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Nonconvex composite multiobjective nonsmooth fractional programming

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Abstract
We consider nonsmooth multiobjective programs where the objective function is a fractional composition of invex functions and locally Lipschitz and Gâteaux differentiable functions. Kuhn-Tucker necessary and sufficient optimality conditions for weakly efficient solutions are presented. We formulate dual problems and establish weak, strong and converse duality theorems for a weakly efficient solution.
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1 Introduction

Recently there has been an increasing interest in developing optimality conditions and duality relations for nonsmooth multiobjective programming problems involving locally Lipschitz functions. Many authors have studied under kinds of generalized convexity, and some results have been obtained. Schaible [1] and Bector *et al.* [2] derived some Kuhn-Tucker necessary and sufficient optimality conditions for the multiobjective fractional programming. By using ρ-invexity of a fractional function, Kim [3] obtained necessary and sufficient optimality conditions and duality theorems for nonsmooth multiobjective fractional programming problems. Lai and Ho [4] established sufficient optimality conditions for multiobjective fractional programming problems involving exponential V-r-invex Lipschitz functions. In [5], Kim and Schaible considered nonsmooth multiobjective programming problems with inequality and equality constraints involving locally Lipschitz functions and presented several sufficient optimality conditions under various invexity assumptions and regularity conditions. Soghra Nobakhtian [6] obtained optimality conditions and a mixed dual model for nonsmooth fractional multiobjective programming problems. Jeyakumar and Yang [7] considered nonsmooth constrained multiobjective optimization problems where the objective function and the constraints are compositions of convex functions and locally Lipschitz and Gâteaux differentiable functions. Lagrangian necessary conditions and new sufficient optimality conditions for efficient and properly efficient solutions were presented. Mishra and Mukherjee [8] extended the work of Jeyakumar and Yang [7] and the constraints are compositions of V-invex functions.

The present article begins with an extension of the results in [7, 8] from the nonfractional to the fractional case. We consider nonsmooth multiobjective programs where the objective functions are fractional compositions of invex functions and locally Lipschitz and

Gâteaux differentiable functions. Kuhn-Tucker necessary conditions and sufficient optimality conditions for weakly efficient solutions are presented. We formulate dual problems and establish weak, strong and converse duality theorems for a weakly efficient solution.

2 Preliminaries

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $\mathbb{R}^n_+$ be its nonnegative orthant. Throughout the paper, the following convention for inequalities will be used for $x, y \in \mathbb{R}^n$:

$$x = y \quad \text{if and only if} \quad x_i = y_i \quad \text{for all } i = 1, 2, \ldots, n;$$
$$x < y \quad \text{if and only if} \quad x_i < y_i \quad \text{for all } i = 1, 2, \ldots, n;$$
$$x \leqq y \quad \text{if and only if} \quad x_i \leqq y_i \quad \text{for all } i = 1, 2, \ldots, n.$$

The real-valued function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be locally Lipschitz if for any $z \in \mathbb{R}^n$ there exists a positive constant K and a neighbourhood N of z such that, for each $x, y \in N$,

$$\left|f(x) - f(y)\right| \leqq K\|x - y\|.$$

The Clarke generalized directional derivative of a locally Lipschitz function f at x in the direction d denoted by $f^\circ(x; d)$ (see, *e.g.*, Clarke [9]) is as follows:

$$f^\circ(x; d) = \limsup_{\substack{y \to x \\ t \downarrow 0}} t^{-1}\big(f(y + td) - f(y)\big).$$

The Clarke generalized subgradient of f at x is denoted by

$$\partial f(x) = \left\{\xi \mid f^0(x; d) \geqq \xi^T d \text{ for all } d \in \mathbb{R}^n\right\}.$$

Proposition 2.1 [9] *Let f, h be Lipschitz near x, and suppose $h(x) \neq 0$. Then $\frac{f}{h}$ is Lipschitz near x, and one has*

$$\partial\left(\frac{f}{h}\right)(x) \subset \frac{h(x)\partial f(x) - f(x)\partial h(x)}{h^2(x)}.$$

If, in addition, $f(x) \geqq 0$, $h(x) > 0$ and if f and $-h$ are regular at x, then equality holds and $\frac{f}{h}$ is regular at x.

