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Global Poincaré inequalities for the composition of the sharp maximal operator and Green's operator with Orlicz norms

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Abstract

In this paper, we establish the global Poincaré-type inequalities for the composition of the sharp maximal operator and Green's operator with Orlicz norm.

Keywords: Poincaré-type inequalities; Orlicz norm; sharp maximal operator; Green's operator

1 Introduction

The L^p-theory of solutions of the homogeneous A-harmonic equation $d^\star A(x,du) = 0$ for differential forms u has been very well developed in recent years. Many L^p-norm estimates and inequalities, including the Poincaré inequalities, for solution of the homogeneous A-harmonic equation have been established; see [1, 2]. The Poincaré inequalities for differential forms is an important tool in analysis and related fields, including partial differential equations and potential theory. However, the study of the nonhomogeneous A-harmonic equations $d^\star A(x,du) = B(x,du)$ has just begun [2–4]. In this paper, we focus on a class of differential forms satisfying the well-known nonhomogeneous A-harmonic equation $d^\star A(x,du) = B(x,du)$.

Let us first introduce some necessary notation and terminology. Ω will refer to a bounded, convex domain in $\mathbb{R}^n$ unless otherwise stated and B is a ball in $\mathbb{R}^n$, $n \geq 2$. We use σB to denote the ball with the same center as B and with $\operatorname{diam}(\sigma B) = \sigma \operatorname{diam}(B)$, $\sigma > 0$. We do not distinguish the balls from cubes in this paper. We use $|E|$ to denote the n-dimensional Lebesgue measure of the set $E \subseteq \mathbb{R}^n$. We say w is a weight if $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $w > 0$ a.e. For a function u, we denote the average of u over B by

$$u_B = \frac{1}{|B|}\int_B u\,dx,$$

where $|B|$ is the volume of B and the μ-average of u over B by

$$u_{B,\mu} = \frac{1}{\mu(B)}\int_B u\,d\mu.$$

Let $\wedge^l = \wedge^l(\mathbb{R}^n)$ be the set of all l-forms in $\mathbb{R}^n$, let $D'(\Omega,\wedge^l)$ be the space of all differential l-forms on Ω, and let $L^p(\Omega,\wedge^l)$ be the l-forms $u(x) = \sum_I u_I(x)\,dx_I$ on Ω satisfying

$\int_\Omega |u_I|^p\,dx < \infty$ for all ordered l-tuples I, $l = 1, 2, \ldots, n$. We denote the exterior derivative by $d : D'(\Omega, \wedge^l) \to D'(\Omega, \wedge^{l+1})$ for $l = 0, 1, \ldots, n-1$, and define the Hodge star operator $\star : \wedge^k \to \wedge^{n-k}$ as follows. If $u = u_I\,dx_I$, $i_1 < i_2 < \cdots < i_k$, is a differential k-form, then $\star u = (-1)^{\sum(I)} u_I\,dx_J$, where $I = (i_1, i_2, \ldots, i_k)$, $J = \{1, 2, \ldots, n\} - I$, and $\sum(I) = \frac{k(k+1)}{2} + \sum_{j=1}^k i_j$. The Hodge codifferential operator

$$d^\star : D'(\Omega, \wedge^{l+1}) \to D'(\Omega, \wedge^l)$$

is given by $d^\star = (-1)^{nl+1} \star d \star$ on $D'(\Omega, \wedge^{l+1})$, $l = 0, 1, \ldots, n-1$. We write

$$\|u\|_{s,\Omega} = \left(\int_\Omega |u|^s\,dx \right)^{1/s}.$$

The well-known nonhomogeneous A-harmonic equation is

$$d^\star A(x, du) = B(x, du), \tag{1}$$

where $A : \Omega \times \wedge^l(\mathbb{R}^n) \to \wedge^l(\mathbb{R}^n)$ and $B : \Omega \times \wedge^l(\mathbb{R}^n) \to \wedge^{l-1}(\mathbb{R}^n)$ satisfy the conditions:

