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Fractional Hermite-Hadamard inequalities for (α, m)-logarithmically convex functions

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Abstract
By means of two fundamental fractional integral identities, we derive two classes of new Hermite-Hadamard type inequalities involving Riemann-Liouville fractional integrals for once and twice differentiable (α, m)-logarithmically convex functions, respectively. The main novelty of this paper is that we use powerful series to describe our estimations.

MSC: 26A33; 26A51; 26D15

Keywords: Hermite-Hadamard type inequalities; Riemann-Liouville fractional integrals; (α, m)-logarithmically convex functions

1 Introduction

Fractional calculus was introduced at the end of the nineteenth century by Liouville and Riemann, but the concept of non-integer calculus, as a generalization of the traditional integer order calculus, was mentioned already in 1695 by Leibnitz and L'Hospital. The subject of fractional calculus has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences and engineering to biological sciences and economics. For more recent development on fractional calculus, one can see the monographs [1–8].

Due to the wide application of fractional integrals and importance of Hermite-Hadamard type inequalities, some authors extended to study fractional Hermite-Hadamard type inequalities according to the Hermite-Hadamard type inequalities for functions of different classes. For example, see for convex functions [9, 10] and nondecreasing functions [11], for m-convex functions [12–14] and (s, m)-convex functions [15], for functions satisfying s-e-condition [16] and the references therein.

Very recently, the authors [17] raised the new concept of (α, m)-logarithmically convex functions and established some interesting Hermite-Hadamard type inequalities of such functions. The main results can be improved if we replace E_i, $i = 1, 2, 3$, by suitable series.

Motivated by [13, 16, 17], we study Hermite-Hadamard type inequalities for (α, m)-logarithmically convex functions involving Riemann-Liouville fractional integrals. Thus, the purpose of this paper is to establish fractional Hermite-Hadamard type inequalities for (α, m)-logarithmically convex functions.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts.

Definition 2.1 (see [3]) Let $f \in L[a,b]$. The symbols ${}_{RL}J^{\alpha}_{a^+}f$ and ${}_{RL}J^{\alpha}_{b^-}f$ denote the left-sided and right-sided Riemann-Liouville fractional integrals of the order $\alpha \in R^+$ and are defined by

$$\left({}_{RL}J^{\alpha}_{a^+}f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)\,dt \quad (0 \le a < x \le b)$$

and

$$\left({}_{RL}J^{\alpha}_{b^-}f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)\,dt \quad (0 \le a \le x < b),$$

respectively. Here $\Gamma(\cdot)$ is the gamma function.

Definition 2.2 (see [17]) The function $f : [0,b] \to R^+$ is said to be (α,m)-logarithmically convex if for every $x, y \in [0,b]$, $(\alpha,m) \in (0,1] \times (0,1]$, and $t \in [0,1]$, we have

$$f\left(tx + (1-t)y\right) \le \left(f(x)\right)^{t^{\alpha}}\left(f(y)\right)^{m(1-t^{\alpha})}.$$

The following inequalities results will be used in the sequel.

Lemma 2.3 (see [16]) *For $\alpha > 0$ and $k > 0$, we have*

$$I(\alpha,k) := \int_0^1 t^{\alpha-1}k^t\,dt = k\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln k)^{i-1}}{(\alpha)_i} < \infty,$$

where

$$(\alpha)_i = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+i-1).$$

Moreover, it holds

$$\left|I(\alpha,k) - k\sum_{i=1}^{m}\frac{(-\ln k)^{i-1}}{(\alpha)_i}\right| \le \frac{|\ln k|}{\alpha\sqrt{2\pi(m-1)}}\left(\frac{|\ln k|e}{m-1}\right)^{m-1}.$$

Lemma 2.4 (see [13]) *For $\alpha > 0$ and $k > 0$, $z > 0$, we have*

$$J(\alpha,k) := \int_0^1 (1-t)^{\alpha-1}k^t\,dt = \sum_{i=1}^{\infty}\frac{(\ln k)^{i-1}}{(\alpha)_i} < \infty,$$

$$H(\alpha,k,z) := \int_0^z t^{\alpha-1}k^t\,dt = z^{\alpha}k^z\sum_{i=1}^{\infty}\frac{(-z\ln k)^{i-1}}{(\alpha)_i} < \infty.$$

Lemma 2.5 *For $t \in [0,1]$, we have*

$$(1-t)^n \le 2^{1-n} - t^n \quad \textit{for } n \in [0,1],$$

$$(1-t)^n \ge 2^{1-n} - t^n \quad \textit{for } n \in [1,\infty).$$

Proof Let $f(t) = t^n + (1-t)^n$ for $n, t \in [0,1]$. Clearly, $f(\cdot)$ is increasing on the interval $[0, \frac{1}{2}]$ and decreasing on the interval $[\frac{1}{2}, 1]$. So, $f(t) \leq f(\frac{1}{2}) = 2^{1-n}$ for all $t \in [0,1]$. Then we have the first statement. Similarly one can obtain the second one. □

3 The first main results

In this section, we apply the fractional integral identity from Sarikaya *et al.* [9] to derive some new Hermite-Hadamard type inequalities for differentiable (α, m)-logarithmically convex functions.

