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Fixed point of nonlinear contractions in modular spaces

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Abstract

Razani *et al.* studied the fixed points of nonlinear and asymptotic contractions in the modular space in 2007. In this paper, we generalize the kind of nonlinear contraction that is the result of Razani *et al.* (Abstr. Appl. Anal. 2007:40575, 2007) and prove the existence and uniqueness of fixed points for the generalized nonlinear contractions in modular spaces.

Keywords: nonlinear contractions; fixed points; modular spaces

1 Introduction

The notion of modular spaces, as a generalization of metric spaces, was introduced by Nakano [1] in 1950 in connection with the theory of order spaces and redefined and generalized by Musielak and Orlicz [2] in 1959. These spaces were developed following the successful theory of Orlicz spaces, which replaces the particular integral form of the nonlinear functional, which controls the growth of members of the space, by an abstractly given functional with some good properties. The monographic exposition of the theory of Orlicz spaces may be found in the book of Krasnosel'skii and Rutickii [3]. For a current review of the theory of Musielak-Orlicz spaces and modular spaces, the reader is referred to the books of Musielak [2] and Kozlowski [4].

Fixed point theorems in modular spaces, generalizing the classical Banach fixed point theorem in metric spaces, have been studied extensively. In 2007, Razani *et al.* [5] studied some fixed points of nonlinear and asymptotic contractions in the modular spaces. In addition, quasi-contraction mappings in modular spaces without Δ_2-condition were considered by Khamsi [6] in 2008. In 2011, Kuaket and Kumam [7] proved the existence of fixed points of asymptotic pointwise contractions in modular spaces. Recently, we proved the fixed points of asymptotic pointwise nonexpansive mappings in modular spaces [8].

In this paper, we introduce the notions of nonlinear contractions in modular spaces and establish their fixed points theorems in modular spaces.

2 Preliminaries

Definition 2.1 Let X be an arbitrary vector space over K (= R or C).

(a) A functional $\rho : X \to [0, \infty]$ is called modular if

   (i) $\rho(x) = 0$ if and only if $x = 0$;

   (ii) $\rho(\alpha x) = \rho(x)$ for $\alpha \in K$ with $|\alpha| = 1$, for all $x \in X$;

   (iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ if $\alpha, \beta \geq 0$, for all $x, y \in X$.

If (iii) is replaced by

(iii′) $\rho(\alpha x + \beta y) \leq \alpha \rho(x) + \beta \rho(y)$, for $\alpha, \beta \geq 0$, $\alpha + \beta = 1$, for all $x, y \in X$,

then the modular ρ is called convex modular.

(b) A modular ρ defines a corresponding modular space, *i.e.*, the space X_ρ given by

$$X_\rho = \left\{ x \in X \mid \rho(\alpha x) \to 0 \text{ as } \alpha \to 0 \right\}.$$

Remark 2.2 Note that ρ is an increasing function. Suppose $0 < a < b$. Then, property (iii) with $y = 0$ shows that $\rho(ax) = \rho(\frac{a}{b}(bx)) \leq \rho(bx)$.

Definition 2.3 Let X_ρ be a modular space.

(a) A sequence $\{x_n\} \subset X_\rho$ is said to be *ρ-convergent* to $x \in X_\rho$, and write $x_n \xrightarrow{\rho} x$ if $\rho(x_n - x) \to 0$ as $n \to \infty$.

(b) A sequence $\{x_n\}$ is called *ρ-Cauchy* whenever $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$.

(c) X_ρ is called *ρ-complete* if any ρ-Cauchy sequence is ρ-convergent.

(d) A subset $B \subset X_\rho$ is called *ρ-closed* if for any sequence $\{x_n\} \subset B$ ρ-convergent to $x \in X_\rho$, we have $x \in B$.

(e) A ρ-closed subset $B \subset X_\rho$ is called *ρ-compact* if any sequence $\{x_n\} \subset B$ has a ρ-convergent subsequence.

(f) ρ is said to satisfy the Δ_2-condition if $\rho(2x_n) \to 0$ whenever $\rho(x_n) \to 0$ as $n \to \infty$.

(g) We say that ρ has the Fâtou property if $\rho(x - y) \leq \liminf_{n \to \infty} \rho(x_n - y_n)$, whenever $x_n \xrightarrow{\rho} x$ and $y_n \xrightarrow{\rho} y$ as $n \to \infty$.