$$\left|A(x,\xi)\right| \le a|\xi|^{p-1}, \qquad A(x,\xi)\cdot\xi \ge |\xi|^p, \qquad \left|B(x,\xi)\right| \le b|\xi|^{p-1} \tag{2}$$

for almost every $x \in \Omega$ and all $\xi \in \wedge^l(\mathbb{R}^n)$. Here, $a, b > 0$ are constants and $1 < p < \infty$ is a fixed exponent associated with (1). If the operator $B = 0$, equation (1) becomes $d^\star A(x, du) = 0$, which is called the (homogeneous) A-harmonic equation. A solution to (1) is an element of the Sobolev space $W^{1,p}_{\mathrm{loc}}(\Omega, \wedge^{l-1})$ such that $\int_\Omega A(x, du)\cdot d\varphi + B(x, du)\cdot\varphi = 0$ for all $\varphi \in W^{1,p}_{\mathrm{loc}}(\Omega, \wedge^{l-1})$ with compact support. Let $A : \Omega \times \wedge^l(\mathbb{R}^n) \to \wedge^l(\mathbb{R}^n)$ be defined by $A(x,\xi) = \xi|\xi|^{p-2}$ with $p > 1$. Then A satisfies the required conditions and $d^\star A(x, du) = 0$ becomes the p-harmonic equation

$$d^\star\left(du|du|^{p-2}\right) = 0 \tag{3}$$

for differential forms. If u is a function (0-form), equation (3) reduces to the usual p-harmonic equation $\operatorname{div}(\nabla u|\nabla u|^{p-2}) = 0$ for functions. A remarkable progress has been made recently in the study of different versions of the harmonic equations, see [1] for more details.

Let $C^\infty(\Omega, \wedge^l)$ be the space of smooth l-forms on Ω and

$$\mathcal{W}(\Omega, \wedge^l) = \left\{u \in L^1_{\mathrm{loc}}(\Omega, \wedge^l) : u \text{ has generalized gradient}\right\}.$$

The harmonic l-fields are defined by

$$\mathcal{H}(\Omega, \wedge^l) = \left\{u \in \mathcal{W}(\Omega, \wedge^l) : du = d^\star u = 0, u \in L^p \text{ for some } 1 < p < \infty\right\}.$$

The orthogonal complement of $\mathcal{H}$ in L^1 is defined by

$$\mathcal{H}^\perp = \left\{u \in L^1 : \langle u, h\rangle = 0 \text{ for all } h \in \mathcal{H}\right\}.$$

Then Green's operator G is defined as

$$G : C^{\infty}(\Omega, \wedge^{l}) \to \mathcal{H}^{\perp} \cap C^{\infty}(\Omega, \wedge^{l})$$

by assigning $G(u)$ to be the unique element of $\mathcal{H}^{\perp} \cap C^{\infty}(\Omega, \wedge^{l})$ satisfying Poisson's equation $\Delta G(u) = u - H(u)$, where H is the harmonic projection operator that maps $C^{\infty}(\Omega, \wedge^{l})$ onto $\mathcal{H}$ so that $H(u)$ is the harmonic part of u. See [5] for more properties of these operators.

In harmonic analysis, a fundamental operator is the Hardy-Littlewood maximal operator. The maximal function is a classical tool in harmonic analysis but recently it has been successfully used in studying Sobolev functions and partial differential equations. For any locally L^s-integrable form $u(y)$, we define the Hardy-Littlewood maximal operator $\mathcal{M}_s$ by

$$\mathcal{M}_s(u) = \mathcal{M}_s(u)(x) = \sup_{r>0} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |u(y)|^s \, dy \right)^{\frac{1}{s}}, \tag{4}$$

where $B(x,r)$ is the ball of radius r, centered at x, $1 \le s < \infty$. We write $\mathcal{M}(u) = \mathcal{M}_1(u)$ if $s = 1$. Similarly, for a locally L^s-integrable form $u(y)$, we define the sharp maximal operator $\mathcal{M}_s^{\#}$ by

$$\mathcal{M}_s^{\#}(u) = \mathcal{M}_s^{\#}(u)(x) = \sup_{r>0} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |u(y) - u_{B(x,r)}|^s \, dy \right)^{\frac{1}{s}}. \tag{5}$$

Some interesting results about these operators have been established, see [3, 4] and [6] for more details.

