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Weighted endpoint estimates for multilinear commutator of singular integral operators with non-smooth kernels

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Abstract
In this paper, we prove the weighted endpoint estimates for multilinear commutator of singular integral operators with non-smooth kernels.

Keywords: multilinear commutator; singular integral operator; BMO

1 Introduction

Let $b \in BMO(R^n)$ and T be the Calderón-Zygmund operator, the commutator $[b, T]$ generated by b and T is defined by $[b, T](f)(x) = b(x)T(f)(x) - T(bf)(x)$. A classical result of Coifman *et al.* (see [1]) proved that the commutator $[b, T]$ is bounded on $L^p(R^n)$ $(1 < p < \infty)$. In [2, 3], the boundedness properties of the commutators for the extreme values of p are obtained. In this paper, we will introduce the multilinear commutator of singular integral operators with non-smooth kernels and prove the weighted boundedness properties of the operator for the extreme cases.

First let us introduce some notations (see [3–12]). In this paper, Q will denote a cube of R^n with sides parallel to the axes. For a cube Q and a function b, let $b_Q = |Q|^{-1}\int_Q b(x)\,dx$ and $b(Q) = \int_Q b(x)\,dx$, the sharp function of b is defined by

$$b^{\#}(x) = \sup_{Q \ni x} \frac{1}{|Q|}\int_Q \left|b(y) - b_Q\right|dy.$$

It is well known that (see [6])

$$b^{\#}(x) = \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|}\int_Q \left|b(y) - c\right|dy.$$

Moreover, for a weight function ω (that is, a non-negative locally integrable function), b is said to belong to $BMO(\omega)$ if $b^{\#} \in L^\infty(\omega)$ and define $\|b\|_{BMO(\omega)} = \|b^{\#}\|_{L^\infty(\omega)}$, if $\omega = 1$, we denote $BMO(\omega) = BMO(R^n)$. It is well known that (see [11])

$$\|b - b_{2^k Q}\|_{BMO} \le Ck\|b\|_{BMO}.$$

The A_p weight is defined by (see [6])

$$A_p = \left\{0 < \omega \in L^1_{\mathrm{loc}}(R^n) : \sup_Q \left(\frac{1}{|Q|}\int_Q \omega(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q \omega(x)^{-1/(p-1)}\,dx\right)^{p-1} < \infty\right\},$$
$$1 < p < \infty$$

and

$$A_1 = \left\{0 < \omega \in L^1_{\mathrm{loc}}(R^n) : \sup_{Q \ni x} \frac{1}{|Q|}\int_Q \omega(y)\,dy \le c\omega(x), \text{a.e.}\right\}.$$

Definition 1 A family of operators D_t, $t > 0$, is said to be an 'approximation to the identity' if, for every $t > 0$, D_t can be represented by the kernel $a_t(x,y)$ in the following sense:

$$D_t(f)(x) = \int_{R^n} a_t(x,y)f(y)\,dy$$

for every $f \in L^p(R^n)$ with $p \ge 1$, and $a_t(x,y)$ satisfies

$$\left|a_t(x,y)\right| \le h_t(x,y) = Ct^{-n/2}s\left(|x-y|^2/t\right),$$

where s is a positive, bounded, and decreasing function satisfying

$$\lim_{r\to\infty} r^{n+\epsilon}s\left(r^2\right) = 0$$

for some $\epsilon > 0$.

Definition 2 A linear operator T is called the singular integral operators with non-smooth kernels if T is bounded on $L^2(R^n)$ and associated with a kernel $K(x,y)$ such that

$$T(f)(x) = \int_{R^n} K(x,y)f(y)\,dy$$

for every continuous function f with compact support, and for almost all x not in the support of f.

(1) There exists an 'approximation to the identity' $\{B_t, t > 0\}$ such that TB_t has associated kernel $k_t(x,y)$ and there exist $c_1, c_2 > 0$ so that

$$\int_{|x-y|>c_1 t^{1/2}} \left|K(x,y) - k_t(x,y)\right|\,dx \le c_2 \quad \text{for all } y \in R^n.$$

(2) There exists an 'approximation to the identity' $\{A_t, t > 0\}$ such that A_tT has associated kernel $K_t(x,y)$ which satisfies

$$\left|K_t(x,y)\right| \le c_4 t^{-n/2} \quad \text{if } |x-y| \le c_3 t^{1/2}$$

and

$$\left|K(x,y) - K_t(x,y)\right| \le c_4 t^{\delta/2}|x-y|^{-n-\delta} \quad \text{if } |x-y| \ge c_3 t^{1/2}$$

for some $c_3, c_4 > 0$, $\delta > 0$.

Given some locally integrable functions b_j ($j = 1, \dots, m$). The multilinear operator associated to T is defined by

$$T_b(f)(x) = \int_{R^n} \left[\prod_{j=1}^{m} \big(b_j(x) - b_j(y)\big) \right] K(x,y) f(y)\, dy.$$

Definition 3 Given the 'approximations to the identity' $\{A_t, t > 0\}$ and a weight function ω.

(1) The weighted BMO space associated with $\{A_t, t > 0\}$ is defined by

$$BMO_A(\omega) = \big\{ f \in L^1_{\mathrm{loc}}\big(R^n\big) : \|f\|_{BMO_A(\omega)} < \infty \big\},$$

where

$$\|f\|_{BMO_A(\omega)} = \sup_Q \frac{1}{\omega(Q)} \int_Q \big| f(x) - A_{t_Q}(f)(x) \big| \omega(x)\, dx,$$

$t_Q = l(Q)^2$ and $l(Q)$ denotes the side length of Q.

(2) The weighted central BMO space associated with $\{A_t, t > 0\}$ is defined by

$$CMO_A(\omega) = \big\{ f \in L^1_{\mathrm{loc}}\big(R^n\big) : \|f\|_{CMO_A(\omega)} < \infty \big\},$$

where

$$\|f\|_{CMO(\omega)} = \sup_{r>1} \frac{1}{\omega(Q(0,r))} \int_Q \big| f(x) - A_{t_Q} f(x) \big| \omega(x)\, dx,$$

and $t_Q = r^2$.

Definition 4 Let $1 < p < \infty$ and ω be a weighted function on R^n. We shall call $B_p(\omega)$ the space of those functions f on R^n, such that

$$\|f\|_{B_p(\omega)} = \sup_{r>1} \big[\omega\big(Q(0,r)\big)\big]^{-1/p} \|f \chi_{Q(0,r)}\|_{L^p(\omega)} < \infty.$$

For $b_j \in BMO(R^n)$ ($j = 1, \dots, m$), set $\|\vec{b}\|_{BMO} = \prod_{j=1}^{m} \|b_j\|_{BMO}$. Given a positive integer m and $1 \le j \le m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \dots, \sigma(j)\}$ of $\{1, \dots, m\}$ of j different elements. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \dots, m\} \backslash \sigma$. For $\vec{b} = (b_1, \dots, b_m)$ and $\sigma = \{\sigma(1), \dots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \dots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and $\|\vec{b}_\sigma\|_{BMO} = \|b_{\sigma(1)}\|_{BMO} \cdots \|b_{\sigma(j)}\|_{BMO}$.

2 Theorems and proofs

We begin with some preliminaries lemmas.