In this paper, we consider the following composite multiobjective fractional programming problem:

$$\begin{aligned} (\mathrm{P}) \quad & \text{Minimize} \quad \left(\frac{f_1(F_1(x))}{h_1(F_1(x))}, \ldots, \frac{f_p(F_p(x))}{h_p(F_p(x))}\right) \\ & \text{subject to} \quad g_j\big(G_j(x)\big) \leqq 0, \quad j = 1, 2, \ldots, m, x \in C, \end{aligned}$$

where

(1) C is an open convex subset of a Banach space X,

(2) f_i, h_i, $i = 1, 2, \dots, p$, and g_j, $j = 1, 2, \dots, m$, are real-valued locally Lipschitz functions on $\mathbb{R}^n$, and F_i and G_j are locally Lipschitz and Gâteaux differentiable functions from X into $\mathbb{R}^n$ with Gâteaux derivatives $F_i'(\cdot)$ and $G_j'(\cdot)$, respectively, but are not necessarily continuously Fréchet differentiable or strictly differentiable [9],
(3) $f_i(x) \geqq 0$, $h_i(x) > 0$, $i = 1, 2, \dots, p$,
(4) $f_i(x)$ and $-h_i(x)$ are regular.

Definition 2.1 A feasible point x_0 is said to be a weakly efficient solution for (P) if there exists no feasible point x for which

$$\frac{f_i(F_i(x))}{h_i(F_i(x))} < \frac{f_i(F_i(x_0))}{h_i(F_i(x_0))}, \quad \forall i = 1, 2, \dots, p.$$

Definition 2.2 [10] A function f is invex on $X_0 \subset \mathbb{R}^n$ if for $x, u \in X_0$ there exists a function $\eta(x, u) : X_0 \times X_0 \to \mathbb{R}^n$ such that

$$f_i(x) - f_i(u) \geqq \xi_i^T \eta(x, u), \quad \forall \xi_i \in \partial f_i(u).$$

Definition 2.3 [10] A function $f : X_0 \to \mathbb{R}^n$ is V-invex on $X_0 \subset \mathbb{R}^n$ if for $x, u \in X_0$ there exist functions $\eta(x, u) : X_0 \times X_0 \to \mathbb{R}^n$ and $\alpha_i(x, u) : X_0 \times X_0 \to \mathbb{R}_+ \setminus \{0\}$ such that

$$f_i(x) - f_i(u) \geqq \alpha_i(x, u) \xi_i^T \eta(x, u), \quad \forall \xi_i \in \partial f(u).$$

The following lemma is needed in necessary optimality conditions, weak duality and converse duality.

Lemma 2.1 [3] *If $f_i \geqq 0$, $h_i > 0$, f_i and $-h_i$ are invex at u with respect to $\eta(x, u)$, and f_i and $-h_i$ are regular at u, then $\frac{f_i}{h_i}$ is V-invex at u with respect to $\bar{\eta}$, where $\bar{\eta}(x, u) = \frac{h_i(u)}{h_i(x)} \eta(x, u)$.*

3 Optimality conditions

Note that if $F : X \to \mathbb{R}^n$ is locally Lipschitz near a point $x \in X$ and Gâteaux differentiable at x and if $f : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz near $F(x)$, then the continuous sublinear function, defined by

$$\pi_x(h) := \max\left\{ \sum_{k=1}^{n} w_k F_k'(x) h \,\middle|\, w \in \partial f\big(F(x)\big) \right\},$$

satisfies the inequality

$$(f \circ F)_+'(x, h) \leqq \pi_x(h), \quad \forall h \in X. \tag{3.1}$$

Recall that $q_+'(x, h) = \lim_{\lambda \downarrow 0} \sup \lambda^{-1}(q(x + \lambda h) - q(x))$is the upper Dini-directional derivative of $q : X \to \mathbb{R}$ at x in the direction of h, and $\partial f(F(x))$ is the Clarke subdifferential of f at $F(x)$. The function $\pi_x(\cdot)$ in (3.1) is called upper convex approximation of $f \circ F$ at x, see [11, 12].