The purpose of this paper is to estimate the global Poincaré-type inequalities for the composition of the sharp maximal operator and Green's operator with Orlicz norm.

2 Definitions and lemmas

We now introduce the following definition and lemmas that will be used in this paper.

Definition 1 We say the weight $w(x)$ satisfies the $A_r(\Omega)$ condition, $r > 1$, write $w \in A_r(\Omega)$ if $w(x) > 1$ a.e., and

$$\sup_{B} \left(\frac{1}{|B|} \int_B w \, dx \right) \left(\frac{1}{|B|} \int_B \left(\frac{1}{w} \right)^{\frac{1}{r-1}} dx \right)^{r-1} < \infty \tag{6}$$

for any ball $B \subset \Omega$.

Definition 2 A proper subdomain $\Omega \subset \mathbb{R}^n$ is called a δ-John domain, $\delta > 0$, if there exists a point $x_0 \in \Omega$ which can be joined with any other point $x \in \Omega$ by a continuous curve $\gamma \subset \Omega$ so that

$$d(\xi, \partial\Omega) \ge \delta |x - \xi|$$

for each $\xi \in \gamma$. Here $d(\xi, \partial\Omega)$ is the Euclidean distance between ξ and $\partial\Omega$.

Lemma 1 [7] *Each Ω has a modified Whitney cover of cubes $\mathcal{V} = \{Q_i\}$ such that*

$$\bigcup_i Q_i = \Omega, \qquad \sum_{Q_i \in \mathcal{V}} \chi_{\sqrt{\frac{5}{4}}Q_i} \le N\chi_\Omega$$

and some $N > 1$, and if $Q_i \cap Q_j \neq \emptyset$, then there exists a cube R (this cube need not be a member of $\mathcal{V}$) in $Q_i \cap Q_j$ such that $Q_i \cup Q_j \subset NR$. Moreover, if Ω is δ-John, then there is a distinguished cube $Q_0 \in \mathcal{V}$ which can be connected with every cube $Q \in \mathcal{V}$ by a chain of cubes $Q_0 = Q_{j_0}, Q_{j_1}, \ldots, Q_{j_k} = Q$ from $\mathcal{V}$ and such that $Q \subset \rho Q_{j_i}$, $i = 0, 1, 2, \ldots, k$, for some $\rho = \rho(n, \delta)$.

3 Poincaré inequalities

In this section, we prove the global Poincaré inequalities for the composition of the sharp maximal operator and Green's operator with L^p norm.

To get our result, we rewrite our Theorem 2 in [4] as follows.

Lemma 2 *Let u be a smooth differential form satisfying A-harmonic equation* (1) *in a bounded domain Ω, let G be Green's operator, and let $\mathcal{M}_s^\sharp$ be the sharp maximal operator defined in* (4) *with $1 < s \le p, q < \infty$. Then there exists a constant C, independent of u, such that*

$$\left(\int_B \left|\mathcal{M}_s^\sharp(G(u)) - \mathcal{M}_s^\sharp(G(u))_B\right|^q d\mu\right)^{1/q} \le C(\delta, \Omega)|B|^{1+\frac{1}{n}-\frac{1}{p}+\frac{1}{q}}\left(\int_{\sigma B} |u|^p d\mu\right)^{1/p}$$

for all balls B with $\sigma B \subset \Omega$, and a constant $\sigma > 1$, where the measure μ is defined by $d\mu = w(x)dx$ and $w(x) \in A_r(\Omega)$ with $w \ge \delta > 0$ for some $r > 1$ and a constant δ.