Lemma 1 ([5, 7]) *Let $\omega \in A_1$, $1 < p \le \infty$, and T be the singular integral operators with non-smooth kernels. Then T is boundedness on $L^p(w)$.*

Lemma 2 *Let $\omega \in A_1$, $\{A_t, t > 0\}$ be an 'approximation to the identity' and $b \in BMO(R^n)$. Then*

(a) *for every* $f \in L^\infty(R^n)$, $1 \le p < \infty$, *and any cube* Q,

$$\left(\frac{1}{|Q|}\int_Q \left|A_{t_Q}\big((b-b_Q)f\big)(y)\right|^p dy\right)^{1/p} \le C\|b\|_{BMO}\|f\|_{L^\infty};$$

(b) *for every* $f \in B_p(\omega)$, $1 \le r < p < \infty$, *and any cube* Q,

$$\left(\frac{1}{\omega(Q)}\int_Q \left|A_{t_Q}\big((b-b_Q)f\big)(y)\right|^r \omega(y)\,dy\right)^{1/r} \le C\|b\|_{BMO}\|f\|_{B_p(\omega)},$$

where $t_Q = l(Q)^2$ *and* $l(Q)$ *denotes the side length of* Q.

Proof (a) Write

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q \left|A_{t_Q}\big((b-b_Q)f\big)(y)\right|^p dy\right)^{1/p} \\
&\quad\le \left(\frac{1}{|Q|}\int_Q\int_{R^n} h_{t_Q}(x,y)^p \left|\big(b(y)-b_Q\big)f(y)\right|^p dy\,dx\right)^{1/p} \\
&\quad\le \left(\frac{1}{|Q|}\int_Q\int_{2Q} h_{t_Q}(x,y)^p \left|\big(b(y)-b_Q\big)f(y)\right|^p dy\,dx\right)^{1/p} \\
&\qquad + \left(\sum_{k=1}^\infty \frac{1}{|Q|}\int_Q\int_{2^{k+1}Q\setminus 2^kQ} h_{t_Q}(x,y)^p \left|\big(b(y)-b_Q\big)f(y)\right|^p dy\,dx\right)^{1/p} \\
&\quad= I_1 + I_2.
\end{aligned}$$

We have, by Hölder's inequality,

$$\begin{aligned}
I_1 &\le \left(\frac{C}{|Q||2Q|}\int_Q\int_{2Q} \left|\big(b(y)-b_Q\big)f(y)\right|^p dy\,dx\right)^{1/p} \\
&\le C\|f\|_{L^\infty}\left(\frac{1}{|2Q|}\int_{2Q}\left|b(y)-b_Q\right|^p dy\right)^{1/p} \\
&\le C\|b\|_{BMO}\|f\|_{L^\infty}.
\end{aligned}$$

For I_2, for $x \in Q$ and $y \in 2^{k+1}Q\setminus 2^kQ$, we have $|x-y| \ge 2^{k-1}t_Q$ and $h_{t_Q}(x,y) \le C\frac{s(2^{2(k-1)})}{|Q|}$. Thus

$$\begin{aligned}
I_2 &\le C\sum_{k=1}^\infty 2^{(k-1)n}s\big(2^{2(k-1)}\big)\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}\left|\big(b(y)-b_Q\big)f(y)\right|^p dy\right)^{1/p} \\
&\le C\|f\|_{L^\infty}\sum_{k=1}^\infty 2^{(k-1)n}s\big(2^{2(k-1)}\big)\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}\left|b(y)-b_{2^{k+1}Q}\right|^p dy\right)^{1/p} \\
&\quad + C\|f\|_{L^\infty}\sum_{k=1}^\infty 2^{(k-1)n}s\big(2^{2(k-1)}\big)|b_Q - b_{2^{k+1}Q}| \\
&\le C\|f\|_{L^\infty}\sum_{k=1}^\infty 2^{(k-1)n}s\big(2^{2(k-1)}\big)(k+1)\|b\|_{BMO} \\
&\le C\|b\|_{BMO}\|f\|_{L^\infty},
\end{aligned}$$

where the last inequality follows from

$$\sum_{k=2}^{\infty} 2^{(k-1)n} s\bigl(2^{2(k-1)}\bigr)(k+1) \le C \sum_{k=2}^{\infty} k 2^{-(k-1)\epsilon} < \infty$$

for some $\epsilon > 0$.

(b) Write

$$\begin{aligned}
&\left(\frac{1}{\omega(Q)} \int_Q \bigl|A_{t_Q}\bigl((b-b_Q)f\bigr)(y)\bigr|^r \omega(y)\,dy\right)^{1/r} \\
&\quad \le \left(\frac{1}{\omega(Q)} \int_Q \int_{R^n} h_{t_Q}(x,y)^r \bigl|\bigl(b(y)-b_Q\bigr)f(y)\bigr|^r \omega(y)\,dy\,dx\right)^{1/r} \\
&\quad \le \left(\frac{1}{\omega(Q)} \int_Q \int_{2Q} h_{t_Q}(x,y)^p \bigl|\bigl(b(y)-b_Q\bigr)f(y)\bigr|^r \omega(y)\,dy\,dx\right)^{1/r} \\
&\qquad + \left(\sum_{k=1}^{\infty} \frac{1}{\omega(Q)} \int_Q \int_{2^{k+1}Q\setminus 2^kQ} h_{t_Q}(x,y)^r \bigl|\bigl(b(y)-b_Q\bigr)f(y)\bigr|^r \omega(y)\,dy\,dx\right)^{1/r} \\
&\quad = I + II.
\end{aligned}$$

For I, since $\omega \in A_1$, ω satisfies the reverse of Hölder's inequality

$$\left(\frac{1}{|Q|} \int_Q \omega(x)^q\,dx\right)^{1/q} \le \frac{C}{|Q|} \int_Q \omega(x)\,dx$$

for some $1 < q < \infty$, and $\frac{\omega(Q_2)}{|Q_2|}\frac{|Q_1|}{\omega(Q_1)} \le C$ for all cubes Q_1, Q_2 with $Q_1 \subset Q_2$, $\omega \in A_{p/ru}$ for $1 < u, v < \infty$ with $u'v = q$ and $p > ru$ (see [6]). We have, by Hölder's inequality,

