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Linear and nonlinear abstract elliptic equations with *VMO* coefficients and applications

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Abstract
In this paper, maximal regularity properties for linear and nonlinear elliptic differential-operator equations with *VMO* (vanishing mean oscillation) coefficients are studied. For linear case, the uniform separability properties for parameter dependent boundary value problems is obtained in L^p spaces. Then the existence and uniqueness of a strong solution of the boundary value problem for a nonlinear equation is established. In application, the maximal regularity properties of the anisotropic elliptic equation and the system of equations with *VMO* coefficients are derived.
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1 Introduction

The goal of the present paper is to study the nonlocal boundary value problems (BVPs) for parameter dependent linear differential-operator equations (DOEs) with discontinuous top-order coefficients,

$$sa(x)u^{(2)}(x) + A(x)u(x) + s^{\frac{1}{2}}A_1(x)u^{(1)}(x) + A_0(x)u(x) + \lambda u(x) = f(x), \tag{1}$$

and the nonlinear equation

$$a(x)u^{(2)}(x) + B\left(x, u, u^{(1)}\right)u(x) = F\left(x, u, u^{(1)}\right),$$

where a is a complex valued function, s is a positive, λ is a complex parameter, and $A = A(x)$, $A_i = A_i(x)$ are linear and B is a nonlinear operator in a Banach space E. Here, the principal coefficients a and A may be discontinuous. More precisely, we assume that a and $A(\cdot)A^{-1}(x_0)$ belong to the operator-valued Sarason class *VMO*. The Sarason class *VMO* was first defined in [1]. In the recent years, there has been considerable interest in elliptic and parabolic equations with *VMO* coefficients. This is mainly due to the fact that the *VMO* class contains as a subspace $C(\bar{\Omega})$ that ensures the extension of L_p-theory of operators with continuous coefficients to discontinuous coefficients (see, *e.g.*, [2–7]). On the other hand, the Sobolev spaces $W^{1,n}(\Omega)$ and $W^{\sigma,\frac{\sigma}{n}}(\Omega)$, $0 < \sigma < 1$, are also contained in *VMO*. Global regularity of the Dirichlet problem for elliptic equations with *VMO* coeffi-

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cients has been studied in [2–4]. We refer to the survey [4], where an excellent presentation and relations with another similar results can be found concerning the regularizing properties of these operators in the framework of Sobolev spaces.

It is known that many classes of PDEs (partial differential equations), pseudo DEs and integro DEs can be expressed in the form of DOEs. Many researchers (see, *e.g.*, [8–24]) investigated similar spaces of functions and classes of PDEs under a single DOE. Moreover, the maximal regularity properties of DOEs with continuous coefficients were studied, *e.g.*, in [10, 19, 20].

Here, the equation with top-order *VMO*-operator coefficients is considered in abstract spaces. We shall prove the uniform separability of the problem (1), *i.e.*, we show that, for each $f \in L^p(0,1;E)$, there exists a unique strong solution u of the problem (1) and a positive constant C depending only on p, γ, E and A such that

$$\sum_{i=0}^{2} s^{\frac{i}{2}} |\lambda|^{1-\frac{i}{2}} \|u^{(i)}\|_{L^p(0,1;E)} + \|Au\|_{L^p(0,1;E)} \leq C\|f\|_{L^p(0,1;E)}.$$

Note that the principal part of a corresponding differential operator is nonselfadjoint. Nevertheless, the sharp uniform coercive estimate for the resolvent and Fredholmness are established. Then the existence and uniqueness of the above nonlinear problem is derived. In application, we study maximal regularity properties of anisotropic elliptic equations in mixed $L^{\mathbf{p}}$ spaces and the systems (finite or infinite) of differential equations with *VMO* coefficients in the scalar L^p space.

Since (1) involves unbounded operators, it is not easy to get representation for the Green function and estimate of solutions. Therefore, we use the modern harmonic analysis elements, *e.g.*, the Hilbert operators and the commutator estimates in E-valued L^p spaces, embedding theorems of Sobolev-Lions spaces and some semigroups estimates to overcome these difficulties. Moreover, we also use our previous results on equations with continuous leading coefficients and the perturbation theory of linear operators to obtain our main assertions.

2 Notations and background

Throughout the paper, we set E to be a Banach space and $\Omega \subset R^n$. $L^p(\Omega;E)$ denotes the space of all strongly measurable E-valued functions that are defined on Ω with the norm

$$\|f\|_p = \|f\|_{L^p(\Omega;E)} = \left(\int_\Omega \|f(x)\|_E^p \, dx\right)^{\frac{1}{p}}, \quad 1 \leq p < \infty.$$

$BMO(E)$ (bounded mean oscillation) (see [21, 25]) is the space of all E-valued local integrable functions with the norm

$$\sup_B \int_B \|f(x) - f_B\|_E \, dx = \|f\|_{*,E} < \infty,$$

where B ranges in the class of the balls in R^n and f_B is the average $\frac{1}{|B|}\int_B f(x)\,dx$.

For $f \in BMO(E)$ and $r > 0$, we set

$$\sup_{\rho \leq r} \int_B \|f(x) - f_B\|_E \, dx = \eta(r),$$

where B ranges in the class of balls with radius ρ.

We will say that a function $f \in BMO(E)$ is in $VMO(E)$ if $\lim_{r\to +0} \eta(r) = 0$. We will call the $\eta(r)$ a VMO modulus of f.

Note that, if $E = \mathbf{C}$, where $\mathbf{C}$ is the set of complex numbers, then $BMO(E)$ and $VMO(E)$ coincide with the John-Nirenberg class BMO and Sarason class VMO, respectively.

The Banach space E is called a *UMD* (unconditional martingale difference)-space if the Hilbert operator

$$(Hf)(x) = \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y}\, dy$$

is bounded in $L_p(R,E)$, $p \in (1,\infty)$ (see, *e.g.*, [26, 27]). *UMD* spaces include, *e.g.*, L_p, l_p spaces and Lorentz spaces L_{pq}, $p,q \in (1,\infty)$.

Let

$$S_\varphi = \{\lambda \in \mathbf{C}, |\arg\lambda| \le \varphi\} \cup \{0\}, \quad 0 \le \varphi < \pi.$$

A linear operator A is said to be a φ-positive (or positive) in a Banach space E with the bound $M > 0$ if $D(A)$ is dense on E and

$$\|(A+\lambda I)^{-1}\|_{L(E)} \le M(1+|\lambda|)^{-1}$$

for $\lambda \in S_\varphi$, $\varphi \in (0,\pi]$, I is an identity operator in E and $L(E)$ is the space of bounded linear operators in E. Sometimes instead of $A + \lambda I$, it will be written as $A + \lambda$ and denoted by A_λ. It is known [22, §1.15.1] that there exist fractional powers A^θ of the positive operator A. Let $E(A^\theta)$ denote the space $D(A^\theta)$ with the graphical norm

$$\|u\|_{E(A^\theta)} = \left(\|u\|^p + \|A^\theta u\|^p\right)^{\frac{1}{p}}, \quad 1 \le p < \infty, -\infty < \theta < \infty.$$

Let E_1 and E_2 be two Banach spaces. A set $W \subset L(E_1,E_2)$ is called R-bounded (see [24, 26]) if there is a positive constant C such that for all $T_1, T_2, \ldots, T_m \in W$ and $u_1, u_2, \ldots, u_m \in E_1$, $m \in N$

$$\int_0^1 \left\| \sum_{j=1}^m r_j(y) T_j u_j \right\|_{E_2} dy \le C \int_0^1 \left\| \sum_{j=1}^m r_j(y) u_j \right\|_{E_1} dy,$$

where $\{r_j\}$ is a sequence of independent symmetric $\{-1,1\}$-valued random variables on $[0,1]$.

Let $S(R^n;E)$ denote the Schwarz class, *i.e.*, the space of all E-valued rapidly decreasing smooth functions on R^n. Let F denote the Fourier transformation. A function $\Psi \in L^\infty(R^n;B(E_1,E_2))$ is called a Fourier multiplier from $L_p(R^n;E_1)$ to $L_p(R^n;E_2)$ if the map $u \to \Lambda_\Psi u = F^{-1}\Psi(\xi)Fu$, $u \in S(R^n;E_1)$ is well defined and extends to a bounded linear operator

$$\Lambda_\Psi : L_p(R^n;E_1) \to L_p(R^n;E_2).$$

The set of all multipliers from $L_p(R^n;E_1)$ to $L_p(R^n;E_2)$ will be denoted by $M_p^p(E_1,E_2)$. For $E_1 = E_2 = E$, it will be denoted by $M_p^p(E)$.

Let

$$U_n = \{\beta = (\beta_1, \beta_2, \dots, \beta_n) \in N^n : \beta_k \in \{0,1\}\}.$$

Definition 1 A Banach space E is said to be a space satisfying the multiplier condition if, for any $\Psi \in C^{(n)}(R^n; L(E))$, the R-boundedness of the set $\{\xi^\beta D_\xi^\beta \Psi(\xi) : \xi \in R^n \backslash 0, \beta \in U_n\}$ implies that Ψ is a Fourier multiplier in $L_p(R^n; E)$, *i.e.*, $\Psi \in M_p^p(E)$ for any $p \in (1, \infty)$.