Theorem 1 *Let $u \in L^t_{\mathrm{loc}}(\Omega, \wedge^l)$, $l = 1, 2, \ldots, n$, be a smooth differential form satisfying A-harmonic equation* (1), *let G be Green's operator, and let $\mathcal{M}_s^\sharp$ be the sharp maximal operator defined in* (4) *with $1 < s < t < \infty$. Then there exists a constant $C(n, t, \delta_0, N, \Omega)$, independent of u, such that*

$$\left(\int_\Omega \left|\mathcal{M}_s^\sharp(G(u)) - \left(\mathcal{M}_s^\sharp(G(u))\right)_{Q_0}\right|^t d\mu\right)^{1/t} \le C(n, t, \delta_0, N, \Omega)\left(\int_\Omega |u|^t d\mu\right)^{1/t} \tag{7}$$

for any bounded and convex δ-John domain $\Omega \subset \mathbb{R}^n$, where the fixed cube $Q_0 \subset \Omega$, the constant $N > 1$ appeared in Lemma 1, *and the measure μ is defined by $d\mu = w(x)\,dx$ and $w(x) \in A_r(\Omega)$ with $w \ge \delta_0 > 0$ for some $r > 1$ and a constant δ_0.*

Proof First, we use Lemma 1 for the bounded and convex δ-John domain Ω. There is a modified Whitney cover of cubes $\mathcal{V} = \{Q_i\}$ for Ω such that $\Omega = \bigcup Q_i$, and $\sum_{Q_i \in \mathcal{V}} \chi_{\sqrt{\frac{5}{4}}Q_i} \le N\chi_\Omega$ for some $N > 1$. Moreover, there is a distinguished cube $Q_0 \in \mathcal{V}$ which can be connected with every cube $Q \in \mathcal{V}$ by a chain of cubes $Q_0 = Q_{j_0}, Q_{j_1}, \ldots, Q_{j_k} = Q$ from $\mathcal{V}$ and such that $Q \subset \rho Q_{j_i}$, $i = 0, 1, 2, \ldots, k$, for some $\rho = \rho(n, \delta)$. Then, by the elementary inequality $(a + b)^t \le 2^t(|a|^t + |b|^t)$, $t \ge 0$, we have

$$\begin{aligned}
&\left(\int_{\Omega} \left|\mathcal{M}_s^{\sharp}(G(u)) - \left(\mathcal{M}_s^{\sharp}(G(u))\right)_{Q_0}\right|^t d\mu\right)^{1/t} \\
&\le \left(\sum_{Q_i \in \mathcal{V}} \left(2^t \int_{Q_i} \left|\mathcal{M}_s^{\sharp}(G(u)) - \left(\mathcal{M}_s^{\sharp}(G(u))\right)_{Q_i}\right|^t d\mu\right.\right. \\
&\quad \left.\left. + 2^t \int_{Q_i} \left|\left(\mathcal{M}_s^{\sharp}(G(u))\right)_{Q_i} - \left(\mathcal{M}_s^{\sharp}(G(u))\right)_{Q_0}\right|^t d\mu\right)\right)^{1/t} \\
&\le C_1(t)\left(\left(\sum_{Q_i \in \mathcal{V}} \int_{Q_i} \left|\mathcal{M}_s^{\sharp}(G(u)) - \left(\mathcal{M}_s^{\sharp}(G(u))\right)_{Q_i}\right|^t d\mu\right)^{1/t}\right. \\
&\quad \left. + \left(\sum_{Q_i \in \mathcal{V}} \int_{Q_i} \left|\left(\mathcal{M}_s^{\sharp}(G(u))\right)_{Q_i} - \left(\mathcal{M}_s^{\sharp}(G(u))\right)_{Q_0}\right|^t d\mu\right)^{1/t}\right). \qquad (8)
\end{aligned}$$

The first sum in (8) can be estimated by using Lemma 2.