Lemma 3.1 (see [9]) *Let $f : [a,b] \to R$ be a differentiable mapping on (a,b). If $f' \in L[a,b]$, then the following equality for fractional integrals holds*:

$$\begin{aligned} &\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\bigl[{}_{RL}J_{a^+}^{\alpha} f(b) + {}_{RL}J_{b^-}^{\alpha} f(a)\bigr] \\ &\quad = \frac{b-a}{2}\int_0^1 \bigl[(1-t)^\alpha - t^\alpha\bigr] f'\bigl(ta+(1-t)b\bigr)\,dt. \end{aligned} \tag{1}$$

By using Lemma 3.1, one can extend to the following result.

Lemma 3.2 *Let $f : [a,b] \to R$ be a differentiable mapping on (a,b) with $a < mb \leq b$. If $f' \in L[a,b]$, then the following equality for fractional integrals holds*:

$$\begin{aligned} &\frac{f(a)+f(mb)}{2} - \frac{\Gamma(\alpha+1)}{2(mb-a)^\alpha}\bigl[{}_{RL}J_{a^+}^{\alpha} f(mb) + {}_{RL}J_{mb^-}^{\alpha} f(a)\bigr] \\ &\quad = \frac{mb-a}{2}\int_0^1 \bigl[(1-t)^\alpha - t^\alpha\bigr] f'\bigl(ta+m(1-t)b\bigr)\,dt. \end{aligned} \tag{2}$$

Proof This is just Lemma 3.1 on the interval $[a, mb] \subset [a,b]$. □

By using Lemma 3.2, we can obtain the main results in this section.

Theorem 3.3 *Let $f : [0,b] \to R$ be a differentiable mapping. If $|f'|$ is measurable and (α, m)-logarithmically convex on $[a,b]$ for some fixed $\alpha \in (0,1]$, $0 \leq a < mb \leq b$, then the following inequality for fractional integrals holds*:

$$\left|\frac{f(a)+f(mb)}{2} - \frac{\Gamma(\alpha+1)}{2(mb-a)^\alpha}\bigl[{}_{RL}J_{a^+}^{\alpha} f(mb) + {}_{RL}J_{mb^-}^{\alpha} f(a)\bigr]\right| \leq I_k,$$

where

$$\begin{aligned} I_k &= \frac{(mb-a)|f'(b)|^m}{\alpha}\sum_{i=1}^{\infty}\left[\frac{(-\ln k)^{i-1}\alpha^i}{[\alpha;i]}\left(k - \frac{k^{2-\alpha}}{2^{i\alpha}}\right)\right. \\ &\quad \left. + \frac{\alpha^i(-\ln k)^{i-1}}{[\alpha;i-1]}\left(\frac{k^{2-\alpha}}{2^{i\alpha+1}} + \frac{k^{2-\alpha}}{2^{i\alpha-\alpha+2}} - \frac{k}{2}\right)\right] \quad \textit{for } k \neq 1, \\ I_k &= \frac{|f'(b)|^m(mb-a)(2^\alpha-1)}{(\alpha+1)2^\alpha} \quad \textit{for } k = 1, \\ k &= \frac{|f'(a)|}{|f'(b)|^m} \end{aligned}$$

and

$$[\alpha;0] := 1, \qquad [\alpha;i] := (\alpha+1)(2\alpha+1)\cdots(i\alpha+1), \quad i \in N.$$