Definition 2 The φ-positive operator A is said to be an R-positive in a Banach space E if there exists $\varphi \in [0, \pi)$ such that the set $L_A = \{A(A + \lambda)^{-1} : \lambda \in S_\varphi\}$ is R-bounded.

A linear operator $A(x)$ is said to be positive in E uniformly in x if $D(A(x))$ is independent of x, $D(A(x))$ is dense in E and

$$\|(A(x) + \lambda)^{-1}\| \le M(1 + |\lambda|)^{-1}$$

for all $\lambda \in S(\varphi)$, $\varphi \in [0, \pi)$.

Let $\sigma_\infty(E_1, E_2)$ denote the space of all compact operators from E_1 to E_2. For $E_1 = E_2 = E$, it is denoted by $\sigma_\infty(E)$.

Assume E_0 and E are two Banach spaces and E_0 is continuously and densely embedded into E. Let m be a natural number. $W^{m,p}(\Omega; E_0, E)$ (the so-called Sobolev-Lions type space) denotes a space of all functions $u \in L^p(\Omega; E_0)$ possessing the generalized derivatives $D_k^m u = \frac{\partial^m u}{\partial x_k^m}$ such that $D_k^m u \in L^p(\Omega; E)$ endowed with the norm

$$\|u\|_{W^{m,p}(\Omega;E_0,E)} = \|u\|_{L^p(\Omega;E_0)} + \sum_{k=1}^n \|D_k^m u\|_{L^p(\Omega;E)} < \infty.$$

For $E_0 = E$, the space $W^{m,p}(\Omega; E_0, E)$ will be denoted by $W^{m,p}(\Omega; E)$. It is clear that

$$W^{m,p}(\Omega; E_0, E) = W^{m,p}(\Omega; E) \cap L^p(\Omega; E_0).$$

Let s be a positive parameter. We define in $W^{m,p}(\Omega; E_0, E)$ the following parameter-dependent norm:

$$\|u\|_{W_s^{m,p}(\Omega;E_0,E)} = \|u\|_{L^p(\Omega;E_0)} + \sum_{k=1}^n \|s D_k^m u\|_{L^p(\Omega;E)}.$$

The space $B_{p,p}^s(\Gamma; E_0, E)$ is defined as a trace space for $W^{m,p}(\Omega; E_0, E)$ as in a scalar case (see, *e.g.*, [22, §3.6.1]).

Function $u \in W^{2,p}(0,1; E(A), E, L_k) = \{u \in W^{2,p}(0,1; E(A), E), L_k u = 0\}$ satisfying the equation (1) a.e. on $(0,1)$ is said to be a solution of the problem (1) on $(0,1)$.

From [28] we have

Theorem A$_1$ *Suppose the following conditions are satisfied*:

(1) *E is a Banach space satisfying the multiplier condition with respect to $p \in (1, \infty)$ and A is an R-positive operator in E*;

(2) $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ *are n tuples of nonnegative integer numbers such that* $\varkappa = \frac{|\alpha|}{m} \leq 1$ *and* $0 < \mu \leq 1 - \varkappa$;

(3) $\Omega \in R^n$ *is a region such that there exists a bounded linear extension operator from* $W^{m,p}(\Omega; E(A), E)$ *to* $W^{m,p}(R^n; E(A), E)$.

Then the embedding

$$D^\alpha W^{m,p}\big(\Omega; E(A), E\big) \subset L^p\big(\Omega; E\big(A^{1-\varkappa-\mu}\big)\big)$$

is continuous and there exists a positive constant C_μ *such that*

$$s^{\frac{|\alpha|}{m}} \big\| D^\alpha u \big\|_{L^p(\Omega; E(A^{1-\varkappa-\mu}))} \leq C_\mu \big[h^\mu \|u\|_{W_s^{m,p}(\Omega; E(A), E)} + h^{-(1-\mu)} \|u\|_{L^p(\Omega; E)} \big]$$

for all $u \in W^{m,p}(\Omega; E(A), E)$ *and* $0 < h \leq h_0 < \infty$.

Theorem $\mathbf{A}_2$ *Suppose all the conditions of Theorem* A_1 *are satisfied. Assume* Ω *is a bounded region in* R^n *and* $A^{-1} \in \sigma_\infty(E)$. *Then, for* $0 < \mu \leq 1 - \varkappa$, *the embedding*

$$D^\alpha W^{m,p}\big(\Omega; E(A), E\big) \subset L^p\big(\Omega; E\big(A^{1-\varkappa-\mu}\big)\big)$$

is compact.

In a similar way as in [3, Theorem 2.1], we have the following result.

Lemma $\mathbf{A}_1$ *Let E be a Banach space and* $f \in VMO(E)$. *The following conditions are equivalent*:

(1) $f \in VMO(E)$;

(2) f *is in the BMO closure of the set of uniformly continuous functions which belong to VMO*;

(3) $\lim_{y \to 0} \|f(x-y) - f(x)\|_{*,E} = 0$.

For $f \in L^p(\Omega; E)$, $p \in (1, \infty)$, $a \in L^\infty(R^n)$, *consider the commutator operator*

$$H[a,f](x) = a(x)Hf(x) - H(af)(x) = \lim_{\varepsilon \to 0} \int_{|x-y| > \varepsilon,} \frac{[a(x) - a(y)]}{x - y} f(y)\, dy.$$

From [21, Theorem 1] and [29, Corollary 2.7], we have

Theorem $\mathbf{A}_3$ *Let E be a UMD space and* $a \in VMO \cap L^\infty(R^n)$. *Then* $H[a,f]$ *is a bounded operator in* $L^p(R; E)$, $p \in (1, \infty)$.

From Theorem A_3 and the property (2) of Lemma A_1 we obtain, respectively, the following.

Theorem $\mathbf{A}_4$ *Assume all the conditions of Theorem* A_3 *are satisfied. Also, let* $a \in VMO \cap L^\infty(R^n)$ *and let* η *be the VMO modulus of a. Then, for any* $\varepsilon > 0$, *there exists a positive number* $\delta = \delta(\varepsilon, \eta)$ *such that*

$$\big\| H[a,f] \big\|_{L^p(0,r;E)} \leq M\varepsilon \|f\|_{L^p(0,r;E)}, \quad r \in (0, \delta).$$

Theorem A$_5$ *Let E be a UMD space and $A(\cdot)$ uniformly R-positive in E. Moreover, let $A(\cdot)A^{-1}(x_0) \in L_\infty(R;L(E)) \cap BMO(L(E))$, $x_0 \in R$. Then the commutator operator*

$$H[A,f](x) = A(x)A^{-1}(x_0)Hf(x) - H\big(A(x)A^{-1}(x_0)f\big)(x) = \lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon,} \frac{[A(x)A^{-1}(x_0) - A(y)A^{-1}(x_0)]}{x-y} f(y)\,dy$$

is bounded in $L^p(R;E)$, $p \in (1,\infty)$.

Theorem A$_6$ *Assume all the conditions of Theorem* A$_5$ *are satisfied and η is a VMO modulus of $A(\cdot)A^{-1}(x_0)$.*

Then, for any $\varepsilon > 0$, there exists a positive number $\delta = \delta(\varepsilon,\eta)$ such that

$$\big\|H[A,f]\big\|_{L^p(\Omega_r;E)} \le M\varepsilon\|f\|_{L^p(\Omega_r;E)}, \quad r \in (0,\delta).$$

Consider the nonlocal BVP for a parameter dependent DOE with constant coefficients

$$(L+\lambda)u = sau^{(2)}(x) + (A+\lambda)u(x) = f(x), \quad x \in (0,1)$$
$$L_k u = \sum_{i=0}^{m_k} s^{\mu_i}\big[\alpha_{ki}u^{(i)}(0) + \beta_{ki}u^{(i)}(1)\big] = f_k, \quad k = 1,2, \tag{2}$$

where $m_k \in \{0,1\}$, a, α_{ki}, β_{ki} are complex numbers, s is a positive parameter, λ is a complex spectral parameter, $\mu_i = \frac{i}{2} + \frac{1}{2p}$, $\theta_k = \frac{m_k}{2} + \frac{1}{2p}$, $A_\lambda = A + \lambda$ and A is a linear operator in E. Let ω_1, ω_2 be roots of the equation $a\omega^2 + 1 = 0$ and let

$$\alpha_k = \alpha_{km_k}, \qquad \beta_k = \beta_{km_k}, \qquad \mu = \begin{vmatrix} (-\omega_1)^{m_1}\alpha_1 & \beta_1\omega_1^{m_1} \\ (-\omega_2)^{m_2}\alpha_2 & \beta_2\omega_2^{m_2} \end{vmatrix}.$$