$$\begin{aligned}
I &\le \left(\frac{C}{\omega(Q)|Q|} \int_Q \int_{2Q} \bigl|\bigl(b(y)-b_Q\bigr)f(y)\bigr|^r \omega(y)\,dy\,dx\right)^{1/r} \\
&\le C\left(\frac{1}{\omega(Q)} \int_{2Q} \bigl|\bigl(b(y)-b_Q\bigr)f(y)\bigr|^r \omega(y)\,dy\right)^{1/r} \\
&\le C\left[\frac{|2Q|}{\omega(Q)} \left(\frac{1}{|2Q|} \int_{2Q} \bigl|b(y)-b_Q\bigr|^{ru'} \omega(y)^{u'}\,dy\right)^{1/u'} \left(\frac{1}{|2Q|} \int_{2Q} \bigl|f(y)\bigr|^{ru}\,dy\right)^{1/u}\right]^{1/r} \\
&\le C\left(\frac{|2Q|}{\omega(Q)}\right)^{1/r} \left(\frac{1}{|2Q|} \int_{2Q} \bigl|b(y)-b_Q\bigr|^{ru'v'}\,dy\right)^{1/ru'v'} \left(\frac{1}{|2Q|} \int_{2Q} \omega(y)^{u'v}\,dy\right)^{1/ru'v} \\
&\quad \times \left(\frac{1}{|2Q|} \int_{2Q} \bigl|f(y)\bigr|^{ru}\,dy\right)^{1/ru} \\
&\le C\|b\|_{BMO} \left(\frac{|2Q|}{\omega(2Q)}\right)^{1/r} \left(\frac{\omega(2Q)}{|2Q|}\right)^{1/r} \left(\frac{1}{|2Q|} \int_{2Q} \bigl|f(y)\bigr|^{ru} \omega(y)^{\frac{ru}{p}} \omega(y)^{-\frac{ru}{p}}\,dy\right)^{1/ru} \\
&\le C\|b\|_{BMO} \left(\frac{1}{|2Q|} \int_{2Q} \Bigl(\bigl|f(y)\bigr|^{ru} \omega(y)^{\frac{ru}{p}}\Bigr)^{\frac{p}{ru}}\,dy\right)^{1/p} \left(\frac{1}{|2Q|} \int_{2Q} \omega(y)^{-\frac{ru}{p}\frac{p}{p-ru}}\,dy\right)^{(p-ru)/pru} \\
&\le C\|b\|_{BMO} \left(\frac{1}{|2Q|}\right)^{1/p} \|f\chi_{2Q}\|_{L^p(\omega)} \left(\frac{1}{|2Q|} \int_{2Q} \omega(y)\,dy\right)^{-1/p}
\end{aligned}$$

$$\begin{aligned}
&\times \left[\left(\frac{1}{|2Q|} \int_{2Q} \omega(y)\,dy \right) \left(\frac{1}{|2Q|} \int_{2Q} \omega(y)^{-\frac{1}{\frac{p}{ru}-1}}\,dy \right)^{\frac{p}{ru}-1} \right]^{1/p} \\
&\le C\|b\|_{BMO}\omega(2Q)^{-1/p}\|f\chi_{2Q}\|_{L^p(\omega)} \\
&\le C\|b\|_{BMO}\|f\|_{B_p(\omega)};
\end{aligned}$$

$$\begin{aligned}
II &\le C\left(\frac{|Q|}{\omega(Q)}\right)^{1/r} \sum_{k=1}^{\infty} 2^{(k-1)n} s\bigl(2^{2(k-1)}\bigr) \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} \bigl|\bigl(b(y)-b_Q\bigr)f(y)\bigr|^r \omega(y)\,dy \right)^{1/r} \\
&\le C\left(\frac{|Q|}{\omega(Q)}\right)^{1/r} \sum_{k=1}^{\infty} 2^{(k-1)n} s\bigl(2^{2(k-1)}\bigr) \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} \bigl|b(y)-b_{2^{k+1}Q}\bigr|^{ru'} \omega(y)^{u'}\,dy \right)^{1/ru'} \\
&\quad \times \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} \bigl|f(y)\bigr|^{ru}\,dy \right)^{1/ru} \\
&\le C\left(\frac{|Q|}{\omega(Q)}\right)^{1/r} \sum_{k=1}^{\infty} 2^{(k-1)n} s\bigl(2^{2(k-1)}\bigr) \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} \bigl|b(y)-b_{2^{k+1}Q}\bigr|^{ru'v'}\,dy \right)^{1/ru'v'} \\
&\quad \times \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} \omega(y)^{u'v}\,dy \right)^{1/ru'v} \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} f(y)^{ru}\,dy \right)^{1/ru} \\
&\le C\|b\|_{BMO} \sum_{k=1}^{\infty} 2^{(k-1)n} s\bigl(2^{2(k-1)}\bigr)(k+1) \left(\frac{|Q|}{\omega(Q)} \cdot \frac{\omega(2^{k+1}Q)}{|2^{k+1}Q|} \right)^{1/r} \\
&\quad \times \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} f(y)^{ru}\,dy \right)^{1/ru} \\
&\le C \sum_{k=1}^{\infty} 2^{(k-1)n} s\bigl(2^{2(k-1)}\bigr)(k+1)\|b\|_{BMO}\omega\bigl(2^{k+1}Q\bigr)^{-1/p}\|f\chi_{2^{k+1}Q}\|_{L^p(\omega)} \\
&\le C\|b\|_{BMO}\|f\|_{B_p(w)}.
\end{aligned}$$

This completes the proof. □

Theorem 1 *Let T be the singular integral operators with non-smooth kernels, $\omega \in A_1$ and $\vec{b} = (b_1, \dots, b_m)$ with $b_j \in BMO(R^n)$ for $1 \le j \le m$. Then T_b is bounded from $L^\infty(\omega)$ to $BMO_A(\omega)$.*

Proof It suffices to prove, for $f \in C_0^\infty(R^n)$, the following inequality holds:

$$\frac{1}{\omega(Q)} \int_Q \bigl|T_b(f)(x) - A_{t_Q}T_b(f)(x)\bigr|\omega(x)\,dx \le C\|f\|_{L^\infty(\omega)}.$$

We fix a cube $Q = Q(x_0, d)$. We decompose f into $f = f_1 + f_2$ with $f_1 = f\chi_Q$, $f_2 = f\chi_{(R^n \setminus Q)}$.

When $m = 1$, set $(b_1)_Q = |Q|^{-1}\int_Q b_1(y)\,dy$, we have

$$\begin{aligned}
T_{b_1}(f)(x) &= \int_{R^n} \bigl[\bigl(b_1(x)-(b_1)_Q\bigr) - \bigl(b_1(y)-(b_1)_Q\bigr)\bigr]K(x,y)f(y)\,dy \\
&= \bigl(b_1(x)-(b_1)_Q\bigr)\int_{R^n} K(x,y)f(y)\,dy - \int_{R^n} \bigl(b_1(y)-(b_1)_Q\bigr)K(x,y)f(y)\,dy
\end{aligned}$$

and

$$A_{t_Q}T_{b_1}(f)(x) = \bigl(b_1(x)-(b_1)_Q\bigr)\int_{R^n} K_t(x,y)f(y)\,dy - \int_{R^n} \bigl(b_1(y)-(b_1)_Q\bigr)K_t(x,y)f(y)\,dy.$$

Then

$$\begin{aligned}
&\bigl|T_{b_1}(f)(x) - A_{t_Q}T_{b_1}(f)(x)\bigr| \\
&\quad\le \left|\bigl(b_1(x) - (b_1)_Q\bigr)\int_{R^n} K(x,y)f(y)\,dy\right| \\
&\qquad + \left|\int_{R^n}\bigl(b_1(y) - (b_1)_Q\bigr)K(x,y)f_1(y)\,dy\right| \\
&\qquad + \left|\bigl(b_1(x) - (b_1)_Q\bigr)\int_{R^n} K_t(x,y)f(y)\,dy\right| \\
&\qquad + \left|\int_{R^n}\bigl(b_1(y) - (b_1)_Q\bigr)K_t(x,y)f_1(y)\,dy\right| \\
&\qquad + \left|\int_{R^n}\bigl(b_1(y) - (b_1)_Q\bigr)\bigl(K(x,y) - K_t(x,y)\bigr)f_2(y)\,dy\right| \\
&\quad= I_1(x) + I_2(x) + I_3(x) + I_4(x) + I_5(x).
\end{aligned}$$