Proof (i) Case 1: $k \neq 1$. By Definition 2.2, Lemma 2.4 and Lemma 2.5, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(mb)}{2} - \frac{\Gamma(\alpha+1)}{2(mb-a)^\alpha}\left[{}_{RL}J^\alpha_{a^+}f(mb) + {}_{RL}J^\alpha_{mb^-}f(a)\right]\right| \\
&\quad\le \frac{mb-a}{2}\int_0^1 \left|(1-t)^\alpha - t^\alpha\right|\left|f'\bigl(ta+m(1-t)b\bigr)\right|dt \\
&\quad\le \frac{mb-a}{2}\int_0^1 \left|(1-t)^\alpha - t^\alpha\right|\left|f'(a)\right|^{t^\alpha}\left|f'(b)\right|^{m-mt^\alpha}dt \\
&\quad\le \frac{(mb-a)|f'(b)|^m}{2}\int_0^1 \left|(1-t)^\alpha - t^\alpha\right|k^{t^\alpha}dt \\
&\quad\le \frac{(mb-a)|f'(b)|^m}{2}\int_0^{\frac{1}{2}} \left|(1-t)^\alpha - t^\alpha\right|k^{t^\alpha}dt \\
&\qquad + \frac{(mb-a)|f'(b)|^m}{2}\int_{\frac{1}{2}}^1 \left|(1-t)^\alpha - t^\alpha\right|k^{t^\alpha}dt \\
&\quad\le \frac{(mb-a)|f'(b)|^m}{2}\int_0^{\frac{1}{2}} \bigl((1-t)^\alpha - t^\alpha\bigr)\bigl(k^{t^\alpha}+k^{(1-t)^\alpha}\bigr)dt \\
&\quad\le \frac{(mb-a)|f'(b)|^m}{2}\int_0^{\frac{1}{2}} \bigl((1-t)^\alpha k^{t^\alpha}+(1-t)^\alpha k^{(1-t)^\alpha}\bigr)dt \\
&\qquad + \frac{(mb-a)|f'(b)|^m}{2}\int_0^{\frac{1}{2}} \bigl(-t^\alpha k^{t^\alpha}-t^\alpha k^{(1-t)^\alpha}\bigr)dt \\
&\quad\le \frac{(mb-a)|f'(b)|^m}{2}\int_0^{\frac{1}{2}} (1-t)^\alpha k^{t^\alpha}dt \\
&\qquad + \frac{(mb-a)|f'(b)|^m}{2}\int_0^{\frac{1}{2}} (1-t)^\alpha k^{(1-t)^\alpha}dt \\
&\qquad - \frac{(mb-a)|f'(b)|^m}{2}\int_0^{\frac{1}{2}} t^\alpha k^{t^\alpha}dt - \frac{(mb-a)|f'(b)|^m}{2}\int_0^{\frac{1}{2}} t^\alpha k^{(1-t)^\alpha}dt \\
&\quad\le \frac{(mb-a)|f'(b)|^m}{2}\int_0^{\frac{1}{2}} \bigl(2^{1-\alpha}-t^\alpha\bigr)k^{t^\alpha}dt + \frac{(mb-a)|f'(b)|^m}{2}\int_{\frac{1}{2}}^1 t^\alpha k^{t^\alpha}dt \\
&\qquad - \frac{(mb-a)|f'(b)|^m}{2}\int_0^{\frac{1}{2}} t^\alpha k^{t^\alpha}dt - \frac{(mb-a)|f'(b)|^m}{2}\int_{\frac{1}{2}}^1 (1-t)^\alpha k^{t^\alpha}dt \\
&\quad\le \frac{(mb-a)|f'(b)|^m}{2}\int_0^{\frac{1}{2}} \bigl(2^{1-\alpha}-t^\alpha\bigr)k^{t^\alpha}dt + \frac{(mb-a)|f'(b)|^m}{2}\int_{\frac{1}{2}}^1 t^\alpha k^{t^\alpha}dt \\
&\qquad - \frac{(mb-a)|f'(b)|^m}{2}\int_0^{\frac{1}{2}} t^\alpha k^{t^\alpha}dt - \frac{(mb-a)|f'(b)|^m}{2}\int_{\frac{1}{2}}^1 \bigl(1-t^\alpha\bigr)k^{t^\alpha}dt \\
&\quad\le \frac{2(mb-a)|f'(b)|^m}{2}\int_{\frac{1}{2}}^1 t^\alpha k^{t^\alpha}dt - \frac{2(mb-a)|f'(b)|^m}{2}\int_0^{\frac{1}{2}} t^\alpha k^{t^\alpha}dt
\end{aligned}$$