(h) A subset $B \subset X_\rho$ is said to be *ρ-bounded* if

$$\operatorname{diam}_\rho(B) < \infty,$$

where $\operatorname{diam}_\rho(B) = \sup\{\rho(x - y); x, y \in B\}$ is called the *ρ-diameter* of B.

(i) Define the *ρ-distance* between $x \in X_\rho$ and $B \subset X_\rho$ as

$$\operatorname{dis}_\rho(x, B) = \inf\left\{ \rho(x - y); y \in B \right\}.$$

(j) Define the *ρ-ball*, $B_\rho(x, r)$, centered at $x \in X_\rho$ with radius r as

$$B_\rho(x, r) = \left\{ y \in X_\rho; \rho(x - y) \leq r \right\}.$$

Theorem 2.4 [5, RNMP] *Let X_ρ be a ρ-complete modular space, where ρ satisfies the Δ_2-condition. Assume that $\psi : R^+ \to [0, \infty)$ is an increasing and upper semi-continuous function satisfying*

$$\psi(t) < t, \quad \forall t > 0. \tag{2.1}$$

Let B be a ρ-closed subset of X_ρ, and let $T : B \to B$ be a mapping such that there exist $c, l \in R^+$ with $c > l$,

$$\rho\big(c(Tx - Ty)\big) \leq \psi\big(\rho\big(l(x - y)\big)\big) \tag{2.2}$$

for all $x, y \in B$. Then T has a fixed point.

The Banach contraction mapping principle shows the existence and uniqueness of a fixed point in a complete metric space; this has been generalized by many mathematicians such as Arandelovic [9], Edelstein [10], Ciric [11], Rakotch [12], Reich [13], Kirk [14], and so forth. In addition, Boyd and Wong [15] studied mappings, which are nonlinear contractions in the metric space. It is necessary to mention that the applications of contraction, generalized contraction principle for self-mappings, and the applications of nonlinear contractions are well known. In the next section, we will prove the existence fixed points theorems for nonlinear contractions in modular space.

3 Fixed points of nonlinear contractions

In the sequel, we assume that $\psi : R^+ \to [0, \infty)$ is an increasing and upper semi-continuous function satisfying

$$\begin{cases} \psi(t) < t, & \forall t > 0, \\ \psi(0) = 0. \end{cases} \tag{3.1}$$

Lemma 3.1 [16] *Let $t > 0$, $\psi(t) < t$ if and only if $\lim_n \psi^n(t) = 0$, where ψ^n denotes the n-times repeated composition of ψ with itself.*

Theorem 3.2 *Let X_ρ be a ρ-complete modular space, where ρ satisfies the Δ_2-condition. Let ψ be as in the previous definition, let C be a ρ-closed subset of X_ρ, and let $T : C \to C$ be a mapping such that there exist $c, l \in R^+$ with $c > l$,*

$$\rho\big(c(Tx - Ty)\big) \le \psi\bigg(\max\bigg\{\rho\big(l(x-y)\big), \rho\big(l(x-Tx)\big), \rho\big(l(y-Ty)\big),$$

$$\frac{1}{2}\big[\rho\big(l(x-Ty)\big) + \rho\big(l(y-Tx)\big)\big]\bigg\}\bigg) \tag{3.2}$$

for all $x, y \in C$. Then T has a unique fixed point.

Proof Let $x \in X_\rho$. At first, we show that the sequence $\{\rho(c(T^n x - T^{n-1}x))\}$ converges to 0. For $n \in N$, we have

$$\rho\big(c\big(T^n x - T^{n-1}x\big)\big)$$

$$\le \psi\bigg(\max\bigg\{\rho\big(l\big(T^{n-1}x - T^{n-2}x\big)\big), \rho\big(l\big(T^{n-1}x - T^n x\big)\big),$$

$$\rho\big(l\big(T^{n-2}x - T^{n-1}x\big)\big), \frac{1}{2}\big[\rho\big(l\big(T^{n-1}x - T^{n-1}x\big)\big)$$

$$+ \rho\big(l\big(T^{n-2}x - T^n x\big)\big)\big]\bigg\}\bigg). \tag{3.3}$$

Note that

$$\rho\big(l\big(T^{n-2}x - T^n x\big)\big)$$

$$= \rho\big(l\big(T^n x - T^{n-1}x + T^{n-1}x - T^{n-2}x\big)\big)$$

$$\le \rho\big(l\big(T^n x - T^{n-1}x\big)\big) + \rho\big(l\big(T^{n-1}x - T^{n-2}x\big)\big). \tag{3.4}$$