For $I_1(x)$, let $1/p + 1/p' = 1$, $1/q + 1/q' = 1$, by the reverse of Hölder's inequality with $1 < q < \infty$, Lemma 1, and Hölder's inequality, we have

$$\begin{aligned}
&\frac{1}{\omega(Q)}\int_Q\bigl|I_1(x)\bigr|\omega(x)\,dx \\
&\quad\le \frac{C}{\omega(Q)}\left(\int_Q\bigl|b_1(x) - (b_1)_Q\bigr|^{p'}\omega(x)\,dx\right)^{1/p'}\left(\int_{R^n}\bigl|T(f)(x)\bigr|^p\omega(x)\chi_Q(x)\,dx\right)^{1/p} \\
&\quad\le \frac{C}{\omega(Q)}\left(\int_Q\bigl|b_1(x) - (b_1)_Q\bigr|^{p'}\omega(x)\,dx\right)^{1/p'}\left(\int_{R^n}\bigl|f(x)\bigr|^p\omega(x)\chi_Q(x)\,dx\right)^{1/p} \\
&\quad\le \frac{C}{\omega(Q)}\left(\int_Q\bigl|b_1(x) - (b_1)_Q\bigr|^{p'}\omega(x)\,dx\right)^{1/p'}\|f\|_{L^\infty(\omega)}\left(\int_Q\omega(x)\,dx\right)^{1/p} \\
&\quad\le \frac{C}{\omega(Q)}\left[\left(\int_Q\bigl|b_1(x) - (b_1)_Q\bigr|^{p'q'}\,dx\right)^{1/q'}\left(\int_Q\omega(x)^q\,dx\right)^{1/q}\right]^{1/p'}\|f\|_{L^\infty(\omega)}\omega(Q)^{1/p} \\
&\quad\le C\omega(Q)^{1/p-1}|Q|^{1/p'}\|b_1\|_{BMO}\left(\frac{1}{|Q|}\int_Q\omega(x)^q\,dx\right)^{1/p'q}\|f\|_{L^\infty(\omega)} \\
&\quad\le C\|b_1\|_{BMO}\|f\|_{L^\infty(\omega)}.
\end{aligned}$$

For $I_2(x)$, taking $p > 1$, by Hölder's inequality, we have

$$\begin{aligned}
&\frac{1}{\omega(Q)}\int_Q\bigl|I_2(x)\bigr|\omega(x)\,dx \\
&\quad\le \left(\frac{1}{\omega(Q)}\int_{R^n}\bigl|T\bigl(\bigl(b_1 - (b_1)_Q\bigr)f_1\bigr)(x)\bigr|^p\omega(x)\chi_Q(x)\,dx\right)^{1/p} \\
&\quad\le C\omega(Q)^{-1/p}\left(\int_{R^n}\bigl|\bigl(b_1(x) - (b_1)_Q\bigr)f_1(x)\bigr|^p\omega(x)\chi_Q(x)\,dx\right)^{1/p} \\
&\quad\le C\omega(Q)^{-1/p}\left[\left(\int_Q\bigl|b_1(x) - (b_1)_Q\bigr|^{pq'}\,dx\right)^{1/q'}\left(\int_Q\bigl|f(x)\bigr|^{pq}\omega(x)^q\,dx\right)^{1/q}\right]^{1/p}
\end{aligned}$$

$$\begin{aligned}
&\leq C\omega(Q)^{-1/p}\left(\int_Q\left|b_1(x)-(b_1)_Q\right|^{pq'}dx\right)^{1/pq'}\left(\int_Q\left|f(x)\right|^{pq}\omega(x)^q\,dx\right)^{1/pq}\\
&\leq C\omega(Q)^{-1/p}\left(\int_Q\left|b_1(x)-(b_1)_Q\right|^{pq'}dx\right)^{1/pq'}\left(\int_Q\omega(x)^q\,dx\right)^{1/pq}\|f\|_{L^\infty(\omega)}\\
&\leq C\omega(Q)^{-1/p}|Q|^{1/pq'}\|b_1\|_{BMO}|Q|^{1/pq}\left(\frac{1}{|Q|}\int_Q\omega(x)^q\,dx\right)^{1/pq}\|f\|_{L^\infty(\omega)}\\
&\leq C\|b_1\|_{BMO}\left(\frac{|Q|}{\omega(Q)}\right)^{1/p}\left(\frac{1}{|Q|}\int_Q\omega(x)\,dx\right)^{1/p}\|f\|_{L^\infty(\omega)}\\
&\leq C\|b_1\|_{BMO}\|f\|_{L^\infty(\omega)}.
\end{aligned}$$

For $I_3(x)$ and $I_4(x)$, we get, for $1 < p_1, p_2 < \infty$ with $1/p_1 + 1/p_2 + 1/q = 1$,

$$\begin{aligned}
&\frac{1}{\omega(Q)}\int_Q\left|I_3(x)\right|\omega(x)\,dx\\
&\quad\leq\frac{C}{\omega(Q)}\int_Q\left|b_1(x)-(b_1)_Q\right|\left|A_{t_Q}(f)(x)\right|\omega(x)\,dx\\
&\quad\leq C\frac{|Q|}{\omega(Q)}\left(\frac{1}{|Q|}\int_Q\left|b_1(x)-(b_1)_Q\right|^{p_1}dx\right)^{1/p_1}\\
&\qquad\times\left(\frac{1}{|Q|}\int_Q\left|A_{t_Q}(f)(x)\right|^{p_2}dx\right)^{1/p_2}\left(\frac{1}{|Q|}\int_Q\omega(x)^q\,dx\right)^{1/q}\\
&\quad\leq C\frac{|Q|}{\omega(Q)}\|b_1\|_{BMO}\|f\|_{L^\infty(\omega)}\frac{\omega(Q)}{|Q|}\\
&\quad\leq C\|b_1\|_{BMO}\|f\|_{L^\infty(\omega)},\\
&\frac{1}{\omega(Q)}\int_Q\left|I_4(x)\right|\omega(x)\,dx\\
&\quad\leq\frac{1}{\omega(Q)}\int_{R^n}\left|A_{t_Q}\left(\left(b_1-(b_1)_Q\right)f_1\right)(x)\right|\omega(x)\,dx\\
&\quad\leq C\frac{|Q|}{\omega(Q)}\left(\frac{1}{|Q|}\int_Q\left|A_{t_Q}\left(\left(b_1-(b_1)_Q\right)f_1\right)(x)\right|^{q'}dx\right)^{1/q'}\left(\frac{1}{|Q|}\int_Q\omega(x)^q\,dx\right)^{1/q}\\
&\quad\leq C\frac{|Q|}{\omega(Q)}\|b_1\|_{BMO}\|f\|_{L^\infty(\omega)}\frac{\omega(Q)}{|Q|}\\
&\quad\leq C\|b_1\|_{BMO}\|f\|_{L^\infty(\omega)}.
\end{aligned}$$