$$\begin{aligned}
&\sum_{Q_i \in \mathcal{V}} \int_{Q_i} \left|\mathcal{M}_s^{\sharp}(G(u)) - \left(\mathcal{M}_s^{\sharp}(G(u))\right)_{Q_i}\right|^t d\mu \\
&\le C_2(n, t, \delta_0, \Omega) \sum_{Q_i \in \mathcal{V}} \int_{\rho_i Q_i} |u|^t d\mu \\
&\le C_3(n, t, \delta_0, \Omega) \sum_{Q_i \in \mathcal{V}} \int_{\Omega} |u|^t d\mu \\
&\le C_4(n, t, N, \delta_0, \Omega) \int_{\Omega} |u|^t d\mu, \qquad (9)
\end{aligned}$$

where the measure μ is defined by $d\mu = w(x)\,dx$ and $w(x) \in A_r(\Omega)$ with $w \ge \delta_0 > 0$ for some r > 1 and a constant δ_0.

To estimate the second sum in (8), we need to use the property of δ-John domain. Fix a cube $Q_i \in \mathcal{V}$ and let $Q_0 = Q_{j_0}, Q_{j_1}, \ldots, Q_{j_k} = Q_i$ be the chain in Lemma 1. Then we have

$$\left|\left(\mathcal{M}_s^{\sharp}(G(u))\right)_{Q_i} - \left(\mathcal{M}_s^{\sharp}(G(u))\right)_{Q_0}\right| \le \sum_{i=0}^{k-1} \left|\left(\mathcal{M}_s^{\sharp}(G(u))\right)_{Q_{j_i}} - \left(\mathcal{M}_s^{\sharp}(G(u))\right)_{Q_{j_{i+1}}}\right|. \qquad (10)$$

The chain $\{Q_{j_i}\}$ also has the property that for each i, $i = 0, 1, \ldots, k-1$, $Q_{j_i} \cap Q_{j_{i+1}} \ne \emptyset$. Thus, there exists a cube D_i such that $D_i \subset Q_{j_i} \cap Q_{j_{i+1}}$ and $Q_{j_i} \cup Q_{j_{i+1}} \subset ND_i$, $N > 1$. So,

$$\frac{\max\{|Q_{j_i}|, |Q_{j_{i+1}}|\}}{|Q_{j_i} \cap Q_{j_{i+1}}|} \le \frac{\max\{|Q_{j_i}|, |Q_{j_{i+1}}|\}}{|D_i|} \le N. \qquad (11)$$

Note that

$$\mu(Q) = \int_Q d\mu = \int_Q w(x)\,dx \ge \int_Q \delta_0\,dx = \delta_0|Q|. \qquad (12)$$