It is known that if the operator A is φ-positive in E, then operators $\omega_k A_\lambda^{\frac{1}{2}}$, $k = 1,2$ generate analytic semigroups

$$U_{1\lambda s}(x) = e^{\omega_1 s^{-\frac{1}{2}} A_\lambda^{\frac{1}{2}} x}, \qquad U_{2\lambda s}(x) = e^{-\omega_2 s^{-\frac{1}{2}} A_\lambda^{\frac{1}{2}}(1-x)} \quad \textit{for } \lambda \in S(\varphi).$$

Let

$$E_k = \big(E(A),E\big)_{\theta_k,p}.$$

From [19, Theorem 2] and [30, Theorem 3.1], we obtain

Theorem A$_7$ *Assume the following conditions are satisfied*:

(1) *E is a Banach space satisfying the multiplier condition with respect to $p \in (1,\infty)$;*
(2) *A is an R-positive operator in E for $0 \le \varphi < \pi$ and $\mu \ne 0$;*
(3) $\operatorname{Re}\omega_k \ne 0$ *and $\frac{\lambda}{\omega_k} \in S(\varphi)$ for $\lambda \in S(\varphi)$, $k = 1,2$.*

Then

(1) *for* $f \in L_p(0,1;E)$, $f_k \in E_k$, $\lambda \in S(\varphi)$ *and for sufficiently large* $|\lambda|$, *the problem* (2) *has a unique solution* $u \in W^{2,p}(0,1;E(A),E)$. *Moreover, the coercive uniform estimate holds*

$$\sum_{i=0}^{2} s^{\frac{i}{2}} |\lambda|^{1-\frac{i}{2}} \| u^{(i)} \|_{L^p(0,1;E)} + \|Au\|_{L^p(0,1;E)} \le C \left[\|f\|_{L^p(0,1;E)} + \sum_{k=1}^{2} \|f_k\|_{E_k} \right].$$

(2) *For* $f_k = 0$, *the solution is represented as*

$$u(x) = \int_0^1 G_{\lambda s}(x,y) f(y)\, dy, \tag{3}$$

$$G_{\lambda s}(x,y) = \sum_{i,j=1}^{2} \sum_{k=0}^{m_1} \left[B_{ij}(\lambda) \left(s^{-1} A_\lambda \right)^{-\frac{1}{2}(2+m_i-k-1)} U_{j\lambda s}(x) U_{1\lambda s}(1-y) \right] + \sum_{i,j=1}^{2} \sum_{k=0}^{m_2} \left[D_{ij}(\lambda) A_\lambda^{-\frac{1}{2}(2+m_i-k-1)} U_{j\lambda s}(x) U_{2\lambda s}(y) \right] + U_{0\lambda s}(x-y),$$

where $B_{ij}(\lambda)$ *and* $D_{ij}(\lambda)$ *are uniformly bounded operators in* E *and*

$$U_{0\lambda s}(x-y) = \begin{cases} a^{-1} s^{\frac{1}{2}} A_\lambda^{-\frac{1}{2}} U_{1\lambda s}(x-y), & x \ge y, \\ -a^{-1} s^{\frac{1}{2}} A_\lambda^{-\frac{1}{2}} U_{2\lambda s}(x-y), & x \le y. \end{cases}$$

Consider the BVP for a DOE with variable coefficients

$$sa(x)u^{(2)}(x) + \big(A(x) + \lambda\big)u(x) = f(x), \quad x \in (0,1),$$
$$L_k u = \sum_{i=0}^{m_k} s^{\theta_k} \left[\alpha_{ki} u^{(i)}(0) + \beta_{ki} u^{(i)}(1) \right] = 0, \quad k = 1,2, \tag{4}$$

where $a = a(x)$ is a complex valued function, s is a positive parameter, $m_k \in \{0,1\}$, α_{ki}, β_{ki} are complex numbers, λ is a spectral parameter, $\theta_k = \frac{m_k}{2} + \frac{1}{2p}$ and $A(x)$ is a linear operator in E.

Let $\omega_1 = \omega_1(x)$, $\omega_2 = \omega_2(x)$ be roots of the equation $a(x)\omega^2 + 1 = 0$ and let

$$\alpha_k = \alpha_{km_k}, \qquad \beta_k = \beta_{km_k}, \qquad \mu(x) = \begin{vmatrix} (-\omega_1)^{m_1}\alpha_1 & \beta_1 \omega_1^{m_1} \\ (-\omega_2)^{m_2}\alpha_2 & \beta_2 \omega_2^{m_2} \end{vmatrix}.$$

In the next theorem, let us consider the case that principal coefficients are continuous. The well-posedness of this problem occurs in the studying of equations with *VMO* coefficients. From [19, Theorem 3] and [21, Theorem 5.1], we get

Theorem A_8 *Suppose the following conditions are satisfied*:

(1) *E is a Banach space satisfying the multiplier condition with respect to $p \in (1,\infty)$*;
(2) $a \in C[0,1]$, $-a \in S(\varphi)$, $a(0) = a(1)$, *and* $\mu(x) \ne 0$ *for a.e.* $x \in [0,1]$;
(3) $\operatorname{Re} \omega_k(x) \ne 0$ *and* $\frac{\lambda}{\omega_k} \in S(\varphi)$ *for* $\lambda \in S(\varphi)$, $k = 1,2$. *a.e.* $x \in [0,1]$;
(4) *$A(x)$ is a uniformly R-positive operator in E and*

$$A(\cdot)A^{-1}(x_0) \in C\big([0,1];L(E)\big), \quad x_0 \in (0,1), A(0) = A(1).$$

Then, for all $f \in L^p(0,1;E)$, $\lambda \in S(\varphi)$ and for sufficiently large $|\lambda|$, there is a unique solution $u \in W^{2,p}(0,1;E(A),E)$ of the problem (4). *Moreover, the following coercive uniform estimate holds*:

$$\sum_{i=0}^{2} s^{\frac{i}{2}}|\lambda|^{1-\frac{i}{2}}\left\|u^{(i)}\right\|_{L^p(0,1;E)} + \|Au\|_{L^p(0,1;E)} \le C\|f\|_{L^p(0,1;E)}.$$

3 DOEs with VMO coefficients

Consider the principal part of the problem (1)

$$\begin{aligned}
&(L+\lambda)u = sa(x)u^{(2)}(x) + \big(A(x)+\lambda\big)u(x) = f(x), \quad x \in (0,1),\\
&L_k u = \sum_{i=0}^{m_k} s^{\mu_i}\big[\alpha_{ki}u^{(i)}(0) + \beta_{ki}u^{(i)}(1)\big] = 0, \quad k = 1,2.
\end{aligned} \tag{5}$$

Condition 1 Assume the following conditions are satisfied:

(1) E is a *UMD* space, $p \in (1,\infty)$;

(2) $a \in VMO \cap L^{\infty}(R)$, η_1 is a *VMO* modulus of a, $-a \in S(\varphi)$, $\mu(x) \neq 0$;

(3) $\operatorname{Re}\omega_k(x) \neq 0$ and $\frac{\lambda}{\omega_k} \in S(\varphi)$ for $\lambda \in S(\varphi)$, $0 \le \varphi < \pi$, $k = 1,2$. a.e. $x \in [0,1]$;

(4) $A(x)$ is a uniformly R-positive operator in E and

$$A(\cdot)A^{-1}(x_0) \in L_{\infty}\big(0,1;L(E)\big) \cap VMO\big(L(E)\big), \quad x_0 \in (0,1);$$

(5) $a(0) = a(1)$, $A(0) = A(1)$ and η_2 is a *VMO* modulus of $A(\cdot)A^{-1}(x_0)$.

First, we obtain an integral representation formula for solutions.

Lemma 1 *Let Condition* 1 *hold and $f \in L^p(0,1;E)$. Then, for all solutions u of the problem* (5) *belonging to $W^{2,p}(0,1;E(A),E)$, we have*

$$\begin{aligned}
u^{(i)}(x) &= \int_0^1 \Gamma_{i\lambda s}(x,x-y)\big\{\big[a(x)-a(y)\big]u^{(2)}(y)\\
&\quad + \big[A(x)-A(y)\big]u(y) + f(y)\big\}\,dy + f(x),\\
A(x)u(x) &= \int_0^1 \Gamma'_{2\lambda s}(x,x-y)\big\{\big[a(x_0)-a(y)\big]u^{(2)}(y)\\
&\quad + \big[A(x)-A(y)\big]u(y) + f(y)\big\}\,dy + f(x),
\end{aligned} \tag{6}$$

where

$$\begin{aligned}
\Gamma'_{i\lambda s}(x,x-y) &= \sum_{i,j=1}^{2}\sum_{k=0}^{m_1}\big[B'_{ij}(\lambda)\big(s^{-1}A_\lambda\big)^{-\frac{1}{2}(2+m_k-k-i-1)}U_{j\lambda s}(x)U_{1\lambda s}(1-y)\big]\\
&\quad + \sum_{i,j=1}^{2}\sum_{k=0}^{m_2}\big[D'_{ij}(\lambda)\big(s^{-1}A_\lambda\big)^{-\frac{1}{2}(2+m_k-k-i-1)}U_{j\lambda s}(x)U_{2\lambda s}(y)\big]\\
&\quad + U'_{\nu 0\lambda s}(x-y), \quad \nu = 0,1,2,
\end{aligned}$$

here $B'_{ij}(\lambda)$, $D'_{ij}(\lambda)$ are uniformly bounded operators,

$$U'_{\nu 0\lambda s}(x-y) = \begin{cases} \omega_1^i a^{-1} s^{(1-\nu)/2} A_\lambda^{-(1-\nu)/2} U_{1\lambda s}(x-y), & x \geq y, \\ -\omega_2^i a^{-1} s^{(1-\nu)/2} A_\lambda^{-(1-\nu)/2} U_{2\lambda s}(x-y), & x \leq y, \end{cases}$$

and the expression $\Gamma'_{2\lambda}(x, x-y)$ is a scalar multiple of $\Gamma_{2\lambda}(x, x-y)$.