For $I_5(x)$, we have

$$\begin{aligned}
I_5(x)&=\left|\int_{R^n}\left(b_1(y)-(b_1)_Q\right)\left(K(x,y)-K_t(x,y)\right)f_2(y)\,dy\right|\\
&\leq C\sum_{k=0}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}\left|b_1(y)-(b_1)_Q\right|\left|f(y)\right|\frac{d^\delta}{|x_0-y|^{n+\delta}}\,dy\\
&\leq C\sum_{k=1}^{\infty}\frac{d^\delta}{(2^{k-1}d)^{n+\delta}}\left|2^kQ\right|\left(\frac{1}{|2^kQ|}\int_{2^kQ}\left|f(y)\right|^p dy\right)^{1/p}
\end{aligned}$$

$$\times \left(\frac{1}{|2^k Q|} \int_{2^k Q} \left| b_1(y) - (b_1)_Q \right|^{p'} dy \right)^{1/p'}$$
$$\leq C \sum_{k=1}^{\infty} k^m 2^{-k\delta} \|b_1\|_{BMO} \|f\|_{L^\infty(\omega)}$$
$$\leq C \|b_1\|_{BMO} \|f\|_{L^\infty(\omega)},$$

so

$$\frac{1}{\omega(Q)} \int_Q \left| I_5(x) \right| \omega(x)\, dx \leq C \|b_1\|_{BMO} \|f\|_{L^\infty(\omega)}.$$

When $m > 1$, set $\vec{b}_Q = ((b_1)_Q, \dots, (b_m)_Q) \in R^n$, where $(b_j)_Q = |Q|^{-1} \int_Q b_j(y)\, dy$, $1 \leq j \leq m$, we have

$$\begin{aligned} T_b(f)(x) = {} & \prod_{j=1}^{m} \bigl(b_j(x) - (b_j)_Q\bigr) \int_{R^n} K(x,y) f(y)\, dy \\ & + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} (-1)^{m-j} \bigl(b(x) - (b)_Q\bigr)_\sigma \int_{R^n} \bigl(b(y) - (b)_Q\bigr)_{\sigma^c} K(x,y) f(y)\, dy \\ & + (-1)^m \int_{R^n} \prod_{j=1}^{m} \bigl(b_j(y) - (b_j)_Q\bigr) K(x,y) f(y)\, dy \end{aligned}$$

and

$$\begin{aligned} A_{t_Q} T_b(f)(x) = {} & \prod_{j=1}^{m} \bigl(b_j(x) - (b_j)_Q\bigr) \int_{R^n} K_t(x,y) f(y)\, dy \\ & + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} (-1)^{m-j} \bigl(b(x) - (b)_Q\bigr)_\sigma \int_{R^n} \bigl(b(y) - (b)_Q\bigr)_{\sigma^c} K_t(x,y) f(y)\, dy \\ & + (-1)^m \int_{R^n} \prod_{j=1}^{m} \bigl(b_j(y) - (b_j)_Q\bigr) K_t(x,y) f(y)\, dy, \end{aligned}$$

then

$$\begin{aligned} & \left| T_b(f)(x) - A_{t_Q} T_b(f)(x) \right| \\ & \quad \leq \left| \prod_{j=1}^{m} \bigl(b_j(x) - (b_j)_Q\bigr) \int_{R^n} K(x,y) f(y)\, dy \right| \\ & \qquad + \left| \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \bigl(b(x) - (b)_Q\bigr)_\sigma \int_{R^n} \bigl(b(y) - (b)_Q\bigr)_{\sigma^c} K(x,y) f(y)\, dy \right| \\ & \qquad + \left| \int_{R^n} \prod_{j=1}^{m} \bigl(b_j(y) - (b_j)_Q\bigr) K(x,y) f_1(y)\, dy \right| \\ & \qquad + \left| \prod_{j=1}^{m} \bigl(b_j(x) - (b_j)_Q\bigr) \int_{R^n} K_t(x,y) f(y)\, dy \right| \end{aligned}$$

$$\begin{aligned}
&+ \left|\sum_{j=1}^{m-1} \sum_{\sigma\in C_j^m} \bigl(b(x)-(b)_Q\bigr)_\sigma \int_{R^n} \bigl(b(y)-(b)_Q\bigr)_{\sigma^c} K_t(x,y) f(y)\,dy\right| \\
&+ \left|\int_{R^n} \prod_{j=1}^{m} \bigl(b_j(y)-(b_j)_Q\bigr) K_t(x,y) f_1(y)\,dy\right| \\
&+ \left|\int_{R^n} \prod_{j=1}^{m} \bigl(b_j(y)-(b_j)_Q\bigr)\bigl(K(x,y)-K_t(x,y)\bigr) f_2(y)\,dy\right| \\
&= J_1(x)+J_2(x)+J_3(x)+J_4(x)+J_5(x)+J_6(x)+J_7(x).
\end{aligned}$$

For $J_1(x)$, same as $m = 1$, for some $1 < q < \infty$, let $1/q_1 + 1/q_2 + \cdots + 1/q_m + 1/q = 1$, $1/p + 1/p' = 1$, by Hölder's inequality, and the reverse of Hölder's inequality, we get

$$\begin{aligned}
&\frac{1}{\omega(Q)} \int_Q \bigl|J_1(x)\bigr| \omega(x)\,dx \\
&\le \frac{C}{\omega(Q)} \left(\int_Q \bigl|\bigl(b_1(x)-(b_1)_Q\bigr)\cdots\bigl(b_m(x)-(b_m)_Q\bigr)\bigr|^{p'} \omega(x)\,dx\right)^{1/p'} \\
&\quad \times \left(\int_Q \bigl|T(f)(x)\bigr|^p \omega(x)\,dx\right)^{1/p} \\
&\le \frac{C}{\omega(Q)} \left(\int_Q \bigl|b_1(x)-(b_1)_Q\bigr|^{p'} \cdots \bigl|b_m(x)-(b_m)_Q\bigr|^{p'} \omega(x)\,dx\right)^{1/p'} \\
&\quad \times \|f\|_{L^\infty(\omega)} \left(\int_Q \omega(x)\,dx\right)^{1/p} \\
&\le \frac{C}{\omega(Q)} \|f\|_{L^\infty(\omega)} \omega(Q)^{1/p} \prod_{j=1}^{m} \left[\left(\int_Q \bigl|b_j(x)-(b_j)_Q\bigr|^{p'q_j}\,dx\right)^{1/q_j} \left(\int_Q \omega(x)^q\,dx\right)^{1/q}\right]^{1/p'} \\
&\le C\|\vec{b}\|_{BMO} \|f\|_{L^\infty(\omega)} \omega(Q)^{1/p'+1/p-1} |Q|^{1/p'(1/q_1+\cdots+1/q_m+1/q-1)} \\
&\le C\|\vec{b}\|_{BMO} \|f\|_{L^\infty(\omega)}.
\end{aligned}$$