Note that for a set C, int C denotes the interior of C, and $C^+ = \{v \in X' | v(x) \geqq 0, \forall x \in C\}$, denotes the dual cone of C, where X' is the topological dual space of X. It is also worth noting that for a convex set C, the closure of the cone generated by the set C at a point

a, cl cone$(C-a)$, is the tangent cone of C at a, and the dual cone $-(C-a)^+$ is the normal cone of C at a, see [9, 13].

Theorem 3.1 (Necessary optimality conditions) *Suppose that f_i, h_i and g_j are locally Lipschitz functions, and that F_i and G_j are locally Lipschitz and Gâteaux differentiable functions. If $a \in C$ is a weakly efficient solution for* (P), *then there exist Lagrange multipliers $\lambda_i \geqq 0$, $i = 1,2,\dots,p$, and $\mu_j \geqq 0$, $j = 1,2,\dots,m$, not all zero, satisfying*

$$0 \in \sum_{i=1}^{p} \lambda_i T_i(a) F_i'(a) + \sum_{j=1}^{m} \mu_j \partial g_j\big(G_j(a)\big) G_j'(a) - (C-a)^+,$$

$$\mu_j g_j\big(G_j(a)\big) = 0, \quad j = 1,2,\dots,m,$$

$$T_i(a) = \frac{\partial f_i(F_i(a)) - \phi_i(a)\partial h_i(F_i(a))}{h_i(F_i(a))}, \quad \phi_i(a) = \frac{f_i(F_i(a))}{h_i(F_i(a))}.$$

Proof Let $I = \{1,2,\dots,p\}$, $J_p = \{p+j \mid j = 1,2,\dots,m\}$, $J_p(a) = \{p+j \mid g_j(G_j(a)) = 0, j \in \{1,2,\dots,m\}\}$.

For convenience, we define

$$l_k(x) = \begin{cases} (\frac{f_k}{h_k} \circ F_k)(x), & k = 1,2,\dots,p, \\ (g_{k-p} \circ G_{k-p}(x), & k = p+1,\dots,p+m. \end{cases}$$

Suppose that the following system has a solution:

$$d \in \operatorname{cone}(C-a), \qquad \pi_a^k(d) < 0, \qquad k \in I \cup J_p(a), \tag{3.2}$$

where $\pi_a^k(d)$ is given by

$$\pi_a^k(d) = \begin{cases} \max\{\sum_{k=1}^{p} \nu_k F_k'(a) d \mid \nu \in T_k(a)\}, & k \in I, \\ \max\{\sum_{k-p=1}^{m} w_{k-p} G_{k-p}'(a) d \mid w \in \partial g_{k-p}(G_{k-p}(a))\}, & k \in J_p(a). \end{cases}$$

Then the system

$$d \in \operatorname{cone}(C-a), \qquad (l_k)_+'(a;d) < 0, \qquad k \in I \cup J_p(a)$$

has a solution. So, there exists $\alpha_1 > 0$ such that $a + \alpha d \in C$, $l_k(a+\alpha d) < l_k(a)$, $k \in I \cup J_p(a)$, whenever $0 < \alpha \leqq \alpha_1$. Since $l_k(a) < 0$ for $k \in J_p \setminus J_p(a)$ and l_k is continuous in a neighbourhood of a, there exists $\alpha_2 > 0$ such that $l_k(a+\alpha d) < 0$, whenever $0 < \alpha \leqq \alpha_2$, $k \in J_p \setminus J_p(a)$. Let $\alpha^* = \min\{\alpha_1, \alpha_2\}$. Then $a + \alpha d$ is a feasible solution for (P) and $l_k(a+\alpha d) < l_k(a)$, $k \in I$ for sufficiently small α such that $0 < \alpha \leqq \alpha^*$.