By (11), (12) and Lemma 2, we have

$$\begin{aligned}
&\bigl|\bigl(\mathcal{M}_s^\sharp(G(u))\bigr)_{Q_{j_i}} - \bigl(\mathcal{M}_s^\sharp(G(u))\bigr)_{Q_{j_{i+1}}}\bigr|^t \\
&= \frac{1}{\mu(Q_{j_i}\cap Q_{j_{i+1}})}\int_{Q_{j_i}\cap Q_{j_{i+1}}} \bigl|\bigl(\mathcal{M}_s^\sharp(G(u))\bigr)_{Q_{j_i}} - \bigl(\mathcal{M}_s^\sharp(G(u))\bigr)_{Q_{j_{i+1}}}\bigr|^t\,d\mu \\
&\le \frac{1}{\delta_0|Q_{j_i}\cap Q_{j_{i+1}}|}\int_{Q_{j_i}\cap Q_{j_{i+1}}} \bigl|\bigl(\mathcal{M}_s^\sharp(G(u))\bigr)_{Q_{j_i}} - \bigl(\mathcal{M}_s^\sharp(G(u))\bigr)_{Q_{j_{i+1}}}\bigr|^t\,d\mu \\
&\le \frac{N}{\delta_0\max\{|Q_{j_i}|,|Q_{j_{i+1}}|\}}\int_{Q_{j_i}\cap Q_{j_{i+1}}} \bigl|\bigl(\mathcal{M}_s^\sharp(G(u))\bigr)_{Q_{j_i}} - \bigl(\mathcal{M}_s^\sharp(G(u))\bigr)_{Q_{j_{i+1}}}\bigr|^t\,d\mu \\
&\le C_5(n,t,\delta_0,N,\Omega)\sum_{k=i}^{i+1}\frac{1}{|Q_{j_k}|}\int_{Q_{j_k}} \bigl|\mathcal{M}_s^\sharp(G(u)) - \bigl(\mathcal{M}_s^\sharp(G(u))\bigr)_{Q_{j_k}}\bigr|^t\,d\mu \\
&\le C_6(n,t,\delta_0,N,\Omega)\sum_{k=i}^{i+1}\frac{|Q_{j_k}|^{1+\frac{1}{n}}}{|Q_{j_k}|}\int_{\sigma_{j_k}Q_{j_k}}|u|^t\,d\mu \\
&= C_6(n,t,\delta_0,N,\Omega)\sum_{k=i}^{i+1}|Q_{j_k}|^{\frac{1}{n}}\int_{\sigma_{j_k}Q_{j_k}}|u|^t\,d\mu \\
&\le C_7(n,t,\delta_0,N,\Omega)\sum_{k=i}^{i+1}|\Omega|^{\frac{1}{n}}\int_{\Omega}|u|^t\,d\mu \\
&\le C_8(n,t,\delta_0,N,\Omega)\sum_{Q_i\in\mathcal{V}}\int_{\Omega}|u|^t\,d\mu \\
&\le C_9(n,t,\delta_0,N,\Omega)\int_{\Omega}|u|^t\,d\mu.
\end{aligned}\tag{13}$$

Then, by (10), (13) and the elementary inequality $|\sum_{i=1}^M t_i|^s \le M^{s-1}\sum_{i=1}^M |t_i|^s$, we finally obtain

$$\begin{aligned}
&\sum_{Q_i\in\mathcal{V}}\int_{Q_i}\bigl|\bigl(\mathcal{M}_s^\sharp(G(u))\bigr)_{Q_i} - \bigl(\mathcal{M}_s^\sharp(G(u))\bigr)_{Q_0}\bigr|^t\,d\mu \\
&\le C_{10}(n,t,\delta_0,N,\Omega)\sum_{Q_i\in\mathcal{V}}\int_{Q_i}\biggl(\int_{\Omega}|u|^t\,d\mu\biggr)d\mu \\
&= C_{10}(n,t,\delta_0,N,\Omega)\biggl(\sum_{Q_i\in\mathcal{V}}\int_{Q_i}d\mu\biggr)\int_{\Omega}|u|^t\,d\mu \\
&\le C_{11}(n,t,\delta_0,N,\Omega)\biggl(\int_{\Omega}d\mu\biggr)\int_{\Omega}|u|^t\,d\mu \\
&= C_{11}(n,t,\delta_0,N,\Omega)\mu(\Omega)\int_{\Omega}|u|^t\,d\mu \\
&= C_{12}(n,t,\delta_0,N,\Omega)\int_{\Omega}|u|^t\,d\mu.
\end{aligned}\tag{14}$$

Substituting (9) and (14) in (8), we have completed the proof of Theorem 1. □

4 Poincaré inequality with Orlicz norm

In this section, we give a global Poincaré inequality with Orlicz norm for the composition of the sharp maximal operator and Green's operator.

Definition 3 Let φ be a continuously increasing convex function on $[0,\infty)$ with $\varphi(0) = 0$, and let Λ be a domain with $\mu(\Lambda) < \infty$. If u is a measurable function in Λ, then we define the Orlicz norm of u by

$$\|u\|_{L(\varphi,\Lambda,\mu)} = \inf\left\{k > 0 : \frac{1}{\mu(\Lambda)}\int_\Lambda \varphi\left(\frac{|u(x)|}{k}\right) d\mu \leq 1\right\}. \tag{15}$$

A continuously increasing function $\psi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$ is called an Orlicz function. A convex Orlicz function φ is often called a Young function.

