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Refined converses of Jensen's inequality for operators

Jadranka Mićić[1*], Josip Pečarić[2] and Jurica Perić[3]

*Correspondence: jmicic@fsb.hr
[1]Faculty of Mechanical Engineering and Naval Architecture, University of Zagreb, Ivana Lučića 5, Zagreb, 10000, Croatia
Full list of author information is available at the end of the article

Abstract
In this paper converses of a generalized Jensen's inequality for a continuous field of self-adjoint operators, a unital field of positive linear mappings and real-valued continuous convex functions are studied. New refined converses are presented by using the Mond-Pečarić method improvement. Obtained results are applied to refine selected inequalities with power functions.

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1 Introduction

Let T be a locally compact Hausdorff space and let $\mathcal{A}$ be a C^*-algebra of operators on some Hilbert space H. We say that a field $(x_t)_{t\in T}$ of operators in $\mathcal{A}$ is continuous if the function $t \mapsto x_t$ is norm continuous on T. If in addition μ is a Radon measure on T and the function $t \mapsto \|x_t\|$ is integrable, then we can form *the Bochner integral* $\int_T x_t\,d\mu(t)$, which is the unique element in $\mathcal{A}$ such that

$$\varphi\left(\int_T x_t\,d\mu(t)\right) = \int_T \varphi(x_t)\,d\mu(t)$$

for every linear functional φ in the norm dual $\mathcal{A}^*$.

Assume further that there is a field $(\phi_t)_{t\in T}$ of positive linear mappings $\phi_t : \mathcal{A} \to \mathcal{B}$ from $\mathcal{A}$ to another $\mathcal{C}^*$-algebra $\mathcal{B}$ of operators on a Hilbert space K. We recall that a linear mapping $\phi : \mathcal{A} \to \mathcal{B}$ is said to be positive if $\phi(x) \geq 0$ for all $x \geq 0$. We say that such a field $(\phi_t)_{t\in T}$ is continuous if the function $t \mapsto \phi_t(x)$ is continuous for every $x \in \mathcal{A}$. Let the $\mathcal{C}^*$-algebras include the identity operators and let the function $t \mapsto \phi_t(1_H)$ be integrable with $\int_T \phi_t(1_H)\,d\mu(t) = k1_K$ for some positive scalar k. If $\int_T \phi_t(1_H)\,d\mu(t) = 1_K$, we say that a *field* $(\phi_t)_{t\in T}$ is *unital*.

Let $B(H)$ be the C^*-algebra of all bounded linear operators on a Hilbert space H. We define bounds of a self-adjoint operator $x \in B(H)$ by

$$m_x := \inf_{\|\xi\|=1} \langle x\xi, \xi\rangle \quad \text{and} \quad M_x := \sup_{\|\xi\|=1} \langle x\xi, \xi\rangle \tag{1}$$

for $\xi \in H$. If $\mathrm{Sp}(x)$ denotes the spectrum of x, then $\mathrm{Sp}(x) \subseteq [m_x, M_x]$.

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For an operator $x \in B(H)$, we define the operator $|x| := (x^*x)^{1/2}$. Obviously, if x is self-adjoint, then $|x| = (x^2)^{1/2}$.

Jensen's inequality is one of the most important inequalities. It has many applications in mathematics and statistics and some other well-known inequalities are its special cases.

Let f be an operator convex function defined on an interval I. Davis [1] proved the so-called Jensen operator inequality

$$f\bigl(\phi(x)\bigr) \le \phi\bigl(f(x)\bigr), \tag{2}$$

where $\phi\colon \mathcal{A} \to B(K)$ is a unital completely positive linear mapping from a C^*-algebra $\mathcal{A}$ to linear operators on a Hilbert space K, and x is a self-adjoint element in $\mathcal{A}$ with spectrum in I. Subsequently, Choi [2] noted that it is enough to assume that ϕ is unital and positive.