This contradicts the fact that a is a weakly efficient solution for (P). Hence (3.2) has no solution.

Since, for each k, $\pi_a^k(\cdot)$ is sublinear and cone$(C-a)$ is convex, it follows from a separation theorem [12, 14] that there exist $\lambda_i \geqq 0$, $i = 1,\dots,p$, $\mu_j \geqq 0$, $j \in J_p(a)$, not all zero, such that

$$\sum_{i=1}^{p} \lambda_i \pi_a^i(x) + \sum_{j \in J_p(a)} \mu_j \pi_a^j(x) \geqq 0, \quad \forall x \in \operatorname{cone}(C-a).$$

Then, by applying standard arguments of convex analysis (see [15, 16]) and choosing $\mu_j = 0$ whenever $j \in J_p \setminus J_p(a)$, we have

$$0 \in \sum_{i=1}^{p} \lambda_i \partial \pi_a^i(0) + \sum_{j=1}^{m} \mu_j \partial \pi_a^{j+p}(0) - (C-a)^+.$$

So, there exist $\nu_i \in T_i(a)$, $w_j \in \partial g_j(G_j(a))$ satisfying

$$\sum_{i=1}^{p} \lambda_i \nu_i^T F_i'(a) + \sum_{j=1}^{m} \mu_j w_j^T G_j'(a) \in (C-a)^+.$$

Hence, the conclusion holds. □

Under the following generalized Slater condition, we do the following:

$$\exists x_0 \in \operatorname{cone}(C-a), \quad \mu^T G_j'(a) x_0 < 0, \quad \forall \mu \in \partial g_j\bigl(G_j(a)\bigr), \forall j \in J(a),$$

where $J(a) = \{j | g_j(G_j(a)) = 0, j = 1, \ldots, m\}$.

Choosing $q \in \mathbb{R}^p$, $q > 0$ with $\lambda^T q = 1$ and defining $\Lambda = qq^T$, we can select the multipliers $\bar{\lambda} = \Lambda\lambda = qq^T\lambda = q > 0$ and $\bar{\mu} = \Lambda\mu = qq^T\mu \geqq 0$. Hence, the following Kuhn-Tucker type optimality conditions (KT) for (P) are obtained:

$$\text{(KT)} \quad \bar{\lambda} \in \mathbb{R}^p, \bar{\lambda}_i > 0, \bar{\mu} \in \mathbb{R}^m, \bar{\mu}_j \geqq 0, \bar{\mu}_j g_j(G_j(a)) = 0,$$
$$0 \in \sum_{i=1}^{p} \bar{\lambda}_i T_i(a) F_i'(a) + \sum_{j=1}^{m} \bar{\mu}_j \partial g_j(G_j(a)) G_j'(a) - (C-a)^+,$$
$$T_i(a) = \frac{\partial f_i(F_i(a)) - \phi_i(a) \partial h_i(F_i(a))}{h_i(F_i(a))}, \quad \phi_i(a) = \frac{f_i(F_i(a))}{h_i(F_i(a))}.$$

We present new conditions under which the optimality conditions (KT) become sufficient for weakly efficient solutions.