By (3.3) and (3.4), therefore, we have

$$\rho\big(c\big(T^n x - T^{n-1} x\big)\big) \leq \psi\big(\rho\big(l\big(T^{n-1} x - T^{n-2} x\big)\big)\big)$$
$$< \rho\big(l\big(T^{n-1} x - T^{n-2} x\big)\big) < \rho\big(c\big(T^{n-1} x - T^{n-2} x\big)\big). \tag{3.5}$$

Consequently, $\rho(c(T^n x - T^{n-1} x))$ is decreasing and bounded from below ($\rho(x) \geq 0$).
Therefore, $\rho(c(T^n x - T^{n-1} x))$ converges to a.

Now, if $a \neq 0$

$$a = \lim_{n \to \infty} \rho\big(c\big(T^n x - T^{n-1} x\big)\big) \leq \lim_{n \to \infty} \psi\big(\rho\big(l\big(T^{n-1} x - T^{n-2} x\big)\big)\big)$$
$$\leq \lim_{n \to \infty} \psi\big(\rho\big(c\big(T^{n-1} x - T^{n-2} x\big)\big)\big), \tag{3.6}$$

then

$$a \leq \psi(a), \tag{3.7}$$

which is a contradiction, therefore, $a = 0$.

Now, we show that $\{T^n x\}$ is a ρ-Cauchy sequence for $x \in X_\rho$. Suppose that $\{l T^n x\}$ is not a ρ-Cauchy sequence. Then, there are an $\varepsilon > 0$ and sequences of integers $\{m_k\}$, $\{n_k\}$, with $m_k > n_k \geq k$, and such that

$$d_k = \rho\big(l\big(T^{m_k} x - T^{n_k} x\big)\big) \geq \varepsilon \quad \text{for } k = 1, 2, \dots \tag{3.8}$$

we can assume that

$$\rho\big(l\big(T^{m_k - 1} x - T^{n_k} x\big)\big) < \varepsilon. \tag{3.9}$$

Let m_k be the smallest number exceeding n_k, for which (3.8) holds, and

$$\Sigma_k = \big\{m \in N \mid \exists n_k \in N; \rho\big(l\big(T^m x - T^{n_k} x\big)\big) \geq \varepsilon, m > n_k \geq k\big\}. \tag{3.10}$$

Obviously, $\Sigma_k \neq \emptyset$ and since $\Sigma_k \subset N$, then by Well Ordering Principle, the minimum element of Σ_k is denoted by m_k, and clearly (3.9) holds.

Now, we assume that $\alpha_0 \in R^+$ and $\frac{l}{c} + \frac{1}{\alpha_0} = 1$, then we have

$$d_k = \rho\big(l\big(T^{m_k} x - T^{n_k} x\big)\big)$$
$$= \rho\bigg(\frac{lc}{c}\big(T^{m_k} x - T^{n_k+1} x + T^{n_k+1} x - T^{n_k} x\big)\bigg)$$
$$\leq \rho\big(c\big(T^{m_k} x - T^{n_k+1} x\big)\big) + \rho\big(\alpha_0 l\big(T^{n_k+1} x - T^{n_k} x\big)\big)$$
$$\leq \psi\big(\rho\big(l\big(T^{m_k-1} x - T^{n_k} x\big)\big)\big) + \rho\big(\alpha_0 l\big(T^{n_k+1} x - T^{n_k} x\big)\big)$$
$$\leq \rho\big(l\big(T^{m_k-1} x - T^{n_k} x\big)\big) + \rho\big(\alpha_0 l\big(T^{n_k+1} x - T^{n_k} x\big)\big)$$
$$\leq \varepsilon + \rho\big(\alpha_0 l\big(T^{n_k+1} x - T^{n_k} x\big)\big). \tag{3.11}$$