For $J_2(x)$, by Hölder's inequality and the reverse of Hölder's inequality, we have

$$\begin{aligned}
&\frac{1}{\omega(Q)} \int_Q \bigl|J_2(x)\bigr| \omega(x)\,dx \\
&\le \sum_{j=1}^{m-1} \sum_{\sigma\in C_j^m} \frac{C}{\omega(Q)} \left(\int_Q \bigl|\bigl(b(x)-b_Q\bigr)_\sigma\bigr|^{p'} \omega(x)\,dx\right)^{1/p'} \\
&\quad \times \left(\int_Q \bigl|T\bigl((b-b_Q)_{\sigma^c} f\bigr)(x)\bigr|^p \omega(x)\,dx\right)^{1/p} \\
&\le C \sum_{j=1}^{m-1} \sum_{\sigma\in C_j^m} \left(\frac{1}{\omega(Q)} \int_Q \bigl|\bigl(b(x)-b_Q\bigr)_\sigma\bigr|^{p'} \omega(x)\,dx\right)^{1/p'} \\
&\quad \times \left(\frac{1}{\omega(Q)} \int_Q \bigl|T\bigl((b-b_Q)_{\sigma^c} f\bigr)(x)\bigr|^p \omega(x)\,dx\right)^{1/p}
\end{aligned}$$

$$\begin{aligned}
&\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\omega(Q)^{-1/p'}\biggl[\biggl(\int_Q\bigl|(b(x)-b_Q)_\sigma\bigr|^{p'q'}\,dx\biggr)^{1/q'}\biggl(\int_Q\omega(x)^q\,dx\biggr)^{1/q}\biggr]^{1/p'}\\
&\quad\times\omega(Q)^{-1/p}\biggl(\int_{R^n}\bigl|(b(x)-b_Q)_{\sigma^c}f(x)\bigr|^p\omega(x)\chi_Q(x)\,dx\biggr)^{1/p}\\
&\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\omega(Q)^{-1/p'}|Q|^{1/p'q'+1/p'q-1/p'}\omega(Q)^{1/p'}\|\vec{b}_\sigma\|_{BMO}\\
&\quad\times\omega(Q)^{-1/p}\biggl(\int_Q\bigl|(b(x)-b_Q)_{\sigma^c}\bigr|^{pq'}\,dx\biggr)^{1/pq'}\biggl(\int_Q\bigl|f(x)\bigr|^{pq}\omega^q(x)\,dx\biggr)^{1/pq}\\
&\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\|\vec{b}_\sigma\|_{BMO}\|\vec{b}_{\sigma^c}\|_{BMO}\biggl(\frac{|Q|}{\omega(Q)}\biggr)^{1/p}\\
&\quad\times\biggl(\frac{1}{|Q|}\int_Q\omega(x)\,dx\biggr)^{1/p}\|f\|_{L^\infty(\omega)}\\
&\le C\|\vec{b}\|_{BMO}\|f\|_{L^\infty(\omega)}.
\end{aligned}$$

For $J_3(x)$, taking $p>1$, by the $L^p(\omega)$-boundedness of T, we have

$$\begin{aligned}
&\frac{1}{\omega(Q)}\int_Q\bigl|J_3(x)\bigr|\omega(x)\,dx\\
&\le\biggl(\frac{1}{\omega(Q)}\int_{R^n}\bigl|T\bigl((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_1\bigr)(x)\bigr|^p\omega(x)\,dx\biggr)^{1/p}\\
&\le C\omega(Q)^{-1/p}\biggl(\int_{R^n}\bigl|(b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)f_1(x)\bigr|^p\omega(x)\,dx\biggr)^{1/p}\\
&\le C\omega(Q)^{-1/p}|Q|^{1/pq'}\|\vec{b}\|_{BMO}|Q|^{1/pq}\biggl(\frac{1}{|Q|}\int_Q\omega^q\,dx\biggr)^{1/pq}\|f\|_{L^\infty(\omega)}\\
&\le C\|\vec{b}\|_{BMO}\biggl(\frac{|Q|}{\omega(Q)}\biggr)^{1/p}\biggl(\frac{\omega(Q)}{|Q|}\biggr)^{1/p}\|f\|_{L^\infty(\omega)}\\
&\le C\|\vec{b}\|_{BMO}\|f\|_{L^\infty(\omega)}.
\end{aligned}$$

For $J_4(x)$, $J_5(x)$, and $J_6(x)$, choose $1<p,q_j<\infty$, $j=1,\dots,m$, such that $1/p+1/q_1+\cdots+1/q_m+1/q$, by Lemma 2 and similar to the proofs of $J_1(x)$, $J_2(x)$, and $J_3(x)$, we get

$$\begin{aligned}
&\frac{1}{\omega(Q)}\int_Q\bigl|J_4(x)\bigr|\omega(x)\,dx\\
&\le C\frac{|Q|}{\omega(Q)}\prod_{j=1}^m\biggl(\frac{1}{|Q|}\int_Q\bigl|(b_j(x)-(b_j)_Q)\bigr|^{q_j}\,dx\biggr)^{1/q_j}\\
&\quad\times\biggl(\frac{1}{|Q|}\int_Q\bigl|A_{t_Q}(f)(x)\bigr|^p\,dx\biggr)^{1/p}\biggl(\frac{1}{|Q|}\int_Q\omega(x)^q\,dx\biggr)^{1/q}\\
&\le C\|\vec{b}\|_{BMO}\|f\|_{L^\infty(\omega)},\\
&\frac{1}{\omega(Q)}\int_Q\bigl|J_5(x)\bigr|\omega(x)\,dx
\end{aligned}$$

$$
\begin{aligned}
&\leq C\frac{|Q|}{\omega(Q)}\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\left(\frac{1}{|Q|}\int_Q\left|\left(b(x)-b_Q\right)_\sigma\right|^{q'}dx\right)^{1/q'}\\
&\quad\times\left(\frac{1}{|Q|}\int_Q\left|A_{t_Q}\left((b-b_Q)_{\sigma^c}f\right)(x)\right|^p dx\right)^{1/p}\left(\frac{1}{|Q|}\int_Q\omega(x)^q\,dx\right)^{1/q}\\
&\leq C\frac{|Q|}{\omega(Q)}\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\|\vec{b}_\sigma\|_{BMO}\|\vec{b}_{\sigma^c}\|_{BMO}\|f\|_{L^\infty(\omega)}\frac{\omega(Q)}{|Q|}\\
&\leq C\|\vec{b}\|_{BMO}\|f\|_{L^\infty(\omega)},\\
&\frac{1}{\omega(Q)}\int_Q\left|J_6(x)\right|\omega(x)\,dx\\
&\leq C\frac{|Q|}{\omega(Q)}\left(\frac{1}{|Q|}\int_Q\left|A_{t_Q}\left(\left(b_1-(b_1)_Q\right)\cdots\left(b_m-(b_m)_Q\right)f_1\right)(x)\right|^{q'}dx\right)^{1/q'}\\
&\quad\times\left(\frac{1}{|Q|}\int_Q\omega(x)^q\,dx\right)^{1/q}\\
&\leq C\|\vec{b}\|_{BMO}\|f\|_{L^\infty(\omega)}.
\end{aligned}
$$

For $J_7(x)$, note that $|x-y|\geq d=t^{1/2}$, taking $1<q_j<\infty$, $j=1,\ldots,m$ such that $1/q_1+\cdots+1/q_m+1/r=1$, then

$$
\begin{aligned}
J_7(x)&\leq C\sum_{k=0}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}\prod_{j=1}^m\left|\left(b_j(y)-(b_j)_Q\right)\right|\left|f(y)\right|\frac{d^\delta}{|x_0-y|^{n+\delta}}\,dy\\
&\leq C\sum_{k=1}^{\infty}\frac{d^\delta}{(2^{k-1}d)^{n+\delta}}\left|2^kQ\right|\left(\frac{1}{|2^kQ|}\int_{2^kQ}\left|f(y)\right|^r dy\right)^{1/r}\\
&\quad\times\prod_{j=1}^m\left(\frac{1}{|2^kQ|}\int_{2^kQ}\left|b_j(y)-(b_j)_Q\right|^{q_j}dy\right)^{1/q_j}\\
&\leq C\sum_{k=1}^{\infty}2^{-k\delta}\|f\|_{L^\infty(\omega)}\prod_{j=1}^m\left(\frac{1}{|2^kQ|}\int_{2^kQ}\left|b_j(y)-(b_j)_Q\right|^{q_j}dy\right)^{1/q_j}\\
&\leq C\sum_{k=1}^{\infty}k^m2^{-k\delta}\prod_{j=1}^m\|b_j\|_{BMO}\|f\|_{L^\infty(\omega)}\\
&\leq C\|\vec{b}\|_{BMO}\|f\|_{L^\infty(\omega)},
\end{aligned}
$$

so

$$
\frac{1}{\omega(Q)}\int_Q\left|J_7(x)\right|\omega(x)\,dx\leq C\|\vec{b}\|_{BMO}\|f\|_{L^\infty(\omega)}.
$$