In [8], Buckley and Koskela gave the following class of functions.

Definition 4 We say a Young function φ lies in the class $G(p,q,C)$, $1 \leq p < q < \infty$, $C \geq 1$, if (i) $1/C \leq \varphi(t^{1/p})/g(t) \leq C$ and (ii) $1/C \leq \varphi(t^{1/q})/h(t) \leq C$ for all $t > 0$, where g is a convex increasing function and h is a concave increasing function on $[0,\infty)$.

From [8] and [9], we know that the class $G(p,q,C)$ contains some very interesting functions, such as $\varphi(t) = t^p$ and $\varphi(t) = t^p \log_+^\alpha(t)$, $p \geq 1$, $\alpha \in \mathbb{R}$, and each of φ, g and h is doubling in the sense that its values at t and $2t$ are uniformly comparable for all $t > 0$, and the consequent fact that

$$C_1 t^q \leq h^{-1}(\varphi(t)) \leq C_2 t^q, \qquad C_1 t^p \leq g^{-1}(\varphi(t)) \leq C_2 t^p, \tag{16}$$

where C_1 and C_2 are constants.

Now, we are ready to give our another global Poincaré inequality with Orlicz norm.

Theorem 2 *Let φ be a Young function in the class $G(p,q,C_0)$, $1 \leq p < q < \infty$, $C_0 \geq 1$, let $u \in L^t_{\mathrm{loc}}(\Omega, \wedge^l)$, $l = 1,2,\ldots,n$, be a smooth differential form satisfying A-harmonic equation* (1) *in Ω, let G be Green's operator, and let $\mathcal{M}_s^\sharp$ be the sharp maximal operator defined in* (4) *with $1 < s \leq t < \infty$. Then there exists a constant C, independent of u, such that*

$$\left\|\mathcal{M}_s^\sharp(G(u)) - \mathcal{M}_s^\sharp(G(u))_{Q_0}\right\|_{L(\varphi,\Omega,\mu)} \leq C\|u\|_{L(\varphi,\Omega,\mu)}$$

for any bounded and convex δ-John domain $\Omega \subset \mathbb{R}^n$ with $\mu(\Omega) < \infty$, where the fixed cube $Q_0 \subset \Omega$ appeared in Lemma 1, *and the measure μ is defined by $d\mu = w(x)\,dx$ and $w(x) \in A_r(\Omega)$ with $w \geq \delta_0 > 0$ for some $r > 1$ and a constant δ_0.*

Proof Let g, h be the functions in the $G(p,q,C_0)$ condition. Note that φ is an increasing function. Using Theorem 1, (i) in Definition 4, and Jensen's inequality, we obtain

$$\begin{aligned}\varphi&\left(\frac{1}{k}\left(\int_\Omega \left|\mathcal{M}_s^\sharp(G(u)) - \mathcal{M}_s^\sharp(G(u))_{Q_0}\right|^t d\mu\right)^{1/t}\right) \\ &\leq \varphi\left(\frac{1}{k}C_1\left(\int_\Omega |u|^t\,d\mu\right)^{1/t}\right)\end{aligned}$$

$$
\begin{aligned}
&= \varphi\left(\left(\frac{1}{k^t}C_1^t\int_\Omega |u|^t\,d\mu\right)^{1/t}\right)\\
&\le C_0 g\left(\frac{1}{k^t}C_1^t\int_\Omega |u|^t\,d\mu\right)\\
&= C_0 g\left(\int_\Omega \frac{1}{k^t}C_1^t|u|^t\,d\mu\right)\\
&\le C_0\int_\Omega g\left(\frac{1}{k^t}C_1^t|u|^t\right)d\mu.
\end{aligned}
\tag{17}
$$

Again, from (i) in Definition 4, we have

$$
g(x) \le C_0\varphi\bigl(x^{\frac{1}{t}}\bigr).
$$

Thus, we obtain

$$
\int_\Omega g\left(\frac{1}{k^t}C_1^t|u|^t\right)d\mu \le C_0\int_\Omega \varphi\left(\frac{1}{k}C_1|u|\right)d\mu.
\tag{18}
$$