If $k \to \infty$ and by Δ_2-condition, $\rho(\alpha_0 l(T^{n_k+1}x - T^{n_k}x)) \to 0$. Hence, $d_k \to \varepsilon$. Now,

$$
\begin{aligned}
d_k &= \rho\big(l\big(T^{m_k}x - T^{n_k}x\big)\big) \\
&\leq \rho\big(c\big(T^{m_k+1}x - T^{n_k+1}x\big)\big) + \rho\big(2\alpha_0 l\big(T^{m_k}x - T^{m_k+1}x\big)\big) + \rho\big(2\alpha_0 l\big(T^{n_k+1}x - T^{n_k}x\big)\big) \\
&\leq \psi\big(\rho\big(l\big(T^{m_k+1}x - T^{n_k+1}x\big)\big)\big) + \rho\big(2\alpha_0 l\big(T^{m_k}x - T^{m_k+1}x\big)\big) \\
&\quad + \rho\big(2\alpha_0 l\big(T^{n_k+1}x - T^{n_k}x\big)\big).
\end{aligned}
\tag{3.12}
$$

Thus, as $k \to \infty$, we obtain $\varepsilon \leq \psi(\varepsilon)$, which is a contradiction for $\varepsilon > 0$. So, $\{lT^n x\}$ is a ρ-Cauchy sequence, and by Δ_2-condition, $\{T^n x\}$ is a ρ-Cauchy sequence, and X_ρ is ρ-complete, there is a $\omega \in C$ such that $\rho(T^n x - \omega) \to 0$ as $n \to \infty$. Now, it is enough to show that ω is a fixed point of T. Indeed,

$$
\begin{aligned}
\rho\left(\frac{c}{2}(T\omega - \omega)\right) &= \rho\left(\frac{c}{2}\big(T\omega - T^{n+1}x\big) + \frac{c}{2}\big(T^{n+1}x - \omega\big)\right) \\
&\leq \rho\big(c\big(T\omega - T^{n+1}x\big)\big) + \rho\big(c\big(T^{n+1}x - \omega\big)\big) \\
&\leq \psi\big(\rho\big(l\big(\omega - T^n x\big)\big)\big) + \rho\big(c\big(T^{n+1}x - \omega\big)\big) \\
&\leq \rho\big(c\big(\omega - T^n x\big)\big) + \rho\big(c\big(T^{n+1}x - \omega\big)\big).
\end{aligned}
\tag{3.13}
$$

Since $\rho(c(\omega - T^n x)) + \rho(c(T^{n+1}x - \omega)) \to 0$ as $n \to \infty$, therefore, $\rho(\frac{c}{2}(T\omega - \omega)) = 0$ and $T\omega = \omega$.

Next, we prove that T has a unique fixed point. Letting ω^* be another fixed point of T, we have

$$
\rho\big(c\big(\omega - \omega^*\big)\big) = \rho\big(c\big(T\omega - T\omega^*\big)\big) \leq \psi\big(\rho\big(l\big(\omega - \omega^*\big)\big)\big) \leq \rho\big(c\big(\omega - \omega^*\big)\big),
\tag{3.14}
$$

which implies that $\rho(c(\omega - \omega^*)) = 0$, so $\omega = \omega^*$.

The proof is complete. □

The next corollary is immediate consequence of Theorem 3.2.

Corollary 3.3 *Let X_ρ be a ρ-complete modular space, where ρ satisfies the Δ_2-condition. Let C be a ρ-closed subset of X_ρ, and let $T : C \to C$ be a mapping such that there exist $c, k, l \in R^+$ with $c > l$ and $k \in (0, 1)$,*

$$
\begin{aligned}
\rho\big(c(Tx - Ty)\big) \leq k \max\bigg\{ &\rho\big(l(x - y)\big), \rho\big(l(x - Tx)\big), \rho\big(l(y - Ty)\big), \\
&\frac{1}{2}\big[\rho\big(l(x - Ty)\big) + \rho\big(l(y - Tx)\big)\big]\bigg\}
\end{aligned}
$$

for all $x, y \in C$. Then T has a unique fixed point.

Next, we continue to generalize the above consequences.

Firstly, for any $x \in C$, define the orbit

$$
O(x) = \big\{x, Tx, T^2 x, \ldots\big\}
$$

and its ρ-diameter by

$$\delta_\rho(x) = \operatorname{diam}(O(x)) = \sup\{\rho(T^n x - T^m x); n, m \in \{0\} \cup N\}$$

then, define

$$\tilde{\delta}_\rho(x) = \sup\{\rho(c(T^n x - T^m x)); n, m \in \{0\} \cup N\}. \tag{3.15}$$

The following lemmas will be helpful to prove the main result.