Proof Consider the problem (5) for $a(x) = a(x_0)$ and $A(x) = A(x_0)$, *i.e.*,

$$(L_0 + \lambda)u = sa(x_0)u^{(2)}(x) + \big(A(x_0) + \lambda\big)u(x) = f(x), \quad x \in (0,1),$$
$$L_k u = \sum_{i=0}^{m_k} s^{\mu_i}\big[\alpha_{ki} u^{(i)}(0) + \beta_{ki} u^{(i)}(1)\big] = 0, \quad k = 1,2. \tag{7}$$

Let $u \in C^{(2)}([0,1]; E(A))$ be a solution of the problem (7). Taking into account the equality $L_0 u = (L_0 - L)u + Lu$ and Theorem A_7, we get

$$u^{(i)}(x) = \int_0^1 \Gamma_{i\lambda s}(x, x-y)\big\{\big[a(x_0) - a(y)\big]u^{(2)}(y) + \big[A(x_0) - A(y)\big]u(y) + f(y)\big\}\,dy + f(x),$$
$$A(x_0)u(x) = \int_0^1 \Gamma'_{2\lambda s}(x, x-y)\big\{\big[a(x_0) - a(y)\big]u^{(2)}(y) + \big[A(x_0) - A(y)\big]u(y) + f(y)\big\}\,dy + f(x).$$

Setting $x = x_0$ in above, we get (6) for $u \in C^{(2)}([0,1]; E(A))$. Then the density argument, using Theorem A_3, gives the conclusion for

$$u \in W^{2,p}\big(0,1; E(A), E\big), \quad L_k u = 0.$$

Consider the problem (5) on $(0,b)$, *i.e.*,

$$(L_b + \lambda)u = sa(x)u^{(2)}(x) + \big(A(x) + \lambda\big)u(x) = f(x), \quad x \in (0,b),$$
$$L_{bk} u = \sum_{i=0}^{m_k} s^{\mu_i}\big[\alpha_{ki} u^{(i)}(0) + \beta_{ki} u^{(i)}(b)\big] = 0, \quad k = 1,2. \tag{8}$$

□

Theorem 1 *Suppose Condition* 1 *is satisfied. Then there exists a number $b \in (0,1)$ such that the uniform coercive estimate*

$$\sum_{i=0}^{2} s^{\frac{i}{2}} |\lambda|^{1-\frac{i}{2}} \big\|u^{(i)}\big\|_{L^p(0,b;E)} + \|Au\|_{L^p(0,b;E)} \leq C\big\|(L_b + \lambda)u\big\|_{L^p(0,b;E)} \tag{9}$$

holds for large enough $|\lambda|$ and $u \in W^{2,p}(0,b; E(A), E)$, $L_{bk}u = 0$, $\lambda \in S(\varphi)$, where C is a positive constant depending only on p, M_0, η_1, η_2.

Proof By Lemma 1, for any solution $u \in W^{2,p}(0,b;E(A),E)$ of the problem (8), we have

$$u^{(i)}(x) = \int_0^b \Gamma_{ib\lambda s}(x,x-y)\bigl\{\bigl[a(x)-a(y)\bigr]u^{(2)}(y) + \bigl[A(x)-A(y)\bigr]u(y) + f(y)\bigr\}\,dy + f(x), \tag{10}$$

where

$$\begin{aligned}\Gamma_{ib\lambda s}(x,x-y) &= \sum_{i,j=1}^{2}\sum_{k=0}^{m_1}\bigl[B'_{ij}(\lambda)\bigl(s^{-1}A_\lambda\bigr)^{-\frac{1}{2}(2+m_k-k-i-1)}U_{j\lambda}(x)U_{1\lambda}(b-y)\bigr] \\ &\quad + \sum_{i,j=1}^{2}\sum_{k=0}^{m_2}\bigl[D'_{ij}(\lambda)\bigl(s^{-1}A_\lambda\bigr)^{-\frac{1}{2}(2+m_k-k-i-1)}U_{j\lambda}(x)U_{2\lambda}(y)\bigr] \\ &\quad + U'_{\nu 0\lambda s}(x-y), \quad \nu = 0,1,2,\end{aligned} \tag{11}$$

here $B'_{ij}(\lambda)$, $D'_{ij}(\lambda)$ are uniformly bounded operators,

$$U_{1\lambda}(x) = e^{\omega_1 s^{-\frac{1}{2}}A_\lambda^{\frac{1}{2}}x}, \qquad U_{2\lambda}(x) = e^{-\omega_2 s^{-\frac{1}{2}}A_\lambda^{\frac{1}{2}}(b-x)}$$

and

$$U'_{\nu 0\lambda s}(x-y) = \begin{cases} \omega_1^i a^{-1}s^{(1-\nu)/2}A_\lambda^{-(1-\nu)/2}U_{1\lambda s}(x-y), & x \geq y, \\ -\omega_2^i a^{-1}s^{(1-\nu)/2}A_\lambda^{-(1-\nu)/2}U_{2\lambda s}(x-y), & x \leq y,\end{cases} \quad \nu = 0,1,2.$$

Moreover, from (10) and (11), clearly, we get

$$Au(x) = \int_0^b \Gamma'_{b\lambda s}(x,x-y)\bigl\{\bigl[a(x)-a(y)\bigr]u^{(2)}(y) + \bigl[A(x)-A(y)\bigr]u(y) + f(y)\bigr\}\,dy, \tag{12}$$

where the expression $\Gamma'_{b\lambda}(x,x-y)$ differs from $\Gamma_{2b\lambda}(x,x-y)$ only by a constant.

Consider the operators

$$B_{0\lambda}f = \int_0^1 G_{b\lambda s}(x,y)f(y)\,dy, \qquad B_{i\lambda s}f = \int_0^b \Gamma_{ib\lambda s}(x,x-y)f(y)\,dy,$$

$$S_{i\lambda}u = \int_0^b \Gamma_{ib\lambda}(x,x-y)\bigl[a(x)-a(y)\bigr]u^{(2)}(y)\,dy,$$

$$D_{i\lambda s}u = \int_0^b \Gamma_{ib\lambda s}(x,x-y)\bigl[A(x)-A(y)\bigr]u(y)\,dy, \quad i = 0,1,2,$$

$$\Phi_{1\lambda s}u = \int_0^b \Gamma'_{b\lambda s}(x,x-y)\bigl[a(x)-a(y)\bigr]u^{(2)}(y)\,dy,$$

$$\Phi_{2\lambda}u = \int_0^b \Gamma'_{b\lambda}(x,x-y)\bigl[A(x)-A(y)\bigr]u(y)\,dy.$$

Since the operators $B_{0\lambda}$ and $B_{1\lambda}$ are regular on $L^p(0,b;E)$, by using the positivity properties of A and the analyticity of semigroups $U_{k\lambda}(x)$ in a similar way as in [30, Theorem 3.1], we get

$$\|B_{i\lambda}f\|_{L^p(0,b;E)} \le M|\lambda|^{-\frac{2-i}{2}}\|f\|_{L^p(0,b;E)}, \quad i = 0,1. \tag{13}$$

Since the Hilbert operator is bounded in $L^p(R;E)$ for a UMD space E, we have

$$\|B_{2\lambda}f\|_{L^p(0,b;E)} \le M\|f\|_{L^p(0,b;E)}. \tag{14}$$

Thus, by virtue of Theorems A_4, A_6 and in view of (10)-(12) for any $\varepsilon > 0$, there exists a positive number $b = b(\varepsilon, \eta_1, \eta_2)$ such that

$$\begin{aligned} &\|S_{i\lambda}u\|_{L^p(0,b;E)} \le M\varepsilon|\lambda|^{-\frac{2-i}{2}}\big\|u^{(2)}\big\|_{L^p(0,b;E)}, \\ &\|D_{i\lambda}u\|_{L^p(0,b;E)} \le M\varepsilon|\lambda|^{-\frac{2-i}{2}}\big\|A(x_0)u\big\|_{L^p(0,b;E)}, \quad i = 0,1,2, \\ &\|\Phi_{1\lambda}u\|_{L^p(0,b;E)} \le M\varepsilon\big\|u^{(2)}\big\|_{L^p(0,b;E)}, \qquad \|\Phi_{2\lambda}u\|_{L^p(0,b;E)} \le M\varepsilon\big\|A(x_0)u\big\|_{L^p(0,b;E)}. \end{aligned} \tag{15}$$