The following null space condition is as in [7]:

Let $x, a \in X$. Define $K : X \to \mathbb{R}^{n(p+m)} := \pi\mathbb{R}^n$ by $K(x) = (F_1(x), \ldots, F_p(x), G_1(x), \ldots, G_m(x))$. For each $x, a \in X$, the linear mapping $A_{x,a} : X \to \mathbb{R}^{n(p+m)}$ is given by

$$A_{x,a}(y) = \bigl(\alpha_1(x,a) F_1'(a) y, \ldots, \alpha_p(x,a) F_p'(a) y, \beta_1(x,a) G_1'(a) y, \ldots, \beta_m(x,a) G_m'(a) y\bigr),$$

where $\alpha_i(x,a)$, $i = 1, 2, \ldots, p$ and $\beta_j(x,a)$, $j = 1, 2, \ldots, m$, are real positive constants. Let us denote the null space of a function H by $N[H]$.

Recall, from the generalized Farkas lemma [14], that $K(x) - K(a) \in A_{x,a}(X)$ if and only if $A_{x,a}^T(u) = 0 \Rightarrow u^T(K(x) - K(a)) = 0$. This observation prompts us to define the following general null space condition:

For each $x, a \in X$, there exist real constants $\alpha_i(x,a) > 0$, $i = 1, 2, \ldots, p$, and $\beta_j(x,a) > 0$, $j = 1, 2, \ldots, m$, such that

$$N[A_{x,a}] \subset N\bigl[K(x) - K(a)\bigr], \qquad \text{(NC)}$$

where

$$A_{x,a}(y) = \big(\alpha_1(x,a)F_1'(a)y, \dots, \alpha_p(x,a)F_p'(a)y, \beta_1(x,a)G_1'(a)y, \dots, \beta_m(x,a)G_m'(a)y\big).$$

Equivalently, the null space condition means that for each $x, a \in X$, there exist real constants $\alpha_i(x,a) > 0$, $i = 1,2,\dots,p$, and $\beta_j(x,a) > 0$, $i = 1,2,\dots,m$, and $\zeta(x,a) \in X$ such that $F_i(x) - F_i(a) = \alpha_i(x,a)F_i'(a)\zeta(x,a)$ and $G_j(x) - G_j(a) = \beta_j(x,a)G_j'(a)\zeta(x,a)$. For our problem (P), we assume the following generalized null space condition for invex function (GNCI):

For each $x, a \in C$, there exist real constants $\alpha_i(x,a) > 0$, $i = 1,2,\dots,p$, and $\beta_j(x,a) > 0$, $j = 1,2,\dots,m$, and $\zeta(x,a) \in (C - a)$ such that $\eta(F_i(x), F_i(a)) = \alpha_i(x,a)F_i'(a)\zeta(x,a)$ and $\eta(G_j(x), G_j(a)) = \beta_j(x,a)G_j'(a)\zeta(x,a)$.

Note that when $C = X$ and $\eta(F_i(x), F_i(a)) = F_i(x) - F_i(a)$ and $\eta(G_j(x), G_j(a)) = G_j(x) - G_j(a)$, the generalized null space condition for invex function (GNCI) reduces to (NC).

Theorem 3.2 (Sufficient optimality conditions) *For the problem* (P), *assume that* f_i, $-h_i$ *and* g_j *are invex functions and* F_i *and* G_j *are locally Lipschitz and Gâteaux differentiable functions. Let* u *be feasible for* (P). *Suppose that the optimality conditions* (KT) *hold at* u. *If* (GNCI) *holds at each feasible point* x *of* (P), *then* u *is a weakly efficient solution of* (P).

Proof From the optimality conditions (KT), there exist $v_i \in T_i(u)$, $w_j \in \partial g_j(G_j(u))$ such that

$$\sum_{i=1}^{p} \lambda_i v_i^T F_i'(u) + \sum_{j=1}^{m} \mu_j w_j^T G_j'(u) \in (C-u)^+, \qquad \mu_j g_j\big(G_j(u)\big) = 0.$$

Suppose that u is not a weakly efficient solution of (P). Then there exists a feasible $x \in C$ for (P) with

$$\frac{f_i(F_i(x))}{h_i(F_i(x)} < \frac{f_i(F_i(u))}{h_i(F_i(u))}, \quad i = 1,2,\dots,p.$$