Lemma 3.4 *Let X_ρ, C, ψ be as in the definitions above, let $T : C \to C$ be a mapping such that there exist $c, l \in R^+$ with $c > l$,*

$$\rho(c(Tx - Ty)) \leq \psi\big(\max\{\rho(l(x-y)), \rho(l(x-Tx)), \rho(l(y-Ty)),$$
$$\rho(l(x-Ty)), \rho(l(y-Tx))\}\big) \tag{3.16}$$

for all $x, y \in C$. Let $x \in C$ such that $\tilde{\delta}_\rho(x) < \infty$. Then for any $n \geq 1$, one has

$$\tilde{\delta}_\rho(T^n x) \leq \psi^n(\tilde{\delta}_\rho(x)). \tag{3.17}$$

Moreover, one has

$$\rho(c(T^n x - T^{n+m} x)) \leq \psi^n(\tilde{\delta}_\rho(x)) \tag{3.18}$$

for all $n \geq 1$ and $m \in N$.

Proof Letting $n, m \geq 1$, we have

$$\rho(c(T^n x - T^m y)) \leq \psi\big(\max\{\rho(l(T^{n-1}x - T^{m-1}y)), \rho(l(T^{n-1}x - T^n x)),$$
$$\rho(l(T^{m-1}y - T^m y)), \rho(l(T^{n-1}x - T^m y)), \rho(l(T^n x - T^{m-1}y))\}\big)$$

for all $x, y \in C$. This obviously implies the following

$$\tilde{\delta}_\rho(T^n x) \leq \psi(\tilde{\delta}_\rho(T^{n-1} x))$$

for any $n \geq 1$.

Hence, for any $n \geq 1$, we have

$$\tilde{\delta}_\rho(T^n x) \leq \psi^n(\tilde{\delta}_\rho(x)).$$

Moreover, for any $n \geq 1$ and $m \in N$, we have

$$\rho(c(T^n x - T^{n+m} x)) \leq \tilde{\delta}_\rho(T^n x) \leq \psi^n(\tilde{\delta}_\rho(x)). \qquad \square$$

Lemma 3.5 *Let X_ρ, T, C, ψ, x be as in the previous lemma, and let ρ satisfy the Fâtou property. Then $\{T^n x\}$ ρ-converges to $\omega \in C$. Moreover, one has*

$$\rho\big(c\big(T^n x - \omega\big)\big) \le \psi^n\big(\tilde{\delta}_\rho(x)\big) \tag{3.19}$$

for any $n \ge 1$.

Proof From the previous lemma, it is easy to know that $\{T^n x\}$ is ρ-Cauchy. Since C is ρ-closed, then there exists $\omega \in C$ such that $\{T^n x\} \xrightarrow{\rho} \omega$. Since

$$\rho\big(c\big(T^n x - T^{n+m} x\big)\big) \le \psi^n\big(\tilde{\delta}_\rho(x)\big)$$

for any $n \ge 1$, $m \in N$, and ρ satisfies the Fâtou property, we let $m \to \infty$ to get

$$\rho\big(c\big(T^n x - \omega\big)\big) \le \psi^n\big(\tilde{\delta}_\rho(x)\big). \qquad \square$$

Theorem 3.6 *Let X_ρ, ρ, T, C, ψ be as in Lemma* 3.5. *Assume that $\omega \in C$ is the ρ-limit of $\{T^n x\}$, and $\rho(c(\omega - T\omega)) < \infty$, $\rho(c(x - T\omega)) < \infty$. Then ω is a fixed point of T, i.e., $T(\omega) = \omega$. Moreover, if ω^* is another fixed point of T in C such that $\rho(c(\omega - \omega^*)) < \infty$, then we have $\omega = \omega^*$.*

Proof We have

$$\rho\big(c(Tx - T\omega)\big) \le \psi\big(\max\big\{\rho\big(l(x - \omega)\big), \rho\big(l(x - Tx)\big), \rho\big(l(\omega - T\omega)\big),$$
$$\rho\big(l(x - T\omega)\big), \rho\big(l(\omega - Tx)\big)\big\}\big).$$

From (3.15) and the previous results, we get

$$\rho\big(c(Tx - T\omega)\big) \le \psi\big(\max\big\{\tilde{\delta}_\rho(x), \rho\big(c(\omega - T\omega)\big), \rho\big(c(x - T\omega)\big)\big\}\big).$$