Mond, Pečarić, Hansen, Pedersen *et al.* in [3–6] studied another generalization of (2) for operator convex functions. Moreover, Hansen *et al.* [7] presented a general formulation of Jensen's operator inequality for a bounded continuous field of self-adjoint operators and a unital field of positive linear mappings:

$$f\biggl(\int_T \phi_t(x_t)\,d\mu(t)\biggr) \le \int_T \phi_t\bigl(f(x_t)\bigr)\,d\mu(t), \tag{3}$$

where f is an operator convex function.

There is an extensive literature devoted to Jensen's inequality concerning different refinements and extensive results, *e.g.*, see [8–20]. Mićić *et al.* [21] proved that the discrete version of (3) stands without operator convexity of f under a condition on the spectra of operators. Recently, Mićić *et al.* [22] presented a discrete version of refined Jensen's inequality for real-valued continuous convex functions. A continuous version is given below.

Theorem 1 *Let $(x_t)_{t\in T}$ be a bounded continuous field of self-adjoint elements in a unital C^*-algebra $\mathcal{A}$ defined on a locally compact Hausdorff space T equipped with a bounded Radon measure μ. Let m_t and M_t, $m_t \le M_t$, be the bounds of x_t, $t \in T$. Let $(\phi_t)_{t\in T}$ be a unital field of positive linear mappings $\phi_t : \mathcal{A} \to \mathcal{B}$ from $\mathcal{A}$ to another unital C^*-algebra $\mathcal{B}$. Let*

$$(m_x, M_x) \cap [m_t, M_t] = \emptyset, \quad t \in T, \text{ and } a < b,$$

where m_x and M_x, $m_x \le M_x$, are the bounds of the operator $x = \int_T \phi_t(x_t)\,d\mu(t)$ and

$$a = \sup\{M_t : M_t \le m_x, t \in T\}, \qquad b = \inf\{m_t : m_t \ge M_x, t \in T\}.$$

If $f : I \to \mathbb{R}$ is a continuous convex (resp. concave) function provided that the interval I contains all m_t, M_t, then

$$f\biggl(\int_T \phi_t(x_t)\,d\mu(t)\biggr) \le \int_T \phi_t\bigl(f(x_t)\bigr)\,d\mu(t) - \delta_f \bar{x} \le \int_T \phi_t\bigl(f(x_t)\bigr)\,d\mu(t)$$

(*resp.*

$$f\biggl(\int_T \phi_t(x_t)\,d\mu(t)\biggr) \ge \int_T \phi_t\bigl(f(x_t)\bigr)\,d\mu(t) - \delta_f \bar{x} \ge \int_T \phi_t\bigl(f(x_t)\bigr)\,d\mu(t)) \tag{4}$$

holds, where

$$\delta_f \equiv \delta_f(\bar{m},\bar{M}) = f(\bar{m}) + f(\bar{M}) - 2f\left(\frac{\bar{m}+\bar{M}}{2}\right)$$
$$\left(\textit{resp. } \delta_f \equiv \delta_f(\bar{m},\bar{M}) = 2f\left(\frac{\bar{m}+\bar{M}}{2}\right) - f(\bar{m}) - f(\bar{M})\right),$$
$$\bar{x} \equiv \bar{x}_x(\bar{m},\bar{M}) = \frac{1}{2}1_K - \frac{1}{\bar{M}-\bar{m}}\left|x - \frac{\bar{m}+\bar{M}}{2}1_K\right|$$

and $\bar{m} \in [a, m_x]$, $\bar{M} \in [M_x, b]$, $\bar{m} < \bar{M}$, *are arbitrary numbers.*

The proof is similar to [22, Theorem 3] and we omit it.