By (GNCI), there exists $\zeta(x,u) \in (C-u)$, same for each F_i and G_j, such that $\eta(F_i(x), F_i(u)) = \alpha_i(x,u)F_i'(u)\zeta(x,u)$, $i = 1,2,\dots,p$, and $\eta(G_j(x), G_j(u)) = \beta_j(x,u)G_j'(u)\zeta(x,u)$, $j = 1,2,\dots,m$. Hence

$$\begin{aligned}
0 &\geqq \sum_{j=1}^{m} \frac{\mu_j}{\beta_j(x,u)}\big(g_j\big(G_j(x)\big) - g_j\big(G_j(u)\big)\big) \quad \text{(by feasibility)} \\
&\geqq \sum_{j=1}^{m} \frac{\mu_j}{\beta_j(x,u)} w_j^T \eta\big(G_j(x), G_j(u)\big) \quad \text{(by subdifferentiability)} \\
&= \sum_{j=1}^{m} \mu_j w_j^T G_j'(u)\zeta(x,u) \quad \text{(by (GNCI))} \\
&\geqq -\sum_{j=1}^{m} \lambda_i v_i^T F_i'(u)\zeta(x,u) \quad \text{(by a hypothesis)} \\
&= -\sum_{i=1}^{p} \frac{\lambda_i}{\alpha_i(x,u)} v_i^T \eta\big(F_i(x), F_i(u)\big) \quad \text{(by (GNCI))}
\end{aligned}$$

$$= -\sum_{i=1}^{p} \frac{\lambda_i}{\alpha_i(x,u)} \left(\frac{h_i(F_i(x))}{h_i(F_i(u))}\right)\left(\frac{f_i(F_i(x))}{h_i(F_i(x))} - \frac{f_i(F_i(u))}{h_i(F_i(u))}\right) \quad \text{(by subdifferentiability)}$$
$$> 0.$$

This is a contradiction and hence u is a weakly efficient solution for (P). □

4 Duality theorems

In this section, we introduce a dual programming problem and establish weak, strong and converse duality theorems. Now we propose the following dual (D) to (P).

$$\begin{aligned}
\text{(D)} \quad \text{Maximize} \quad & \left(\frac{f_1(F_1(u))}{h_1(F_1(u))}, \dots, \frac{f_p(F_p(u))}{h_p(F_p(u))}\right) \\
\text{subject to} \quad & 0 \in \sum_{i=1}^{p} \lambda_i v_i^T F_i'(u) + \sum_{j=1}^{m} \mu_j w_j^T G_j'(u) - (C-u)^+, \\
& \mu_j g_j\big(G_j(u)\big) \geqq 0, \quad j = 1, 2, \dots, m, \\
& u \in C, \lambda \in \mathbb{R}^p, \lambda_i > 0, \mu_j \in \mathbb{R}^m, \mu_j \geqq 0.
\end{aligned}$$

Theorem 4.1 (Weak duality) *Let x be feasible for* (P), *and let (u, λ, μ) be feasible for* (D). *Assume that* (GNCI) *holds with $\alpha_i(x,u) = \beta_j(x,u) = 1$. Moreover, f_i, $-h_i$ and g_j are invex functions and F_i and G_j are locally Lipschitz and Gâteaux differentiable functions. Then*

$$\left(\frac{f_1(F_1(x))}{h_1(F_1(x))}, \dots, \frac{f_p(F_p(x))}{h_p(F_p(x))}\right)^T - \left(\frac{f_1(F_1(u))}{h_1(F_1(u))}, \dots, \frac{f_p(F_p(u))}{h_p(F_p(u))}\right)^T \notin -\mathbb{R}^p{}_+ \setminus \{0\}.$$