Assume that for $n \ge 1$, we have

$$\rho\big(c\big(T^n x - T\omega\big)\big) \le \max\big\{\psi^n\big(\tilde{\delta}_\rho(x)\big), \psi\big(\rho\big(c(\omega - T\omega)\big)\big), \psi^n\big(\rho\big(c(x - T\omega)\big)\big)\big\}.$$

Then

$$\rho\big(c\big(T^{n+1} x - T\omega\big)\big) \le \psi\big(\max\big\{\rho\big(l\big(T^n x - \omega\big)\big), \rho\big(l\big(T^n x - T^{n+1} x\big)\big), \rho\big(l(\omega - T\omega)\big),$$
$$\rho\big(l\big(T^n x - T\omega\big)\big), \rho\big(l\big(T^{n+1} x - \omega\big)\big)\big\}\big).$$

Hence

$$\rho\big(c\big(T^{n+1} x - T\omega\big)\big) \le \psi\big(\max\big\{\psi^n\big(\tilde{\delta}_\rho(x)\big), \rho\big(c(\omega - T\omega)\big), \rho\big(c\big(T^n x - T\omega\big)\big)\big\}\big).$$

Using our previous assumption, we get

$$\rho\big(c\big(T^{n+1} x - T\omega\big)\big) \le \max\big\{\psi^{n+1}\big(\tilde{\delta}_\rho(x)\big), \psi\big(\rho\big(c(\omega - T\omega)\big)\big), \psi^{n+1}\big(\rho\big(c(x - T\omega)\big)\big)\big\}.$$

So, by induction, we have

$$\rho\big(c\big(T^n x - T\omega\big)\big) \leq \max\big\{\psi^n\big(\tilde{\delta}_\rho(x)\big), \psi\big(\rho\big(c(\omega - T\omega)\big)\big), \psi^n\big(\rho\big(c(x - T\omega)\big)\big)\big\}$$

for any $n \geq 1$. Therefore, we have

$$\limsup_{n\to\infty} \rho\big(c\big(T^n x - T\omega\big)\big) \leq \psi\big(\rho\big(c(\omega - T\omega)\big)\big).$$

Using the Fâtou property, satisfied by ρ, we get

$$\rho\big(c(\omega - T\omega)\big) \leq \liminf_{n\to\infty} \rho\big(c\big(T^n x - T\omega\big)\big) \leq \psi\big(\rho\big(c(\omega - T\omega)\big)\big),$$

so, we get $\rho(c(\omega - T\omega)) = 0$, *i.e.*, $T(\omega) = \omega$.

Let ω^* be another fixed point of T such that $\rho(c(\omega - \omega^*)) < \infty$. Then we get

$$\rho\big(c\big(\omega - \omega^*\big)\big) = \rho\big(c\big(T\omega - T\omega^*\big)\big) \leq \psi\big(\rho\big(c(\omega - \omega^*)\big)\big),$$

which implies $\rho(c(\omega - \omega^*)) = 0$, *i.e.*, $\omega = \omega^*$.

We complete the proof of our theorem. □

The following corollary is an immediate consequence of Theorem 3.6.

Corollary 3.7 [6] *Let X_ρ be a ρ-complete modular space, where ρ satisfies the Fâtou property. Let C be a ρ-closed subset of X_ρ, and let $T : C \to C$ be a mapping such that there exist $c, k, l \in R^+$ with $c > l$ and $k \in (0,1)$,*

$$\rho\big(c(Tx - Ty)\big) \leq k \max\big\{\rho\big(l(x - y)\big), \rho\big(l(x - Tx)\big), \rho\big(l(y - Ty)\big),$$
$$\rho\big(l(x - Ty)\big), \rho\big(l(y - Tx)\big)\big\}$$

for all $x, y \in C$. Assume that $\omega \in C$ is the ρ-limit of $\{T^n x\}$, and $\rho(c(\omega - T\omega)) < \infty$, $\rho(c(x - T\omega)) < \infty$. Then ω is a fixed point of T, i.e., $T(\omega) = \omega$. Moreover, if ω^* is another fixed point of T in C such that $\rho(c(\omega - \omega^*)) < \infty$, then we have $\omega = \omega^*$.

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