$$
\begin{aligned}
&\quad + 2^{1-\alpha}\frac{(mb-a)|f'(b)|^m}{2}\int_0^{\frac{1}{2}} k^{t^\alpha}\,dt - \frac{(mb-a)|f'(b)|^m}{2}\int_{\frac{1}{2}}^1 k^{t^\alpha}\,dt \\
&\le (mb-a)\left|f'(b)\right|^m \int_{\frac{1}{2}}^1 t^\alpha k^{t^\alpha}\,dt - (mb-a)\left|f'(b)\right|^m \int_0^{\frac{1}{2}} t^\alpha k^{t^\alpha}\,dt \\
&\quad + 2^{-\alpha}(mb-a)\left|f'(b)\right|^m \int_0^{\frac{1}{2}} k^{t^\alpha}\,dt - \frac{(mb-a)|f'(b)|^m}{2}\int_{\frac{1}{2}}^1 k^{t^\alpha}\,dt \\
&\le (mb-a)\left|f'(b)\right|^m \frac{1}{\alpha}\int_{\frac{1}{2^\alpha}}^1 t^{(\frac{1}{\alpha}+1)-1}k^t\,dt - (mb-a)|f'(b)|^m\cdot\frac{1}{\alpha}\int_0^{\frac{1}{2^\alpha}} t^{(\frac{1}{\alpha}+1)-1}k^t\,dt \\
&\quad + 2^{-\alpha}(mb-a)\left|f'(b)\right|^m\cdot\frac{1}{\alpha}\int_0^{\frac{1}{2^\alpha}} t^{\frac{1}{\alpha}-1}k^t\,dt - \frac{(mb-a)|f'(b)|^m}{2}\cdot\frac{1}{\alpha}\int_{\frac{1}{2^\alpha}}^1 t^{\frac{1}{\alpha}-1}k^t\,dt \\
&\le \frac{(mb-a)|f'(b)|^m}{\alpha}\left(\int_{\frac{1}{2^\alpha}}^1 t^{(\frac{1}{\alpha}+1)-1}k^t\,dt - \int_0^{\frac{1}{2^\alpha}} t^{(\frac{1}{\alpha}+1)-1}k^t\,dt\right) \\
&\quad + \frac{2^{-\alpha}(mb-a)|f'(b)|^m}{\alpha}\int_0^{\frac{1}{2^\alpha}} t^{\frac{1}{\alpha}-1}k^t\,dt - \frac{(mb-a)|f'(b)|^m}{2\alpha}\int_{\frac{1}{2^\alpha}}^1 t^{\frac{1}{\alpha}-1}k^t\,dt \\
&\le \frac{(mb-a)|f'(b)|^m}{\alpha}\left(\int_0^1 t^{(\frac{1}{\alpha}+1)-1}k^t\,dt - 2\int_0^{\frac{1}{2^\alpha}} t^{(\frac{1}{\alpha}+1)-1}k^t\,dt\right) \\
&\quad + \frac{2^{-\alpha}(mb-a)|f'(b)|^m}{\alpha}\int_0^{\frac{1}{2^\alpha}} t^{\frac{1}{\alpha}-1}k^t\,dt \\
&\quad - \frac{(mb-a)|f'(b)|^m}{2\alpha}\left(\int_0^1 t^{\frac{1}{\alpha}-1}k^t\,dt - \int_0^{\frac{1}{2^\alpha}} t^{\frac{1}{\alpha}-1}k^t\,dt\right) \\
&\le \frac{(mb-a)|f'(b)|^m}{\alpha}\left[k\sum_{i=1}^\infty \frac{(-\ln k)^{i-1}}{(\frac{1}{\alpha}+1)_i} - 2\left(\frac{1}{2^\alpha}\right)^{\frac{1}{\alpha}+1}k^{\frac{1}{2^\alpha}}\sum_{i=1}^\infty \frac{(-\frac{1}{2^\alpha}\ln k)^{i-1}}{(\frac{1}{\alpha}+1)_i}\right] \\
&\quad + \frac{2^{-\alpha}(mb-a)|f'(b)|^m}{\alpha}\left(\frac{1}{2^\alpha}\right)^{\frac{1}{\alpha}}k^{\frac{1}{2^\alpha}}\sum_{i=1}^\infty \frac{(-\frac{1}{2^\alpha}\ln k)^{i-1}}{(\frac{1}{\alpha})_i} \\
&\quad - \frac{(mb-a)|f'(b)|^m}{2\alpha}\left(k\sum_{i=1}^\infty \frac{(-\ln k)^{i-1}}{(\frac{1}{\alpha})_i} - \left(\frac{1}{2^\alpha}\right)^{\frac{1}{\alpha}}k^{\frac{1}{2^\alpha}}\sum_{i=1}^\infty \frac{(-\frac{1}{2^\alpha}\ln k)^{i-1}}{(\frac{1}{\alpha})_i}\right) \\
&\le \frac{(mb-a)|f'(b)|^m}{\alpha}\left[k\sum_{i=1}^\infty \frac{(-\ln k)^{i-1}\alpha^i}{[\alpha;i]} - k^{2^{-\alpha}}\sum_{i=1}^\infty \frac{(-\ln k)^{i-1}\alpha^i}{[\alpha;i]2^{i\alpha}}\right] \\
&\quad + \frac{k^{2^{-\alpha}}(mb-a)|f'(b)|^m}{\alpha}\sum_{i=1}^\infty \frac{\alpha^i(-\ln k)^{i-1}}{[\alpha;i-1]2^{i\alpha+1}} \\
&\quad - \frac{(mb-a)|f'(b)|^m}{2\alpha}\left(k\sum_{i=1}^\infty \frac{\alpha^i(-\ln k)^{i-1}}{[\alpha;i-1]} - k^{2^{-\alpha}}\sum_{i=1}^\infty \frac{\alpha^i(-\ln k)^{i-1}}{[\alpha;i-1]2^{i\alpha-\alpha+1}}\right) \\
&\le \frac{(mb-a)|f'(b)|^m}{\alpha}\sum_{i=1}^\infty\left[\frac{(-\ln k)^{i-1}\alpha^i}{[\alpha;i]}\left(k - \frac{k^{2^{-\alpha}}}{2^{i\alpha}}\right)\right. \\
&\quad \left. + \frac{\alpha^i(-\ln k)^{i-1}}{[\alpha;i-1]}\left(\frac{k^{2^{-\alpha}}}{2^{i\alpha+1}} + \frac{k^{2^{-\alpha}}}{2^{i\alpha-\alpha+2}} - \frac{k}{2}\right)\right].
\end{aligned}
$$

The proof is done.