Proof Since (u, λ, μ) is feasible for (D), there exist $\lambda_i > 0$, $\mu_j \geqq 0$, $v_i \in T_i(u)$, $i = 1, 2, \dots, p$, $w_j \in \partial g_j(G_j(u))$, $j = 1, 2, \dots, m$, satisfying $\mu_j g_j(G_j(u)) \geqq 0$ for $j = 1, 2, \dots, m$ and

$$\sum_{i=1}^{p} \lambda_i v_i^T F_i'(u) + \sum_{j=1}^{m} \mu_j w_j^T G_j'(u) \in (C-u)^+.$$

Suppose that $x \neq u$ and

$$\left(\frac{f_1(F_1(x))}{h_1(F_1(x))}, \dots, \frac{f_p(F_p(x))}{h_p(F_p(x))}\right)^T - \left(\frac{f_1(F_1(u))}{h_1(F_1(u))}, \dots, \frac{f_p(F_p(u))}{h_p(F_p(u))}\right)^T \in -\mathbb{R}^p{}_+ \setminus \{0\}.$$

Then

$$0 > \frac{f_i(F_i(x))}{h_i(F_i(x))} - \frac{f_i(F_i(u))}{h_i(F_i(u))}.$$

By the invexity of f_i and $-h_i$, we have

$$\begin{aligned}
0 &> \frac{h_i(F_i(u))}{h_i(F_i(x))} v_i^T \eta\big(F_i(x), F_i(u)\big) \\
&= \frac{h_i(F_i(u))}{h_i(F_i(x))} v_i^T \alpha_i(x,u) F_i'(u) \zeta(x,u) \quad \text{(by (GNCI))} \\
&> \frac{h_i(F_i(u))}{h_i(F_i(x))} v_i^T F_i'(u) \zeta(x,u) \quad (\text{by } \alpha_i(x,u) = 1)
\end{aligned}$$

since $\frac{h_i(F_i(u))}{h_i(F_i(x))} > 0$ and $\lambda_i > 0$, then

$$\sum_{i=1}^{p} \lambda_i v_i^T F_i'(u)\zeta(x,u) < 0. \tag{4.1}$$

From the feasibility conditions, we get $\mu_j g_j(G_j(x)) \leqq 0$, $\mu_j g_j(G_j(u)) \geqq 0$, and so

$$\sum_{j=1}^{m} \frac{\mu_j}{\beta_j(x,u)} \bigl(g_j\bigl(G_j(x)\bigr) - g_j\bigl(G_j(u)\bigr)\bigr) \leqq 0.$$

Similarly, by the invexity of g_j, positivity of $\beta_j(x,u)$ and by (GNCI), we have

$$\sum_{j=1}^{m} \mu_j w_j^T G_j'(u)\zeta(x,u) \leqq 0. \tag{4.2}$$

By (4.1) and (4.2), we get

$$\left[\sum_{i=1}^{p} \lambda_i v_i^T F_i'(u) + \sum_{j=1}^{m} \mu_j w_j^T G_j'(u)\right]\zeta(x,u) < 0.$$

This is a contradiction. The proof is completed by noting that when $x = u$ the conclusion trivially holds. □

Theorem 4.2 (Strong duality) *For the problem* (P), *assume that the generalized Slater constraint qualification holds. If u is a weakly efficient solution for* (P), *then there exist* $\lambda \in \mathbb{R}^p$, $\lambda_i > 0$, $\mu \in \mathbb{R}^m$, $\mu_j \geqq 0$ *such that* (u,λ,μ) *is a weakly efficient solution for* (D).

