+m}^{n-\sigma_1-1}\phi(n,s+\sigma_1)\bigl|q_1(s+\sigma_1)\bigr|s^r \\ & - \sum_{i=0}^{m-1}\phi_i(n+i+1,n_0)\Delta^{m-i-1}x(n_0), \end{aligned} \tag{32}$$

where $Q_1(n,s)$ is defined by (8). Multiplying $\frac{1}{\phi(n+1,n_0)}$ on both sides of (10), from (32), (17), (5) and (29), we get a contradiction to (7).

(ii) For the case $\sigma_2-\tau > -m$, we have that $n-\sigma_2+\tau-1 < n+m-1$. By (15), we get

$$\begin{aligned} F_2(n,s) = {} & \sum_{s=n_0}^{n+m-1}\bigl[\phi_m(n+m,s)p(s)x(s-\tau)-\phi(n,s+\sigma_2-\tau)q_2(s+\sigma_2-\tau)x^{\beta}(s-\tau)\bigr] \\ & + \sum_{s=n-\sigma_2+\tau}^{n+m-1}\phi(n,s+\sigma_2-\tau)q_2(s+\sigma_2-\tau)x^{\beta}(s-\tau) \\ & - \sum_{s=n_0-\sigma_2+\tau}^{n_0-1}\phi(n,s+\sigma_2-\tau)q_2(s+\sigma_2-\tau)x^{\beta}(s-\tau) \\ & - \sum_{i=0}^{m-1}\phi_i(n+i+1,n_0)\Delta^{m-i-1}\bigl[p(n_0)x(n_0-\tau)\bigr]. \end{aligned} \tag{33}$$

By Fact 1 and (33), and noting that $x(n)\leq cn^r$ for $n\geq n_0$, we have that

$$\begin{aligned} F_2(n,s) \leq {} & \sum_{s=n_0}^{n+m-1} Q_2(n,s) + c^{\beta}\sum_{s=n-\sigma_2+\tau}^{n+m-1}\phi(n,s+\sigma_2-\tau)q_2(s+\sigma_2-\tau)(s-\tau)^{\beta r} \\ & - \sum_{i=0}^{m-1}\phi_i(n+i+1,n_0)\Delta^{m-i-1}\bigl[p(n_0)x(n_0-\tau)\bigr], \end{aligned} \tag{34}$$

where $Q_2(n,s)$ is defined by (9). Multiplying $\frac{1}{\phi(n+1,n_0)}$ on both sides of (10), from (14), (34) and (5), we can get a contradiction to (7).

(iii) Multiplying $\frac{1}{\phi(n+1,n_0)}$ on both sides of (10), from (32), (34), (29) and (30), we derive a contradiction. The proof of Theorem 3 is complete. □

**Theorem 4** *Assume that $q_1(n) \geq 0$, $q_2(n) < 0$, (18) and (19) hold. All solutions satisfying $x(n) = O(n^r)$ are oscillatory if one of the following conditions holds:*

*(i) $\sigma_1 - \tau \geq -m$, $\sigma_2 > -m$, and*

$$\limsup_{n\to\infty} \frac{1}{\phi(n+1,n_0)} \sum_{s=n-\sigma_2}^{n+m-1} \left[\phi(n,s+\sigma_2)\left|q_2(s+\sigma_2)\right|s^{r\beta}\right] < \infty, \tag{35}$$

*(ii) $\sigma_1 - \tau \leq -m$, $\sigma_2 \leq -m$, and*

$$\limsup_{n\to\infty} \frac{1}{\phi(n+1,n_0)} \sum_{s=n+m}^{n-\sigma_1+\tau-1} \left[\phi_m(n+m,s)p(s)(s-\tau)^r\right] < \infty, \tag{36}$$

*(iii) $\sigma_1 - \tau < -m$, $\sigma_2 > -m$, (35) and (36) hold.*

*Proof* The proof is similar to that of Theorem 2 and Theorem 3, and hence it is omitted. □

## 3 Examples

We here work out two simple examples to illustrate the importance of Theorem 1 and Theorem 2.

**Example 1** Consider the following third-order neutral difference equation:

$$\Delta^3\left[x(n) - x(n-1)\right] - \Phi_{1/2}(n-\sigma_1) + \Phi_2(n-\sigma_2) = n^k \sin n, \quad n \geq 0, \tag{37}$$

where $k > 0$ is a constant. It is evident that $m = 3$, $\sigma_1 = \tau = 1$, $\sigma_2 = -2$, $\alpha = 1/2$, $\beta = 2$, $p(n) \equiv 1$, $q_1(n) \equiv -1$, $q_2(n) \equiv 1$ and $f(n) = n^k \sin n$. If we choose $\phi(n,s) = (n-s)^{(3)}$, we have $\phi_3(n,s) = 6$. By the straightforward computation, we have that

$$Q_1(n,s) = -\left[(n-s-1)^{(2)}\right]^2/24, \qquad Q_2(n,s) = \left[(n-s+3)^{(3)}\right]^{-1}.$$

By Theorem 1, we have that every solution of Eq. (37) is oscillatory if

$$\limsup_{n\to\infty} \frac{1}{(n+1)(n+2)(n+3)} \left[\sum_{s=0}^{n+2}(n-s)^{(3)}s^k \sin s + \sum_{s=0}^{n-2} Q_1(n,s)\right] = +\infty,$$

$$\liminf_{n\to\infty} \frac{1}{(n+1)(n+2)(n+3)} \left[\sum_{s=0}^{n+2}(n-s)^{(3)}s^k \sin s - \sum_{s=n_0}^{n-2} Q_1(n,s)\right] = -\infty.$$

It is not difficult to see that the above two inequalities hold for appropriate $k > 0$.

**Example 2** Consider the following third-order neutral difference equation:

$$\Delta^3\left[x(n) - x(n-1)\right] + \Phi_{1/2}(n-\sigma_1) - \Phi_2(n-\sigma_2) = n^k \cos n, \quad n \geq 0, \tag{38}$$

where $k > 0$ is a constant. It is obvious that $m = 3$, $\sigma_1 = -2$, $\sigma_2 = \tau = 1$, $\alpha = 1/2$, $\beta = 2$, $p(n) \equiv 1$, $q_1(n) \equiv 1$, $q_2(n) \equiv -1$ and $f(n) = n^k \cos n$. We also choose $\phi(n,s) = (n-s)^{(3)}$. By the straightforward computation, we have that

$$P_1(n,s) = \left[(n-s+3)^{(3)}\right]^{-1}, \qquad P_2(n,s) = -\left[(n-s-1)^{(3)}\right]^2/24.$$

By Theorem 2, we have that every solution of Eq. (38) is oscillatory if

$$\limsup_{n\to\infty} \frac{1}{(n+1)(n+2)(n+3)}\left[\sum_{s=0}^{n+2}(n-s)^{(3)}s^k\cos s + \sum_{s=0}^{n-1}P_2(n,s)\right] = +\infty,$$

$$\liminf_{n\to\infty} \frac{1}{(n+1)(n+2)(n+3)}\left[\sum_{s=0}^{n+2}(n-s)^{(3)}s^k\cos s - \sum_{s=0}^{n-1}P_2(n,s)\right] = -\infty.$$

It is not difficult to see that the above two inequalities hold for appropriate $k > 0$.

**Competing interests**

The authors declare that they have no competing interests.

**Authors' contributions**

YS framed the problem. YG solved the problem. BZ and HL made necessary changes in the proof of the theorems. All authors read and approved the manuscript.

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