(ii) Case 2: $k = 1$. By Definition 2.2, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(mb)}{2} - \frac{\Gamma(\alpha+1)}{2(mb-a)^{\alpha}}\left[{}_{RL}J_{a^+}^{\alpha}f(mb) + {}_{RL}J_{mb^-}^{\alpha}f(a)\right]\right| \\
&\quad \le \frac{mb-a}{2}\int_0^1 \left|(1-t)^{\alpha} - t^{\alpha}\right|\left|f'\bigl(ta+m(1-t)b\bigr)\right| dt \\
&\quad \le \frac{mb-a}{2}\int_0^1 \left|(1-t)^{\alpha} - t^{\alpha}\right|\left|f'(a)\right|^{t^{\alpha}}\left|f'(b)\right|^{m-mt^{\alpha}} dt \\
&\quad \le \frac{(mb-a)|f'(b)|^m}{2}\int_0^1 \left|(1-t)^{\alpha} - t^{\alpha}\right| dt \\
&\quad \le (mb-a)\left|f'(b)\right|^m \int_0^{\frac{1}{2}} \left|(1-t)^{\alpha} - t^{\alpha}\right| dt \\
&\quad \le \frac{|f'(b)|^m (mb-a)(2^{\alpha}-1)}{(\alpha+1)2^{\alpha}}.
\end{aligned}$$

The proof is completed. □

Theorem 3.4 *Let $f : [0,b] \to R$ be a differentiable mapping and $1 < q < \infty$. If $|f'|^q$ is measurable and (α,m)-logarithmically convex on $[a,b]$ for some fixed $\alpha \in (0,1]$, $0 \le a < mb \le b$, then the following inequality for fractional integrals holds*:

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[{}_{RL}J_{a^+}^{\alpha}f(b) + {}_{RL}J_{b^-}^{\alpha}f(a)\right]\right| \le I_k,$$

where

$$I_k = \frac{(mb-a)|f'(a)|}{2^{\frac{1}{q}}\alpha^{\frac{1}{q}}}\left(\frac{1-2^{-p\alpha}}{p\alpha+1}\right)^{\frac{1}{p}}\left[\sum_{i=1}^{\infty}\frac{(mq\ln|f'(b)| - q\ln|f'(a)|)^{i-1}\alpha^i}{[\alpha; i-1]}\right]^{\frac{1}{q}} \quad \textit{for } k \neq 1,$$

$$I_k = \frac{(b-a)|f'(b)|^m}{2^{\frac{1}{q}}}\left(\frac{1-2^{-p\alpha}}{p\alpha+1}\right)^{\frac{1}{p}} \quad \textit{for } k = 1$$

and $k = \frac{|f'(a)|^q}{|f'(b)|^{mq}}$, $\frac{1}{p} + \frac{1}{q} = 1$.