This completes the proof of Theorem 1. □

Theorem 2 *Let $1<p<\infty$, $\omega\in A_1$ and $\vec{b}=(b_1,\ldots,b_m)$ with $b_j\in BMO(R^n)$ for $1\leq j\leq m$. Then T_b is bounded from $B_p(\omega)$ to $CMO_A(\omega)$.*

Proof It suffices to prove for $f \in C_0^\infty(R^n)$, the following inequality holds:

$$\frac{1}{\omega(Q)}\int_Q \big|T_b(f)(x) - A_{t_Q}T_b(f)(x)\big|\omega(x)\,dx \le C\|f\|_{B_p(\omega)}$$

for any cube $Q = Q(0,d)$ with $d > 1$. Fix a cube $Q = Q(0,d)$ with $d > 1$. Set $f_1 = f\chi_Q$, $f_2 = f\chi_{(R^n\setminus Q)}$ and $\vec{b}_Q = ((b_1)_Q,\ldots,(b_m)_Q) \in R^n$, where $(b_j)_Q = |Q|^{-1}\int_Q |b_j(y)|\,dy$, $1 \le j \le m$, we have

$$\begin{aligned}
&\big|T_b(f)(x) - A_{t_Q}T_b(f)(x)\big| \\
&\quad\le \left|\prod_{j=1}^m \big(b_j(x) - (b_j)_Q\big)\int_{R^n} K(x,y)f(y)\,dy\right| \\
&\qquad + \left|\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\big(b(x) - (b)_Q\big)_\sigma \int_{R^n}\big(b(y) - (b)_Q\big)_{\sigma^c}K(x,y)f(y)\,dy\right| \\
&\qquad + \left|\int_{R^n}\prod_{j=1}^m \big(b_j(y) - (b_j)_Q\big)K(x,y)f_1(y)\,dy\right| \\
&\qquad + \left|\prod_{j=1}^m \big(b_j(x) - (b_j)_Q\big)\int_{R^n} K_t(x,y)f(y)\,dy\right| \\
&\qquad + \left|\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\big(b(x) - (b)_Q\big)_\sigma \int_{R^n}\big(b(y) - (b)_Q\big)_{\sigma^c}K_t(x,y)f(y)\,dy\right| \\
&\qquad + \left|\int_{R^n}\prod_{j=1}^m \big(b_j(y) - (b_j)_Q\big)K_t(x,y)f_1(y)\,dy\right| \\
&\qquad + \left|\int_{R^n}\prod_{j=1}^m \big(b_j(y) - (b_j)_Q\big)\big(K(x,y) - K_t(x,y)\big)f_2(y)\,dy\right| \\
&\quad= L_1(x) + L_2(x) + L_3(x) + L_4(x) + L_5(x) + L_6(x) + L_7(x).
\end{aligned}$$

For $L_1(x)$, we have

$$\begin{aligned}
&\frac{1}{\omega(Q)}\int_Q \big|L_1(x)\big|\omega(x)\,dx \\
&\quad\le \frac{C}{\omega(Q)}\left(\int_Q \big|\big(b_1(x) - (b_1)_Q\big)\cdots\big(b_m(x) - (b_m)_Q\big)\big|^{p'}\omega(x)\,dx\right)^{1/p'} \\
&\qquad\times\left(\int_Q \big|T(f)(x)\big|^p\omega(x)\,dx\right)^{1/p} \\
&\quad\le \frac{C}{\omega(Q)}\left[\left(\int_Q \big|\big(b_1(x) - (b_1)_Q\big)\cdots\big(b_m(x) - (b_m)_Q\big)\big|^{p'q'}dx\right)^{1/q'}\left(\int_Q \omega(x)^q\,dx\right)^{1/q}\right]^{1/p'} \\
&\qquad\times\left(\int_Q \big|f(x)\big|^p\omega(x)\,dx\right)^{1/p} \\
&\quad\le \frac{C}{\omega(Q)}|Q|^{1/p'q'}\|\vec{b}\|_{BMO}|Q|^{1/p'q}\left(\frac{\omega(Q)}{|Q|}\right)^{1/p'}\|f\chi_Q\|_{L^p(\omega)}
\end{aligned}$$

$$\begin{aligned}
&\le C\|\vec{b}\|_{BMO}\omega(Q)^{-1/p}\|f\chi_Q\|_{L^p(\omega)} \\
&\le C\|\vec{b}\|_{BMO}\|f\|_{B_p(\omega)}.
\end{aligned}$$

For $L_2(x)$, taking $1 < s, s' < \infty$, and $1/s + 1/s' = 1$, we have

$$\begin{aligned}
&\frac{1}{\omega(Q)}\int_Q \left|L_2(x)\right|\omega(x)\,dx \\
&\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\left(\frac{1}{\omega(Q)}\int_Q\left|\left(b(x)-b_Q\right)_\sigma\right|^{s'}\omega(x)\,dx\right)^{1/s'} \\
&\quad\times\left(\frac{1}{\omega(Q)}\int_Q\left|T\left((b-b_Q)_{\sigma^c}f\right)(x)\right|^s\omega(x)\,dx\right)^{1/s} \\
&\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\omega(Q)^{-1/s'}\left[\left(\left(\int_Q\left|\left(b(x)-b_Q\right)_\sigma\right|^{s'q'}dx\right)^{1/q'}\left(\int_Q\omega^q\,dx\right)^{1/q}\right]^{1/s'} \\
&\quad\times\omega(Q)^{-1/s}\left(\int_Q\left|\left(b(x)-b_Q\right)_{\sigma^c}f(x)\right|^s\omega(x)\,dx\right)^{1/s} \\
&\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\omega(Q)^{-1/s'}|Q|^{1/s'q'+1/s'q-1/s'}\omega(Q)^{1/s'}\|\vec{b}_\sigma\|_{BMO} \\
&\quad\times\omega(Q)^{-1/s}|Q|^{1/rs}\|\vec{b}_{\sigma^c}\|_{BMO}\left(\int_Q\left|f(x)\right|^p\omega(x)\,dx\right)^{1/p}\left(\int_Q\omega(x)^q\,dx\right)^{(p-s)/pqs} \\
&\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\|\vec{b}_\sigma\|_{BMO}\|\vec{b}_{\sigma^c}\|_{BMO}\omega(Q)^{-1/p}\|f\chi_Q\|_{L^p(\omega)} \\
&\le C\|\vec{b}\|_{BMO}\|f\|_{B_p(\omega)}.
\end{aligned}$$