Proof It follows from Theorem 3.1 that there exist $\lambda \in \mathbb{R}^p$, $\lambda_i > 0$, $\mu \in \mathbb{R}^m$, $\mu_j \geqq 0$ such that

$$0 \in \sum_{i=1}^{p} \lambda_i T_i(u) F_i'(u) + \sum_{j=1}^{m} \mu_j \partial g_j\bigl(G_j(u)\bigr) G_j'(u) - (C-u)^+,$$

$$\mu_j g_j\bigl(G_j(u)\bigr) = 0, \quad j = 1,2,\ldots,m.$$

Then (u,λ,μ) is a feasible solution for (D). By weak duality,

$$\left(\frac{f_1(F_1(x))}{h_1(F_1(x))},\ldots,\frac{f_p(F_p(x))}{h_p(F_p(x))}\right)^T - \left(\frac{f_1(F_1(u))}{h_1(F_1(u))},\ldots,\frac{f_p(F_p(u))}{h_p(F_p(u))}\right)^T \notin -\mathbb{R}^p{}_+ \setminus \{0\}.$$

Since (u,λ,μ) is a feasible solution for (D), (u,λ,μ) is a weakly efficient solution for (D). Hence the result holds. □

Theorem 4.3 (Converse duality) *Let* (u,λ,μ) *be a weakly efficient solution of* (D), *and let a be a feasible solution of* (P). *Assume that* f_i, $-h_i$ *and* g_j *are invex functions and* F_i *and* G_j *are locally Lipschitz and Gâteaux differentiable functions. Moreover,* (GNCI) *holds with* $\alpha_i(x,u) = \beta_j(x,u) = 1$. *Then u is a weakly efficient solution of* (P).

Proof Suppose, contrary to the result, that u is not a weakly efficient solution of (P). Then there exists $x \in D$ such that

$$\frac{f_i(F_i(x))}{h_i(F_i(x))} < \frac{f_i(F_i(u))}{h_i(F_i(u))}.$$

Since f_i, $-h_i$ are invex functions, for each $\nu_i \in T_i(x)$, we have

$$0 > \frac{h_i(F_i(u))}{h_i(F_i(x))} \nu_i^T \eta\bigl(F_i(x), F_i(u)\bigr).$$

Since (u, λ, μ) are feasible for (P), we get

$$\begin{aligned} 0 &> \sum_{i=1}^{p} \lambda_i \nu_i^T \eta\bigl(F_i(x), F_i(u)\bigr) \\ &= \sum_{i=1}^{p} \lambda_i \nu_i^T \alpha_i(x,u) F_i'(u) \zeta(x,u) \quad \text{(by (GNCI))} \\ &= \sum_{i=1}^{p} \lambda_i \nu_i^T F_i'(u) \zeta(x,u) \quad (\text{by } \alpha_i(x,u) = 1). \end{aligned} \tag{4.3}$$

From the hypothesis $\mu_j g_j(G_j(x)) \leqq \mu_j g_j(G_j(u))$, g_j is an invex function and for each $w_j \in \partial g_j(G_j(x))$, it follows that

$$\begin{aligned} 0 &\geqq \mu_j w_j^T \eta\bigl(G_j(x), G_j(u)\bigr) \\ &= \mu_j w_j^T \beta(x,u) G_j'(u) \zeta(x,u) \quad \text{(by (GNCI))} \\ &= \mu_j w_j^T G_j'(u) \zeta(x,u) \quad (\text{by } \beta_j(x,u) = 1) \end{aligned}$$

and $\mu_j \geqq 0$, $j = 1, 2, \ldots, m$, then we have

$$\sum_{j=1}^{m} \mu_j w_j^T G_j'(u) \zeta(x,u) \leqq 0. \tag{4.4}$$

From (4.3) and (4.4), we get

$$\left[\sum_{i=1}^{p} \lambda_i \nu_i^T F_i'(u) + \sum_{j=1}^{m} \mu_j w_j^T G_j'(u) \right] \zeta(x,u) < 0.$$

This is a contradiction. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
DSK presented necessary and sufficient optimality conditions, formulated Mond-Weir type dual problem and established weak, strong and converse duality theorems for nonconvex composite multiobjective nonsmooth fractional programs. HJK carried out the optimality and duality studies, participated in the sequence alignment and drafted the manuscript. All authors read and approved the final manuscript.

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