Proof (i) Case 1: $k \neq 1$. By Definition 2.2, Lemma 2.4, and using the Hölder inequality, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(mb)}{2} - \frac{\Gamma(\alpha+1)}{2(mb-a)^{\alpha}}\left[{}_{RL}J_{a^+}^{\alpha}f(b) + {}_{RL}J_{mb^-}^{\alpha}f(a)\right]\right| \\
&\quad \le \frac{mb-a}{2}\int_0^1 \left|(1-t)^{\alpha} - t^{\alpha}\right|\left|f'\bigl(ta+m(1-t)b\bigr)\right| dt \\
&\quad \le \frac{mb-a}{2}\left[\int_0^1 \left|(1-t)^{\alpha} - t^{\alpha}\right|^p dt\right]^{\frac{1}{p}}\left[\int_0^1 \left|f'\bigl(ta+m(1-t)b\bigr)\right|^q dt\right]^{\frac{1}{q}} \\
&\quad = \frac{mb-a}{2}\left[2\int_0^{\frac{1}{2}} \bigl((1-t)^{\alpha} - t^{\alpha}\bigr)^p dt\right]^{\frac{1}{p}}\left[\int_0^1 \left|f'\bigl(ta+m(1-t)b\bigr)\right|^q dt\right]^{\frac{1}{q}} \\
&\quad \le \frac{mb-a}{2^{1-\frac{1}{p}}}\left[\int_0^{\frac{1}{2}} \bigl((1-t)^{p\alpha} - t^{p\alpha}\bigr) dt\right]^{\frac{1}{p}}\left[\int_0^1 \left|f'\bigl(ta+m(1-t)b\bigr)\right|^q dt\right]^{\frac{1}{q}}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{mb-a}{2^{1-\frac{1}{p}}}\left(\frac{1-2^{-p\alpha}}{p\alpha+1}\right)^{\frac{1}{p}}\left[\int_0^1 \left|f'\bigl(ta+m(1-t)b\bigr)\right|^q\,dt\right]^{\frac{1}{q}} \\
&\le \frac{mb-a}{2^{\frac{1}{q}}}\left(\frac{1-2^{-p\alpha}}{p\alpha+1}\right)^{\frac{1}{p}}\left[\int_0^1 \left|f'(a)\right|^{qt^\alpha}\left|f'(b)\right|^{mq-mqt^\alpha}\,dt\right]^{\frac{1}{q}} \\
&\le \frac{(mb-a)|f'(b)|^m}{2^{\frac{1}{q}}}\left(\frac{1-2^{-p\alpha}}{p\alpha+1}\right)^{\frac{1}{p}}\left[\int_0^1 k^{t^\alpha}\,dt\right]^{\frac{1}{q}} \\
&\le \frac{(mb-a)|f'(b)|^m}{2^{\frac{1}{q}}\alpha^{\frac{1}{q}}}\left(\frac{1-2^{-p\alpha}}{p\alpha+1}\right)^{\frac{1}{p}}\left[k\sum_{i=1}^{\infty}\frac{(-\ln k)^{i-1}}{(\frac{1}{\alpha})_i}\right]^{\frac{1}{q}} \\
&\le \frac{(mb-a)|f'(a)|}{2^{\frac{1}{q}}\alpha^{\frac{1}{q}}}\left(\frac{1-2^{-p\alpha}}{p\alpha+1}\right)^{\frac{1}{p}}\left[\sum_{i=1}^{\infty}\frac{(-\ln k)^{i-1}}{(\frac{1}{\alpha})_i}\right]^{\frac{1}{q}} \\
&\le \frac{(mb-a)|f'(a)|}{2^{\frac{1}{q}}\alpha^{\frac{1}{q}}}\left(\frac{1-2^{-p\alpha}}{p\alpha+1}\right)^{\frac{1}{p}}\left[\sum_{i=1}^{\infty}\frac{(mq\ln|f'(b)|-q\ln|f'(a)|)^{i-1}\alpha^i}{[\alpha;i-1]}\right]^{\frac{1}{q}}.
\end{aligned}
$$

The proof is done.

(ii) Case 2: $k = 1$. By Definition 2.2, Lemma 2.4, and using the Hölder inequality, we have

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\bigl[{}_{RL}J_{a^+}^{\alpha}f(b)+{}_{RL}J_{b^-}^{\alpha}f(a)\bigr]\right| \\
&\quad\le \frac{b-a}{2^{\frac{1}{q}}}\left(\frac{1-2^{-p\alpha}}{p\alpha+1}\right)^{\frac{1}{p}}\left[\int_0^1 \left|f'(a)\right|^{qt^\alpha}\left|f'(b)\right|^{mq-mqt^\alpha}\,dt\right]^{\frac{1}{q}} \\
&\quad\le \frac{(b-a)|f'(b)|^m}{2^{\frac{1}{q}}}\left(\frac{1-2^{-p\alpha}}{p\alpha+1}\right)^{\frac{1}{p}}.
\end{aligned}
$$

The proof is done. □

4 The second main results

In this section, we apply a fractional integral identity to derive some new Hermite-Hadamard type inequalities for twice differentiable (α, m)-logarithmically convex functions.

We need the following result.

Lemma 4.1 (see [15]) *Let $f : [a, b] \to R$ be a twice differentiable mapping on (a, b) with $a < mb \le b$. If $f'' \in L^1[a, b]$, then the following equality for fractional integrals holds*:

$$
\begin{aligned}
&\frac{f(a)+f(mb)}{2}-\frac{\Gamma(\alpha+1)}{2(mb-a)^\alpha}\bigl[{}_{RL}J_{a^+}^{\alpha}f(mb)+{}_{RL}J_{mb^-}^{\alpha}f(a)\bigr] \\
&\quad= \frac{(mb-a)^2}{2}\int_0^1 \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}f''\bigl(ta+m(1-t)b\bigr)\,dt.
\end{aligned}
$$

Now we are ready to present the main results in this section.

Theorem 4.2 *Let $f:[0,b]\to R$ be a differentiable mapping. If $|f''|$ is measurable and (α,m)-logarithmically convex on $[a,b]$ for some fixed $\alpha\in(0,1]$, $0\le a<mb\le b$, then the following inequality for fractional integrals holds*:

$$\left|\frac{f(a)+f(mb)}{2}-\frac{\Gamma(\alpha+1)}{2(mb-a)^\alpha}\left[{}_{RL}J^\alpha_{a^+}f(mb)+{}_{RL}J^\alpha_{mb^-}f(a)\right]\right|\le I_k,$$

where

$$I_k=\frac{|f''(a)|(mb-a)^2(2^\alpha-1)}{2^{\alpha+1}(\alpha+1)}\sum_{i=1}^{\infty}\frac{(m\alpha\ln|f''(b)|-\alpha\ln|f''(a)|)^{i-1}}{[\alpha;i-1]}\quad \textit{for } k\ne 1,$$

$$I_k=\frac{(b-a)^2|f''(b)|^m}{2(\alpha+1)}\left(1-\frac{2}{p\alpha+p+1}\right)^{\frac{1}{p}}\quad \textit{for } k=1$$

and $k=\frac{|f''(a)|}{|f''(b)|^m}$.