Combining (17) and (18) yields

$$
\begin{aligned}
&\varphi\left(\frac{1}{k}\left(\int_\Omega \bigl|\mathcal{M}_s^\sharp\bigl(G(u)\bigr) - \mathcal{M}_s^\sharp\bigl(G(u)\bigr)_{Q_0}\bigr|^t\,d\mu\right)^{1/t}\right)\\
&\quad\le C_0^2\int_\Omega \varphi\left(\frac{1}{k}C_1|u|\right)d\mu\\
&\quad= C_2\int_\Omega \varphi\left(\frac{1}{k}C_1|u|\right)d\mu.
\end{aligned}
\tag{19}
$$

Now, using Jensen's inequality for h^{-1}, (16) and (ii) in Definition 4, and noticing that φ is doubling, we see

$$
\begin{aligned}
&\int_\Omega \varphi\left(\frac{|\mathcal{M}_s^\sharp(G(u)) - \mathcal{M}_s^\sharp(G(u))_{Q_0}|}{k}\right)d\mu\\
&\quad= h\left(h^{-1}\left(\int_\Omega \varphi\left(\frac{|\mathcal{M}_s^\sharp(G(u)) - \mathcal{M}_s^\sharp(G(u))_{Q_0}|}{k}\right)d\mu\right)\right)\\
&\quad\le h\left(\int_\Omega h^{-1}\left(\varphi\left(\frac{|\mathcal{M}_s^\sharp(G(u)) - \mathcal{M}_s^\sharp(G(u))_{Q_0}|}{k}\right)\right)d\mu\right)\\
&\quad\le h\left(C_3\int_\Omega \left(\frac{|\mathcal{M}_s^\sharp(G(u)) - \mathcal{M}_s^\sharp(G(u))_{Q_0}|}{k}\right)^t d\mu\right)\\
&\quad\le C_0\varphi\left(\left(C_3\int_\Omega \left(\frac{|\mathcal{M}_s^\sharp(G(u)) - \mathcal{M}_s^\sharp(G(u))_{Q_0}|}{k}\right)^t d\mu\right)^{\frac{1}{t}}\right)\\
&\quad= C_0\varphi\left(\frac{1}{k}\left(C_3\int_\Omega \bigl(\bigl|\mathcal{M}_s^\sharp\bigl(G(u)\bigr) - \mathcal{M}_s^\sharp\bigl(G(u)\bigr)_{Q_0}\bigr|\bigr)^t\,d\mu\right)^{\frac{1}{t}}\right)\\
&\quad\le C_4\varphi\left(\frac{1}{k}\left(\int_\Omega \bigl(\bigl|\mathcal{M}_s^\sharp\bigl(G(u)\bigr) - \mathcal{M}_s^\sharp\bigl(G(u)\bigr)_{Q_0}\bigr|\bigr)^t\,d\mu\right)^{\frac{1}{t}}\right).
\end{aligned}
\tag{20}
$$

Substituting (19) into (20) and using the fact that φ is doubling, we get

$$\int_\Omega \varphi\left(\frac{|\mathcal{M}_s^\sharp(G(u)) - \mathcal{M}_s^\sharp(G(u))_{Q_0}|}{k}\right) d\mu \tag{21}$$

$$\le C_5 \int_\Omega \varphi\left(\frac{1}{k} C_1 |u|\right) d\mu$$

$$\le C_6 \int_\Omega \varphi\left(\frac{1}{k}|u|\right) d\mu. \tag{22}$$

Therefore, from Definition 3, we have

$$\left\|\mathcal{M}_s^\sharp\big(G(u)\big) - \mathcal{M}_s^\sharp\big(G(u)\big)_{Q_0}\right\|_{L(\varphi,\Omega,\mu)} \le C_6 \|u\|_{L(\varphi,\Omega,\mu)}. \qquad \square$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors completed the paper together. All authors read and approved the final manuscript.

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