For $L_3(x)$, we have

$$\begin{aligned}
&\frac{1}{\omega(Q)}\int_Q\left|L_3(x)\right|\omega(x)\,dx \\
&\le C\left(\frac{1}{\omega(Q)}\int_{R^n}\left|T\left(\left(b_1-(b_1)_Q\right)\cdots\left(b_m-(b_m)_Q\right)f_1\right)(x)\right|^s\omega(x)\,dx\right)^{1/s} \\
&\le C\omega(Q)^{-1/s}\left(\int_Q\left|\left(b_1(x)-(b_1)_Q\right)\cdots\left(b_m(x)-(b_m)_Q\right)f(x)\right|^s\omega(x)\,dx\right)^{1/s} \\
&\le C\omega(Q)^{-1/p}\|\vec{b}\|_{BMO}\|f\chi_Q\|_{L^p(\omega)} \\
&\le C\|\vec{b}\|_{BMO}\|f\|_{B_p(\omega)}.
\end{aligned}$$

For $L_4(x)$, $L_5(x)$, and $L_6(x)$, by Lemma 2, we have

$$\begin{aligned}
&\frac{1}{\omega(Q)}\int_Q\left|L_4(x)\right|\omega(x)\,dx \\
&\le C\left(\frac{1}{\omega(Q)}\int_Q\left|\left(b_1(x)-(b_1)_Q\right)\cdots\left(b_m(x)-(b_m)_Q\right)\right|^{s'}\omega(x)\,dx\right)^{1/s'}
\end{aligned}$$

$$\begin{aligned}
&\times \left(\frac{1}{\omega(Q)} \int_Q \left| A_{t_Q}(f)(x) \right|^s \omega(x)\, dx \right)^{1/s} \\
&\leq C \left(\frac{1}{\omega(Q)} \right)^{1/s'} \left[\left(\int_Q \left| \left(b_1(x) - (b_1)_Q\right) \cdots \left(b_m(x) - (b_m)_Q\right) \right|^{s'q'} dx \right)^{1/q'} \right. \\
&\quad \times \left. \left(\int_Q \omega(x)^q \, dx \right)^{1/q} \right]^{1/s'} \|f\|_{B_p(w)} \\
&\leq C \left(\frac{1}{\omega(Q)} \right)^{1/s'} |Q|^{1/s'q'} \|\vec{b}\|_{BMO} |Q|^{1/s'q} \left(\frac{\omega(Q)}{|Q|} \right)^{1/s'} \|f\|_{B_p(w)} \\
&\leq C \|\vec{b}\|_{BMO} \|f\|_{B_p(\omega)};
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{\omega(Q)} \int_Q \left| L_5(x) \right| \omega(x)\, dx \\
&\leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \left(\frac{1}{\omega(Q)} \int_Q \left| \left(b(x) - b_Q\right)_\sigma \right|^{s'} \omega(x)\, dx \right)^{1/s'} \\
&\quad \times \left(\frac{1}{\omega(Q)} \int_Q \left| A_{t_Q}\left((b - b_Q)_{\sigma^c} f\right)(x) \right|^s \omega(x)\, dx \right)^{1/s} \\
&\leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \omega(Q)^{-1/s'} \left[\left(\int_Q \left| \left(b(x) - b_Q\right)_\sigma \right|^{s'q'} dx \right)^{1/q'} \left(\int_Q \omega^q \, dx \right)^{1/q} \right]^{1/s'} \\
&\quad \times \|\vec{b}_{\sigma^c}\|_{BMO} \|f\|_{B_p(\omega)} \\
&\leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \omega(Q)^{-1/s'} |Q|^{1/s'q' + 1/s'q - 1/s'} \omega(Q)^{1/s'} \|\vec{b}_\sigma\|_{BMO} \|\vec{b}_{\sigma^c}\|_{BMO} \|f\|_{B_p(\omega)} \\
&\leq C \|\vec{b}_\sigma\|_{BMO} \|f\|_{B_p(\omega)};
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{\omega(Q)} \int_Q \left| L_6(x) \right| \omega(x)\, dx \\
&\leq \left(\frac{1}{\omega(Q)} \int_{R^n} \left| A_{t_Q}\left(\left(b_1 - (b_1)_Q\right) \cdots \left(b_m - (b_m)_Q\right) f_1 \right)(x) \right|^s \omega(x)\, dx \right)^{1/s} \\
&\leq C \|\vec{b}\|_{BMO} \|f\|_{B_p(\omega)}.
\end{aligned}$$

For $L_7(x)$, note that $|x - y| \geq d = t^{1/2}$, taking $1 < u < p$, then

$$\begin{aligned}
L_7(x) &\leq C \int_{Q^c} \prod_{j=1}^m \left| b_j(y) - (b_j)_Q \right| \left| f(y) \right| \frac{d^\delta}{|x_0 - y|^{n+\delta}} \, dy \\
&\leq C \sum_{k=0}^\infty \int_{2^{k+1}Q \setminus 2^k Q} \prod_{j=1}^m \left| b_j(y) - (b_j)_Q \right| \left| f(y) \right| \frac{d^\delta}{|x_0 - y|^{n+\delta}} \, dy \\
&\leq C \sum_{k=1}^\infty \frac{d^\delta}{(2^{k-1} d)^{n+\delta}} \left| 2^k Q \right| \left(\frac{1}{|2^k Q|} \int_{2^k Q} \left| f(y) \right|^u dy \right)^{1/u} \\
&\quad \times \left(\frac{1}{|2^k Q|} \int_{2^k Q} \prod_{j=1}^m \left| b_j(y) - (b_j)_Q \right|^{u'} dy \right)^{1/u'}
\end{aligned}$$

$$
\begin{aligned}
&\le C\|\vec{b}\|_{BMO}\sum_{k=1}^{\infty}k^m 2^{-k\delta}\left(\frac{1}{|2^kQ|}\right)^{1/u} \\
&\quad\times\left[\left(\int_{2^kQ}\left|f(y)\right|^p\omega(y)\,dy\right)^{\frac{u}{p}}\left(\int_{2^kQ}\omega(y)^{-\frac{u}{p-u}}\,dy\right)^{\frac{p-u}{p}}\right]^{1/u} \\
&\le C\|\vec{b}\|_{BMO}\sum_{k=1}^{\infty}k^m 2^{-k\delta}\left(\frac{1}{|2^kQ|}\right)^{1/u}\|f\chi_{2^kQ}\|_{L^p(\omega)}\left(\frac{\omega(2^kQ)}{|2^kQ|}\right)^{-1/p}\left|2^kQ\right|^{(\frac{p}{u}-1)\frac{1}{p}} \\
&\quad\times\left[\left(\frac{1}{|2^kQ|}\int_{2^kQ}\omega(y)\,dy\right)\left(\frac{1}{|2^kQ|}\int_{2^kQ}\omega(y)^{-\frac{1}{\frac{p}{u}-1}}\,dy\right)^{\frac{p}{u}-1}\right]^{1/p} \\
&\le C\|\vec{b}\|_{BMO}\sum_{k=1}^{\infty}k^m 2^{-k\delta}\omega\left(2^kQ\right)^{-1/p}\|f\chi_{2^kQ}\|_{L^p(\omega)} \\
&\le C\|\vec{b}\|_{BMO}\|f\|_{B_p(\omega)},
\end{aligned}
$$

so

$$
\frac{1}{\omega(Q)}\int_Q\left|L_7(x)\right|\omega(x)\,dx\le C\|\vec{b}\|_{BMO}\|f\|_{B_p(\omega)}.
$$

This completes the proof of Theorem 2. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors completed the paper, and read and approved the final manuscript.

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