Proof (i) Case 1: $k\ne 1$. By Definition 2.2, Lemma 2.3 and Lemma 2.5, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(mb)}{2}-\frac{\Gamma(\alpha+1)}{2(mb-a)^\alpha}\left[{}_{RL}J^\alpha_{a^+}f(mb)+{}_{RL}J^\alpha_{mb^-}f(a)\right]\right|\\
&\quad\le\frac{(mb-a)^2}{2}\int_0^1\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\left|f''\bigl(ta+m(1-t)b\bigr)\right|dt\\
&\quad\le\frac{(mb-a)^2}{2}\int_0^1\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\left|f''(a)\right|^{t^\alpha}\left|f''(b)\right|^{m-mt^\alpha}dt\\
&\quad\le\frac{(mb-a)^2|f''(b)|^m}{2(\alpha+1)}\int_0^1\left[1-\bigl(2^{-\alpha}-t^{\alpha+1}\bigr)-t^{\alpha+1}\right]k^{t^\alpha}dt\\
&\quad\le\frac{(mb-a)^2|f''(b)|^m(2^\alpha-1)}{2^{\alpha+1}(\alpha+1)}\int_0^1k^{t^\alpha}dt\\
&\quad\le\frac{(mb-a)^2|f''(b)|^m(2^\alpha-1)}{2^{\alpha+1}\alpha(\alpha+1)}\int_0^1t^{\frac{1}{\alpha}-1}k^t\,dt\\
&\quad\le\frac{(mb-a)^2|f''(b)|^m(2^\alpha-1)}{2^{\alpha+1}\alpha(\alpha+1)}k\sum_{i=1}^{\infty}\frac{(-\ln k)^{i-1}}{(\frac{1}{\alpha})_i}\\
&\quad\le\frac{|f''(a)|(mb-a)^2(2^\alpha-1)}{2^{\alpha+1}\alpha(\alpha+1)}\sum_{i=1}^{\infty}\frac{(m\ln|f''(b)|-\ln|f''(a)|)^{i-1}\alpha^i}{[\alpha;i-1]}\\
&\quad\le\frac{|f''(a)|(mb-a)^2(2^\alpha-1)}{2^{\alpha+1}(\alpha+1)}\sum_{i=1}^{\infty}\frac{(m\alpha\ln|f''(b)|-\alpha\ln|f''(a)|)^{i-1}}{[\alpha;i-1]}.
\end{aligned}$$

The proof is done.

(ii) Case 2: $k=1$. By Definition 2.2, Lemma 2.3, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(mb)}{2}-\frac{\Gamma(\alpha+1)}{2(mb-a)^\alpha}\left[{}_{RL}J^\alpha_{a^+}f(mb)+{}_{RL}J^\alpha_{mb^-}f(a)\right]\right|\\
&\quad\le\frac{(mb-a)^2}{2}\int_0^1\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\left|f''(a)\right|^{t^\alpha}\left|f''(b)\right|^{m-mt^\alpha}dt
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{(mb-a)^2|f''(b)|^m}{2(\alpha+1)} \int_0^1 \left[1-\left(2^{-\alpha}-t^{\alpha+1}\right)-t^{\alpha+1}\right] dt \\
&\leq \frac{(mb-a)^2|f''(b)|^m(2^\alpha-1)}{2^{\alpha+1}(\alpha+1)}.
\end{aligned}
$$

The proof is done. □

Theorem 4.3 *Let $f:[0,b]\to R$ be a differentiable mapping and $1<q<\infty$. If $|f''|^q$ is measurable and (α,m)-logarithmically convex on $[a,b]$ for some fixed $\alpha\in(0,1]$, $0\leq a<mb\leq b$, then the following inequality for fractional integrals holds*:

$$
\left|\frac{f(a)+f(mb)}{2}-\frac{\Gamma(\alpha+1)}{2(mb-a)^\alpha}\left[{}_{RL}J_{a^+}^\alpha f(mb)+{}_{RL}J_{mb^-}^\alpha f(a)\right]\right|\leq I_k,
$$

where

$$
I_k=\frac{(mb-a)^2|f''(a)|}{2(\alpha+1)}\left(1-\frac{1}{2^{p\alpha}}\right)^{\frac{1}{p}}\left[\sum_{i=1}^\infty\frac{(mq\alpha\ln|f'(b)|-q\alpha\ln|f'(a)|)^{i-1}}{[\alpha;i-1]}\right]^{\frac{1}{q}} \quad \text{for } k\neq 1,
$$

$$
I_k=\frac{(mb-a)^2|f''(b)|}{2(\alpha+1)}\left(1-\frac{1}{2^{p\alpha}}\right)^{\frac{1}{p}} \quad \text{for } k=1
$$

and $k=\frac{|f''(a)|^q}{|f''(b)|^{mq}}$, $\frac{1}{p}+\frac{1}{q}=1$.