Hence, the estimates (13)-(15) imply (9). □

Theorem 2 *Assume Condition* 1 *holds*. *Let* $u \in W^{2,p}(0,1;E(A),E)$ *be a solution of* (4). *Then*, *for sufficiently large* $|\lambda|$, $\lambda \in S(\varphi)$, *the following coercive uniform estimate holds*:

$$\sum_{i=0}^{2} s^{\frac{i}{2}}|\lambda|^{1-\frac{i}{2}}\big\|u^{(i)}\big\|_{L^p(0,1;E)} + \|Au\|_{L^p(0,1;E)} \le C\big[\big\|(L+\lambda)u\big\|_{L^p(0,1;E)} + \|u\|_{L^p(0,1;E)}\big]. \tag{16}$$

Proof This fact is shown by a covering and flattening argument, in a similar way as in Theorem A_8. Particularly, by partition of unity, the problem is localized. Choosing diameters of supports for corresponding finite functions, by using Theorem 1, Theorems A_4, A_6, A_7 and embedding Theorem A_1 (see the same technique for DOEs with continuous coefficients [19, 20]), we obtain the assertion.

Let Q_s denote the operator in $L^p(0,1;E)$ generated by the problem (4) for $\lambda = 0$, *i.e.*,

$$D(Q_s) = W^{2,p}\big(0,1;E(A),E,L_k\big), \qquad Q_s u = sa(x)u^{(2)} + A(x)u.$$ □

Theorem 3 *Assume Condition* 1 *holds*. *Then*, *for all* $f \in L^p(0,1;E)$, $\lambda \in S(\varphi)$ *and for large enough* $|\lambda|$, *the problem* (5) *has a unique solution* $u \in W^{2,p}(0,1;E(A),E)$. *Moreover*, *the following coercive uniform estimate holds*:

$$\sum_{i=0}^{2} s^{\frac{i}{2}}|\lambda|^{1-\frac{i}{2}}\big\|u^{(i)}\big\|_{L^p(0,1;E)} + \|Au\|_{L^p(0,1;E)} \le C\|f\|_{L^p(0,1;E)}. \tag{17}$$

Proof First, let us show that the operator $Q + \lambda$ has a left inverse. Really, it is clear that

$$\|u\|_{L^p(0,1;E)} = \frac{1}{|\lambda|}\big[\big\|(Q_s+\lambda)u\big\|_{L^p(0,1;E)} + \|Q_s u\|_{L^p(0,1;E)}\big].$$

By Theorem A_1 for $u \in W^{2,p}(0,1;E(A),E)$, we have

$$\|Q_s u\|_{L^p(0,1;E)} \le C\|u\|_{W_s^{2,p}(0,1;E(A),E)}.$$

Then, by virtue of (16) and in view of the above relations, we infer, for all $u \in W^{2,p}(0,1;E(A),E)$ and sufficiently large $|\lambda|$, that there is a small ε and $C(\varepsilon)$ such that

$$\sum_{i=0}^{2} s^{\frac{i}{2}} |\lambda|^{1-\frac{i}{2}} \left\| u^{(i)} \right\|_{L^p(0,1;E)} + \|Au\|_{L^p(0,1;E)} \\ \leq C\left[\left\| (Q_s + \lambda)u \right\|_{L^p(0,1;E)} + \varepsilon \|u\|_{W^{2,p}(0,1;E(A),E)} + C(\varepsilon) \left\| (Q_s + \lambda)u \right\|_{L^p(0,1;E)} \right]. \tag{18}$$

From the estimate (18) for $u \in W^{2,p}(0,1;E(A),E)$, we get

$$\sum_{i=0}^{2} s^{\frac{i}{2}} |\lambda|^{1-\frac{i}{2}} \left\| u^{(i)} \right\|_{L^p(0,1;E)} + \|Au\|_{L^p(0,1;E)} \leq C \left\| (Q + \lambda)u \right\|_{L^p(0,1;E)}. \tag{19}$$

The estimate (19) implies that (4) has a unique solution and the operator $Q_s + \lambda$ has a bounded inverse in its rank space. We need to show the rank space coincides with all the space $L^p(0,1;E)$. It suffices to prove that there is a solution $u \in W^{2,p}(0,1;E(A),E)$ for all $f \in L^p(0,1;E)$. This fact can be derived in a standard way, approximating the equation with a similar one with smooth coefficients [19, 20]. More precisely, by virtue of [23, Theorem 3.4], *UMD* spaces satisfy the multiplier condition. Moreover, by the part (2) of Lemma A_1, given $a \in VMO$ with VMO modules $\eta(r)$, we can find a sequence of mollifiers functions $\{a_h\}$ converging to a in BMO as $h \to 0$ with a VMO modulus η_h such that $\eta_h(r) \leq \eta(r)$. In a similar way, it can be derived for the operator function $A(x)A^{-1}(x_0) \in VMO(L(E))$. □

Result 1 Theorem 3 implies that the resolvent $(Q_s + \lambda)^{-1}$ satisfies the following sharp uniform estimate:

$$\sum_{i=0}^{2} s^{\frac{i}{2}} |\lambda|^{1-\frac{i}{2}} \left\| \frac{d^i}{dx^i} (Q_s + \lambda)^{-1} \right\|_{L(L^p(0,1;E))} + \left\| A(Q_s + \lambda)^{-1} \right\|_{L(L^p(0,1;E))} \leq C \tag{20}$$

for $|\arg \lambda| \leq \varphi$, $\varphi \in (0,\pi)$ and $s > 0$.

The estimate (20) particularly implies that the operator Q is uniformly positive in $L^p(0,1;E)$ and generates analytic semigroups for $\varphi \in (\frac{\pi}{2},\pi)$ (see, *e.g.*, [22, §1.14.5]).

Remark 1 Conditions $a(0) = a(1)$, $A(0) = A(1)$ arise due to nonlocality of the boundary conditions (4). If the boundary conditions are local, then the conditions mentioned above are not required any more.

Consider the problem (1), where $L_k u$ is the same boundary condition as in (4). Let O_s denote a differential operator generated by the problem (1). We will show the separability and Fredholmness of (1).

Theorem 4 *Assume*

(1) *Condition* 1 *holds*;

(2) *for any* $\varepsilon > 0$, *there is* $C(\varepsilon) > 0$ *such that for a.e.* $x \in (0,1)$ *and for* $0 < \nu_0 < 1$, $0 < \nu_1 < \frac{1}{2}$,

$$\left\| A_0(x)u \right\|_E \leq \varepsilon \|Au\|_E + C(\varepsilon)\|u\|_E, \quad u \in E(A),$$

$$\left\| A_1(x)u \right\|_E \leq \varepsilon \|u\|_{(E(A),E)_{\frac{1}{2},\infty}} + C(\varepsilon)\|u\|, \quad u \in \left(E(A),E\right)_{\frac{1}{2},\infty}.$$

Then, for all $f \in L^p(0,1;E)$ and for large enough $|\lambda|$, $\lambda \in S(\varphi)$, there is a unique solution $u \in W^{2,p}(0,1;E(A),E)$ of the problem (1) *and the following coercive uniform estimate holds*:

$$\sum_{i=0}^{2} s^{\frac{i}{2}} |\lambda|^{1-\frac{i}{2}} \left\| u^{(i)} \right\|_{L^p(0,1;E)} + \|Au\|_{L^p(0,1;E)} \le C\|f\|_{L^p(0,1;E)}. \tag{21}$$

Proof It is sufficient to show that the operator $O_s + \lambda$ has a bounded inverse $(O_s + \lambda)^{-1}$ from $L^p(0,1;E)$ to $W^{2,p}(0,1;E(A),E)$. Put $O_s u = Q_s u + Q_0 u$, where

$$Q_0 u = s^{\frac{1}{2}} A_1 u^{(1)} + A_0 u, u \in W^{2,p}\big(0,1;E(A),E,L_k\big).$$

By the second assumption and Theorem A_1, there is a small ε and $C(\varepsilon)$ such that

$$\begin{aligned} \|A_0 u\|_{L^p(0,1;E)} &\le C\left\|A^{1-\nu_0} u\right\|_{L^p(0,1;E)} \\ &\le \varepsilon\|u\|_{W_s^{2,p}(0,1;E(A),E)} + C(\varepsilon)\|u\|_{L^p(0,1;E)}, \\ \left\|s^{\frac{1}{2}} A_1 u^{(1)}\right\|_{L^p(0,1;E)} &\le C\left\|s^{\frac{1}{2}} A^{\frac{1}{2}-\nu_1} u\right\|_{L^p(0,1;E)} \\ &\le \varepsilon\|u\|_{W_s^{2,p}(0,1;E(A),E)} + C(\varepsilon)\|u\|_{L^p(0,1;E)}. \end{aligned} \tag{22}$$