Proof (i) Case 1: $k\neq 1$. By Definition 2.2, Lemma 2.3, Lemma 2.5, and using the Hölder inequality, we have

$$
\begin{aligned}
&\left|\frac{f(a)+f(mb)}{2}-\frac{\Gamma(\alpha+1)}{2(mb-a)^\alpha}\left[{}_{RL}J_{a^+}^\alpha f(mb)+{}_{RL}J_{mb^-}^\alpha f(a)\right]\right| \\
&\leq \frac{(mb-a)^2}{2}\int_0^1\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\left|f''\left(ta+m(1-t)b\right)\right| dt \\
&\leq \frac{(mb-a)^2}{2(\alpha+1)}\left(\int_0^1\left(1-(1-t)^{\alpha+1}-t^{\alpha+1}\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''\left(ta+m(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{(mb-a)^2}{2(\alpha+1)}\left(1-\frac{1}{2^{p\alpha}}\right)^{\frac{1}{p}}\left(\int_0^1\left|f''(a)\right|^{qt^\alpha}\left|f''(b)\right|^{mq-mqt^\alpha} dt\right)^{\frac{1}{q}} \\
&\leq \frac{(mb-a)^2|f''(b)|^m}{2(\alpha+1)}\left(1-\frac{1}{2^{p\alpha}}\right)^{\frac{1}{p}}\left(\int_0^1 k^{t^\alpha} dt\right)^{\frac{1}{q}} \\
&\leq \frac{(mb-a)^2|f''(a)|}{2(\alpha+1)\alpha^{\frac{1}{q}}}\left(1-\frac{1}{2^{p\alpha}}\right)^{\frac{1}{p}}\left(\sum_{i=1}^\infty\frac{(-\ln k)^{i-1}}{(\frac{1}{\alpha})_i}\right)^{\frac{1}{q}} \\
&\leq \frac{(mb-a)^2|f''(a)|}{2(\alpha+1)\alpha^{\frac{1}{q}}}\left(1-\frac{1}{2^{p\alpha}}\right)^{\frac{1}{p}}\left[\sum_{i=1}^\infty\frac{(mq\ln|f'(b)|-q\ln|f'(a)|)^{i-1}\alpha^i}{[\alpha;i-1]}\right]^{\frac{1}{q}} \\
&\leq \frac{(mb-a)^2|f''(a)|}{2(\alpha+1)}\left(1-\frac{1}{2^{p\alpha}}\right)^{\frac{1}{p}}\left[\sum_{i=1}^\infty\frac{(mq\alpha\ln|f'(b)|-q\alpha\ln|f'(a)|)^{i-1}}{[\alpha;i-1]}\right]^{\frac{1}{q}},
\end{aligned}
$$

where we use the following inequality:

$$\begin{aligned}\left(1-(1-t)^{\alpha+1}-t^{\alpha+1}\right)^p &\le 1-\left[(1-t)^{\alpha+1}+t^{\alpha+1}\right]^p\\ &\le 1-\left(2^{-\alpha}\right)^p\\ &= 1-\frac{1}{2^{p\alpha}}.\end{aligned}$$

The proof is done.

(ii) Case 2: $k = 1$. By Definition 2.2, Lemma 2.3, and using the Hölder inequality, we have

$$\begin{aligned}&\left|\frac{f(a)+f(mb)}{2}-\frac{\Gamma(\alpha+1)}{2(mb-a)^\alpha}\left[{}_{RL}J^\alpha_{a^+}f(mb)+{}_{RL}J^\alpha_{mb^-}f(a)\right]\right|\\ &\quad\le\frac{(mb-a)^2}{2}\int_0^1\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\left|f''\left(ta+m(1-t)b\right)\right|dt\\ &\quad\le\frac{(mb-a)^2}{2(\alpha+1)}\left(\int_0^1\left(1-(1-t)^{\alpha+1}-t^{\alpha+1}\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''\left(ta+m(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}}\\ &\quad\le\frac{(mb-a)^2}{2(\alpha+1)}\left(1-\frac{1}{2^{p\alpha}}\right)^{\frac{1}{p}}\left(\int_0^1\left|f''(a)\right|^{qt^\alpha}\left|f''(b)\right|^{mq-mqt^\alpha}dt\right)^{\frac{1}{q}}\\ &\quad\le\frac{(mb-a)^2|f''(b)|^m}{2(\alpha+1)}\left(1-\frac{1}{2^{p\alpha}}\right)^{\frac{1}{p}}.\end{aligned}$$

The proof is done. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
This work was carried out in collaboration between all authors. JRW raised these interesting problems in this research. JD and JRW proved the theorems, interpreted the results and wrote the article. All authors defined the research theme, read and approved the manuscript.

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