In view of estimates (17) and (22), we have

$$\begin{aligned} &\|A_0 u\|_{L^p(0,1;E)} < \delta_1 \|Q_s u\|_{L^p(0,1;E)}, \\ &\left\|s^{\frac{1}{2}} A_1 u^{(1)}\right\|_{L^p(0,1;E)} < \delta_1 \|Q_s u\|_{L^p(0,1;E)} \end{aligned} \tag{23}$$

for $u \in W^{2,p}(0,1;E(A),E)$ and $\delta_k < 1$. By Theorem 3, the operator $Q_s + \lambda$ has a bounded inverse $(Q_s + \lambda)^{-1}$ from $L^p(0,1;E)$ to $W^{2,p}(0,1;E(A),E)$ for sufficiently large $|\lambda|$. So, (23) implies the following uniform estimate:

$$\left\|Q_0(Q_s + \lambda)^{-1}\right\|_{L(L^p(0,1;E))} < 1.$$

It is clear that

$$\begin{aligned} &(O_s + \lambda) = \left[I + Q_0(Q_s + \lambda)^{-1}\right](Q_s + \lambda), \\ &(O_s + \lambda)^{-1} = (Q + \lambda)^{-1}\left[I + Q_0(Q_s + \lambda)^{-1}\right]^{-1}. \end{aligned}$$

Then, by above relation and by virtue of Theorem 3, we get the assertion. □

Theorem 4 implies the following result.

Result 2 Suppose all the conditions of Theorem 4 are satisfied. Then the resolvent $(O_s + \lambda)^{-1}$ of the operator O_s satisfies the following sharp uniform estimate:

$$\sum_{i=0}^{2} s^{\frac{i}{2}} |\lambda|^{1-\frac{i}{2}} \left\| \frac{d^i}{dx^i}(O_s + \lambda)^{-1} \right\|_{L(L^p(0,1;E))} + \left\|A(O_s + \lambda)^{-1}\right\|_{L(L^p(0,1;E))} \le C$$

for $|\arg\lambda| \le \varphi$, $\varphi \in [0,\pi)$ and $s > 0$.

Consider the problem (1) for $\lambda = 0$, *i.e.*,

$$Lu = sa(x)u^{(2)}(x) + A(x)u(x) + A_1(x)u^{(1)} + A_0(x)u = f(x), \quad x \in (0,1),$$
$$L_k u = \sum_{i=0}^{m_k} s^{\mu_i}\bigl[\alpha_{ki}u^{(i)}(0) + \beta_{ki}u^{(i)}(1)\bigr] = 0, \quad k = 1,2. \tag{24}$$

Theorem 5 *Assume all the conditions of Theorem* 4 *hold and* $A^{-1} \in \sigma_\infty(E)$. *Then the problem* (24) *is Fredholm from* $W^{2,p}(0,1;E(A),E)$ *into* $L^p(0,1;E)$.

Proof Theorem 4 implies that the operator $O_s + \lambda$ has a bounded inverse $(O_s + \lambda)^{-1}$ from $L^p(0,1;E)$ to $W^{2,p}(0,1;E(A),E)$ for large enough $|\lambda|$; that is, the operator $O_s + \lambda$ is Fredholm from $W^{2,p}(0,1;E(A),E)$ into $L^p(0,1;E)$. Then, by virtue of Theorem A_2 and by perturbation theory of linear operators, we obtain the assertion. □

4 Nonlinear DOEs with VMO coefficients

Let, at first, consider the linear BVP in a moving domain $(0,b(s))$,

$$a(x)u^{(2)}(x) + A(x)u(x) + A_1(x)u^{(1)}(x) + A_0(x)u(x) = f(x), \tag{25}$$
$$L_j u = \sum_{i=0}^{m_j}\bigl[\alpha_{ji}u^{(i)}(0) + \beta_{ji}u^{(i)}(b)\bigr] = 0, \quad j = 1,2,$$

where a is a complex valued function, and $A = A(x)$, $A_i = A_i(x)$ are possible operators in a Banach space E, where $b(s)$ is a positive continuous independent of u.

Theorem 4 implies the following result.

Result 3 Let all the conditions of Theorem 4 be satisfied. Then the problem (25), for $f \in L^p(0,b(s);E)$, $p \in (1,\infty)$, $\lambda \in S_\varphi$ and for large enough $|\lambda|$, has a unique solution $u \in W^{2,p}(0,b;E(A),E)$ and the following coercive uniform estimate holds:

$$\sum_{i=0}^{2} |\lambda|^{1-\frac{i}{2}} \bigl\|u^{(i)}\bigr\|_{L^p(0,b;E)} + \|Au\|_{L^p(0,b;E)} \le \|f\|_{L^p(0,b;E)}.$$

Proof Really, under the substitution $\tau = xb(s)$, the moving boundary problem (25) maps to the following BVP with a parameter in a fixed domain $(0,1)$:

$$b^{-2}(s)\tilde{a}(\tau)u^{(2)}(\tau) + (\tilde{A} + \lambda)u(\tau) + b^{-1}(s)\tilde{A}_1(\tau)u^{(1)}(\tau) + \tilde{A}_2(\tau)u(\tau) = f(\tau),$$
$$\sum_{i=0}^{m_j} b^{-i}(s)\bigl[\alpha_{ji}u^{(i)}(0) + \beta_{ji}u^{(i)}(1)\bigr] = 0, \quad j = 1,2,$$

where

$$\tau \in (0,1), \quad \tilde{A} = A\bigl(\tau b^{-1}(s)\bigr), \qquad \tilde{A}_i = A_i\bigl(\tau b^{-1}(s)\bigr).$$

Then, by virtue of Theorem 4, we obtain the assertion. □

Consider the nonlinear problem

$$a(x)u^{(2)}(x) + B(x,u,u_x)u(x) = F(x,u,u_x), \tag{26}$$
$$\sum_{i=0}^{m_k}\left[\alpha_{ki}u^{(i)}(0) + \beta_{ki}u^{(i)}(a)\right] = 0, \quad k = 1,2,$$

where $m_k \in \{0,1\}$, α_{ki}, β_{ki} are complex numbers, $x \in (0,b)$, where b is a positive number in $(0,b_0]$.

In this section, we will prove the existence and uniqueness of a maximal regular solution of the nonlinear problem (26). Assume A is a φ-positive operator in a Banach space E. Let

$$X = L^p(0,b;E), \qquad Y = W^{2,p}\bigl(0,b;E(A),E\bigr),$$
$$E_j = \bigl(E(A),E\bigr)_{\sigma_j,p}, \quad \sigma_j = \frac{1+jp}{2p}, \qquad X_0 = \prod_{j=0}^{1} E_j.$$

Remark 2 By using [22, §1.8], we obtain that the embedding $D^jY \in E_j$ is continuous and there exists the constant C_1 such that for $w \in Y$, $W = \{w_0,w_1\}$, $w_j = D^jw(\cdot)$, $j = 0,1$,

$$\|w\|_{X_0,\infty} = \prod_{j=0}^{1}\|D^jw\|_{C([0,b],E_j)} = \sup_{x\in[0,b]}\prod_{j=0}^{1}\|D^jw(x)\|_{E_j} \le C_1\|w\|_Y.$$

Condition 2 Assume the following are satisfied:

(1) $\eta = (-1)^{m_1}\alpha_1\beta_2 - (-1)^{m_2}\alpha_2\beta_1 \neq 0$ and $a(x)$ is a positive continuous function on $[0,b]$, $a(0) = a(b)$;

(2) E is a UMD space and $p \in (1,\infty)$;

(3) $F(x,\upsilon_0,\upsilon_1) : [0,b] \times X_0 \to E$ is a measurable function for each $\upsilon_i \in E_i$, $i = 0,1$; $F(x,\cdot,\cdot)$ is continuous with respect to $x \in [0,b]$ and $f(x) = F(x,0) \in X$. Moreover, for each $R > 0$, there exists μ_R such that

$$\|F(x,U) - F(x,\bar U)\|_E \le \mu_R\|U - \bar U\|_{X_0},$$

where $U = \{u_0,u_1\}$ and $\bar U = \{\bar u_0,\bar u_1\}$ for a.a. $x \in [0,b]$, $u_i,\bar u_i \in E_i$ and

$$\|U\|_{X_0} \le R, \qquad \|\bar U\|_{X_0} \le R.$$

(4) for $U = \{u_0,u_1\} \in X_0$, the operator $B(x,U)$ is R-positive in E uniformly with respect to $x \in [0,b]$; $B(x,U)B^{-1}(x^0,U) \in C([0,b];B(E))$, where the domain definition $D(B(x,U))$ does not depend on x and U; $B(x,W) : (0,b) \times X_0 \to B(E(A),E)$ is continuous, where $A = A(x) = B(x,W)$ for fixed $W = \{w_0,w_1\} \in X_0$;

(5) for each $R > 0$, there is a positive constant $L(R)$ such that $\|[B(x,U) - B(x,\bar U)]\upsilon\|_E \le L(R)\|U - \bar U\|_{X_0}\|A\upsilon\|_E$ for $x \in (0,b)$, $U,\bar U \in X_0$, $\|U\|_{X_0}, \|\bar U\|_{X_0} \le R$ and $\upsilon \in D(A)$ and $A(0) = A(b)$.

Theorem 6 *Let Condition* 1 *hold. Then there is* $b \in (0,b_0]$ *such that the problem* (26) *has a unique solution that belongs to the space* $W_p^2(0,b;E(A),E)$.

Proof Consider the linear problem

$$-a(x)w^{(2)}(x) + \big(A(x) + d\big)w(x) = f(x), \tag{27}$$
$$L_k w = \sum_{i=0}^{m_k} \alpha_{ki} w^{(i)}(0) + \beta_{ki} w^{(i)}(b) = 0,$$

where

$$f(x) = F(x, 0), \quad x_0 \in (0, b).$$

By virtue of Result 3, the problem (27) has a unique solution for all $f \in X$ and for sufficiently large $d > 0$ that satisfies the following

$$\|w\|_Y \le C_0 \|f\|_X,$$

where the constant C does not depend on $f \in X$ and $b \in (0, b_0]$. We want to solve the problem (26) locally by means of maximal regularity of the linear problem (27) via the contraction mapping theorem. For this purpose, let w be a solution of the linear BVP (27). Consider a ball

$$B_r = \big\{\upsilon \in Y, L_k\upsilon = 0, k = 1, 2, \|\upsilon - w\|_Y \le r\big\}.$$

For $\upsilon \in B_r$, consider the linear problem

$$-a(x)u^{(2)}(x) + Au(x) + du = F(x, V) + \big[B(0, W) - B(x, V)\big]\upsilon, \tag{28}$$
$$L_k u = \sum_{i=0}^{m_k} \alpha_{ki} u^{(i)}(0) + \beta_{ki} u^{(i)}(b) = 0,$$

where

$$V = \big\{\upsilon, \upsilon^{(1)}\big\}, \qquad W = \big\{w, w^{(1)}\big\}.$$

Define a map Q on B_r by $Q\upsilon = u$, where u is a solution of the problem (28). We want to show that $Q(B_r) \subset B_r$ and that Q is a contraction operator provided b is sufficiently small and r is chosen properly. For this aim, by using maximal regularity properties of the problem (28), we have

$$\|Q\upsilon - w\|_Y = \|u - w\|_Y \le C_0\big\{\big\|F(x, V) - F(x, 0)\big\|_X + \big\|\big[B(0, W) - B(x, V)\big]\upsilon\big\|_X\big\}.$$

By assumption (5), we have

$$\begin{aligned}
&\big\|\big[B(0, W)\upsilon - B(x, V)\big]\upsilon\big\|_X \\
&\quad \le \sup_{x\in[0,b]}\big\{\big\|\big[B(0, W) - B(x, W)\big]\upsilon\big\|_{B(X_0,X)} + \big\|B(x, W) - B(x, V)\big\|_{B(X_0,X)}\|\upsilon\|_Y\big\} \\
&\quad \le \big[\delta(b) + L(R)\|W - V\|_{\infty,X_0}\big]\big[\|\upsilon - w\|_Y + \|w\|_Y\big]
\end{aligned}$$

$$\le \{\delta(b) + L(R)[C_1\|\upsilon - w\|_Y + \|\upsilon - w\|_Y][\|\upsilon - w\|_Y + \|w\|_Y]\}$$
$$\le \{\delta(b) + L(R)[C_1 r + r][r + \|w\|_Y]\},$$

where

$$\delta(b) = \sup_{x\in[0,b]} \|[B(0,W) - B(x,W)]\|_{B(X_0,X)}.$$

Bearing in mind

$$\begin{aligned}\|F(x,V) - F(x,0,)\|_E &\le \delta(b) + \|F(x,V) - F(x,W)\|_E + \|F(x,W) - F(x,0)\|_E \\ &\le \delta(b) + \mu_R[\|\upsilon - w\|_Y + \|w\|_Y]\mu_R C_1[\|\upsilon - w\|_Y + \|w\|_Y] \\ &\le \mu_R[C_1 r + \|w\|_Y],\end{aligned}$$

where $R = C_1 r + \|w\|_Y$ is a fixed number. In view of the above estimates, by a suitable choice of μ_R, L_R and for sufficiently small $b \in [0; b_0)$, we have

$$\|Q\upsilon - w\|_Y \le r,$$

i.e.,

$$Q(B_r) \subset B_r.$$

Moreover, in a similar way, we obtain

$$\begin{aligned}\|Q\upsilon - Q\bar{\upsilon}\|_Y \le C_0\{&\mu_R C_1 + M_a + L(R)[\|\upsilon - w\|_Y + C_1 r] \\ &+ L(R)C_1[r + \|w\|_Y]\|\upsilon - \bar{\upsilon}\|_Y\} + \delta(b).\end{aligned}$$

By a suitable choice of μ_R, L_R and for sufficiently small $b \in (0, b_0)$, we obtain $\|Q\upsilon - Q\bar{\upsilon}\|_Y < \eta\|\upsilon - \bar{\upsilon}\|_Y$, $\eta < 1$, *i.e.*, Q is a contraction operator. Eventually, the contraction mapping principle implies a unique fixed point of Q in B_r which is the unique strong solution $u \in Y$. □

5 Boundary value problems for anisotropic elliptic equations with VMO coefficients

The Fredholm property of BVPs for elliptic equations with parameters in smooth domains were studied, *e.g.*, in [8, 10], also, for nonsmooth domains, these questions were investigated, *e.g.*, in [31].

Let $\Omega \subset R^n$ be an open connected set with a compact C^{2m}-boundary $\partial\Omega$. Let us consider the nonlocal boundary value problems on a cylindrical domain $G = (0,1) \times \Omega$ for the following anisotropic elliptic equation with *VMO* top-order coefficients:

$$\begin{aligned}(L+\lambda)u = sa(x)\frac{\partial^2 u}{\partial x^2} + s^{\frac{1}{2}} d_1(x,y)\frac{\partial u}{\partial x} + d_0(x,y)u \\ + \sum_{|\alpha|\le 2m} a_\alpha(y) D_y^\alpha u + \lambda u = f(x,y), \quad x \in (0,1), y \in \Omega,\end{aligned} \tag{29}$$

$$L_k u = \sum_{i=0}^{m_k} s^{\mu_i} \left[\alpha_{ki} u_x^{(i)}(0,y) + \beta_{ki} u_x^{(i)}(1,y) \right] = 0, \quad k = 1,2, \tag{30}$$

$$B_j u = \sum_{|\beta| \le m_j} b_{j\beta}(y) D_y^{\beta} u(x,y) = 0, \quad x \in (0,1), y \in \partial\Omega, j = 1,2,\ldots,m, \tag{31}$$

where s is a positive parameter, a, d_i are complex valued functions, α_{ki} and β_{ki} are complex numbers,

$$D_j = -i\frac{\partial}{\partial y_j}, \qquad m_k \in \{0,1\}, \qquad y = (y_1,\ldots,y_n), \qquad \mu_i = \frac{i}{2} + \frac{1}{2p}.$$

If $G = (0,1) \times \Omega$, $\mathbf{p} = (p_1, p)$, $L^{\mathbf{p}}(G)$ will denote the space of all $\mathbf{p}$-summable scalar-valued functions with a mixed norm (see, *e.g.*, [32, §1]), *i.e.*, the space of all measurable functions f defined on G, for which

$$\|f\|_{L^{\mathbf{p}}(G)} = \left(\int_0^1 \left(\int_{\Omega} |f(x,y)|^{p_1} \, dy \right)^{\frac{p}{p_1}} dx \right)^{\frac{1}{p}} < \infty.$$

Analogously, $W^{2,2m,\mathbf{p}}(G)$ denotes the anisotropic Sobolev space with the corresponding mixed norm [32, §10].

Theorem 7 *Let the following conditions be satisfied*;

(1) $a, d_0 \in VMO \cap L^{\infty}(R)$, $a(0) = a(1)$, $-a \in S(\varphi)$, $\mu(x) \neq 0$, *a.e.* $x \in [0,1]$;

(2) $\operatorname{Re} \omega_k \neq 0$ *and* $\frac{\lambda}{\omega_k} \in S(\varphi)$ *for* $\lambda \in S(\varphi)$, $0 \le \varphi < \pi$, $k = 1,2$ *a.e.* $x \in [0,1]$;

(3) $d_1 \in L^{\infty}$, $d_1(\cdot,y) d_0^{\frac{1}{2}-\nu}(\cdot) \in L^{\infty}(0,1)$ *for a.e.* $y \in \Omega$ *and* $0 < \nu < \frac{1}{2}$;

(4) $a_\alpha \in VMO \cap L^{\infty}(R^n)$ *for each* $|\alpha| = 2m$ *and* $a_\alpha \in [L^{\infty} + L^{\gamma_k}](\Omega)$ *for each* $|\alpha| = k < 2m$ *with* $r_k \ge q$ *and* $2m - k > \frac{l}{r_k}$;

(5) $b_{j\beta} \in C^{2m-m_j}(\partial\Omega)$ *for each* j, β *and* $m_j < 2m$, $\sum_{j=1}^m b_{j\beta}(y') \sigma_j \neq 0$, *for* $|\beta| = m_j$, $y' \in \partial G$, *where* $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in R^n$ *is a normal to* ∂G;

(6) *for* $y \in \bar{\Omega}$, $\xi \in R^n$, $\nu \in S(\varphi)$, $\varphi \in (0,\pi)$, $|\xi| + |\nu| \neq 0$ *let* $\nu + \sum_{|\alpha|=2m} a_\alpha(y)\xi^\alpha \neq 0$;

(7) *for each* $y_0 \in \partial\Omega$, *a local BVP in local coordinates corresponding to* y_0

$$\nu + \sum_{|\alpha|=2m} a_\alpha(y_0) D^\alpha \vartheta(y) = 0,$$

$$B_{j0}\vartheta = \sum_{|\beta|=m_j} b_{j\beta}(y_0) D^\beta u(y) = h_j, \quad j = 1,2,\ldots,m$$

has a unique solution $\vartheta \in C_0(R_+)$ *for all* $h = (h_1, h_2, \ldots, h_n) \in R^n$, *and for* $\xi' \in R^{n-1}$ *with* $|\xi'| + |\nu| \neq 0$.

Then

(a) *for all* $f \in L^{\mathbf{p}}(G)$, $\lambda \in S(\varphi)$ *and sufficiently large* $|\lambda|$, *the problem* (29)-(31) *has a unique solution* u *belonging to* $W^{2,2m,\mathbf{p}}(G)$ *and the following coercive uniform estimate holds*:

$$\sum_{i=0}^{2} s^{\frac{i}{2}} |\lambda|^{1-\frac{i}{2}} \left\| \frac{\partial^i u}{\partial^i x} \right\|_{L^{\mathbf{p}}(G)} + \sum_{|\beta|=2m} \left\| D_y^\beta u \right\|_{L^{\mathbf{p}}(G)} \le C\|f\|_{L^{\mathbf{p}}(G)};$$

(b) *for* $\lambda = 0$, *the problem* (29)-(31) *is Fredholm in* $L^{\mathbf{p}}(G)$.

Proof Let $E = L^{p_1}(\Omega)$. Then by virtue of [26], the part (1) of Condition 1 is satisfied. Consider the operator A acting in $L^{p_1}(\Omega)$ defined by

$$D(A) = W^{2m,p_1}(\Omega; B_j u = 0), \qquad Au = \sum_{|\alpha| \le 2m} a_\alpha(y) D^\alpha u(y).$$

For $x \in \Omega$, also consider operators in $L^{p_1}(\Omega)$

$$D(A_i) = W^{2m,p_1}(\Omega; B_j u = 0),$$
$$A_0(x)u = d_0(x,y)u(y), \qquad A_1(x)u = d_1(x,y)u(y).$$

The problem (29)-(31) can be rewritten in the form (1), where $u(x) = u(x,\cdot)$, $f(x) = f(x,\cdot)$ are functions with values in $E = L^{p_1}(\Omega)$. By virtue of [13], the problem

$$\nu u(y) + \sum_{|\alpha| \le 2m} a_\alpha(y) D_y^\alpha u(y) = f(y),$$
$$B_j u = \sum_{|\beta| \le m_j} b_{j\beta}(y) D_y^\beta u(y) = 0, \quad j = 1, 2, \ldots, m$$

has a unique solution for $f \in L^{p_1}(\Omega)$ and $\arg \nu \in S(\varphi)$, $|\nu| \to \infty$. Moreover, in view of [10, Theorem 8.2], the operator A is R-positive in $L^{p_1}(\Omega)$, *i.e.*, Condition 1 holds. Moreover, it is known that the embedding $W^{2m,p_1}(\Omega) \subset L^{p_1}(\Omega)$ is compact (see, *e.g.*, [22, Theorem 3.2.5]). Then, by using interpolation properties of Sobolev spaces (see, *e.g.*, [22, §4]), it is clear that the condition (2) of Theorem 4 is fulfilled too. Then from Theorems 4, 5, the assertions are obtained. □

6 Systems of differential equations with VMO coefficients

Consider the nonlocal BVPs for infinity systems of parameter-differential equations with principal *VMO* coefficients,

$$sa(x)u_m^{(2)}(x) + \sum_{j=1}^{N} s^{\frac{1}{2}} b_{mj}(x) u_j^{(1)}(x) + \sum_{j=1}^{N} d_{mj}(x) u_j(x) + \big(d_m(x) + \lambda\big) u_m(x)$$
$$= f_m(x), \quad x \in (0,1), m = 1, 2, \ldots, N, \tag{32}$$

$$L_k u = \sum_{i=0}^{m_k} s^{\mu_i} \big[\alpha_{ki} u_m^{(i)}(0) + \beta_{ki} u_m^{(i)}(1)\big] = 0, \quad k = 1, 2, \tag{33}$$

where s is a positive parameter, a, b_{mj}, d_{mj} are complex valued functions, N is a finite or infinite natural number, α_{ki} and β_{ki} are complex numbers, $\mu_i = \frac{i}{2} + \frac{1}{2p}$.

Let

$$d(x) = \{d_m(x)\}, \quad d_m > 0, \qquad u = \{u_m\}, \qquad Du = \{d_m u_m\}, \quad m = 1, 2, \ldots, \infty,$$

$$l_q(D) = \left\{ u : u \in l_q(N), \|u\|_{l_q(D)} = \|Du\|_{l_q} = \left(\sum_{m=1}^{N} |d_m u_m|^q \right)^{\frac{1}{q}} < \infty \right\},$$
$$x \in (0,1), 1 < q < \infty.$$

Let Q denote the operator in $L^p(0,1;l_q)$ generated by the problem (32)-(33). Let

$$B = L\big(L^p(0,1;l_q)\big).$$

Theorem 8 *Suppose the following conditions are satisfied*:

(1) $a \in VMO \cap L^\infty(R)$, $a(0) = a(1)$, $-a \in S(\varphi)$, $\mu(x) \neq 0$ *a.e.* $x \in (0,1)$;

(2) $\operatorname{Re} \omega_k(x) \neq 0$ *and* $\frac{\lambda}{\omega_k} \in S(\varphi)$ *for* $\lambda \in S(\varphi)$, *a.e.* $x \in (0,1)$, $0 \leq \varphi < \pi$, $k = 1,2$;

(3) $d_j \in VMO \cap L^\infty(R)$, $b_{mj}, d_{mj} \in L^\infty(0,1)$, $p \in (1,\infty)$;

(4) *there are* $0 < \nu_0 < 1$, $0 < \nu_1 < \frac{1}{2}$ *such that*

$$\sup_m \sum_{j=1}^N b_{mj}(x) d_j^{-(\frac{1}{2}-\nu_1)}(x) < M,$$

$$\sup_m \sum_{j=1}^N d_{mj}(x) d_j^{-(1-\nu_0)}(x) < M \quad \text{for a.e. } x \in (0,1).$$

Then, for all $f(x) = \{f_m(x)\}_1^N \in L^p(0,1;l_q)$, $\lambda \in S(\varphi)$ *and for sufficiently large* $|\lambda|$, *the problem* (32)-(33) *has a unique solution* $u = \{u_m(x)\}_1^\infty$ *belonging to* $W^{2,p}((0,1), l_q(D), l_q)$ *and the following coercive estimate holds*:

$$\sum_{i=0}^{2} s^{\frac{i}{2}} |\lambda|^{1-\frac{i}{2}} \left\| \frac{d^i u}{dx^i} \right\|_{L^p(0,1;l_q)} + \|Au\|_{L^p(0,1;l_q)} \leq C\|f\|_{L^p(0,1;l_q)}. \tag{34}$$

Proof Really, let $E = l_q$, A and $A_k(x)$ be infinite matrices such that

$$A = \big[d_m(x)\delta_{jm}\big], \qquad A_0(x) = \big[d_{mj}(x)\big], \qquad A_1(x) = \big[b_{mj}(x)\big], \quad m,j = 1,2,\ldots,\infty.$$

It is clear that the operator A is R-positive in l_q. Therefore, by Theorem 4, the problem (32)-(33) has a unique solution $u \in W^{2,p}((0,1); l_q(D), l_q)$ for all $f \in L^p((0,1); l_q)$, $\lambda \in S(\varphi)$ the estimate (34) holds. □

Remark 3 There are many positive operators in different concrete Banach spaces. Therefore, putting concrete Banach spaces and concrete positive operators (*i.e.*, pseudo-differential operators or finite or infinite matrices for instance) instead of E and A, respectively, by virtue of Theorem 4, 5, we can obtain a different class of maximal regular BVPs for partial differential or pseudo-differential equations or their finite and infinite systems with *VMO* coefficients.

Competing interests

The author declares that he has no competing interests.

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