On the other hand, Mond, Pečarić, Furuta *et al.* in [6, 23–27] investigated converses of Jensen's inequality. For presenting these results, we introduce some abbreviations. Let $f : [m,M] \to \mathbb{R}$, $m < M$. Then a linear function through $(m,f(m))$ and $(M,f(M))$ has the form $h(z) = k_f z + l_f$, where

$$k_f := \frac{f(M) - f(m)}{M - m} \quad \text{and} \quad l_f := \frac{Mf(m) - mf(M)}{M - m}. \tag{5}$$

Using the Mond-Pečarić method, in [27] the following generalized converse of Jensen's operator inequality (2) is presented

$$F\big[\phi\big(f(A)\big), g\big(\phi(A)\big)\big] \le \max_{m \le z \le M} F\big[k_f z + l_f, g(z)\big]1_{\tilde{n}}, \tag{6}$$

for a convex function f defined on an interval $[m,M]$, $m < M$, where g is a real-valued continuous function on $[m,M]$, $F(u,v)$ is a real-valued function defined on $U \times V$, operator monotone in u, $U \supset f[m,M]$, $V \supset g[m,M]$, $\phi : H_n \to H_{\tilde{n}}$ is a unital positive linear mapping and A is a self-adjoint operator with spectrum contained in $[m,M]$.

A continuous version of (6) and in the case of $\int_T \phi_t(1_H)\,d\mu(t) = k1_K$ for some positive scalar k, is presented in [28]. Recently, Mićić *et al.* [29] obtained better bound than the one given in (6) as follows.

Theorem 2 [29, Theorem 2.1] *Let $(x_t)_{t\in T}$ be a bounded continuous field of self-adjoint elements in a unital C^*-algebra $\mathcal{A}$ with the spectra in $[m,M]$, $m < M$, defined on a locally compact Hausdorff space T equipped with a bounded Radon measure μ, and let $(\phi_t)_{t\in T}$ be a unital field of positive linear maps $\phi_t : \mathcal{A} \to \mathcal{B}$ from $\mathcal{A}$ to another unital C^*-algebra $\mathcal{B}$. Let m_x and M_x, $m_x \le M_x$, be the bounds of the self-adjoint operator $x = \int_T \phi_t(x_t)\,d\mu(t)$ and $f : [m,M] \to \mathbb{R}$, $g : [m_x,M_x] \to \mathbb{R}$, $F : U \times V \to \mathbb{R}$, where $f([m,M]) \subseteq U$, $g([m_x,M_x]) \subseteq V$ and F is bounded.*

If f is convex and F is an operator monotone in the first variable, then

$$F\left[\int_T \phi_t\big(f(x_t)\big)\,d\mu(t), g\left(\int_T \phi_t(x_t)\,d\mu(t)\right)\right] \le C_1 1_K \le C 1_K, \tag{7}$$

where constants $C_1 \equiv C_1(F,f,g,m,M,m_x,M_x)$ *and* $C \equiv C(F,f,g,m,M)$ *are*

$$\begin{aligned} C_1 &= \sup_{m_x \le z \le M_x} F\bigl[k_f z + l_f, g(z)\bigr] \\ &= \sup_{\frac{M-M_x}{M-m} \le p \le \frac{M-m_x}{M-m}} F\bigl[pf(m) + (1-p)f(M), g\bigl(pm + (1-p)M\bigr)\bigr], \\ C &= \sup_{m \le z \le M} F\bigl[k_f z + l_f, g(z)\bigr] \\ &= \sup_{0 \le p \le 1} F\bigl[pf(m) + (1-p)f(M), g\bigl(pm + (1-p)M\bigr)\bigr]. \end{aligned}$$

If f *is concave, then reverse inequalities are valid in* (7) *with* inf *instead of* sup *in bounds* C_1 *and* C.

In this paper, we present refined converses of Jensen's operator inequality. Applying these results, we further refine selected inequalities with power functions.

2 Main results

In the following we assume that $(x_t)_{t\in T}$ is a bounded continuous field of self-adjoint elements in a unital C^*-algebra $\mathcal{A}$ with the spectra in $[m,M]$, $m < M$, defined on a locally compact Hausdorff space T equipped with a bounded Radon measure μ and that $(\phi_t)_{t\in T}$ is a unital field of positive linear mappings $\phi_t : \mathcal{A} \to \mathcal{B}$ between C^*-algebras.

For convenience, we introduce abbreviations $\widetilde{x}$ and δ_f as follows:

$$\widetilde{x} \equiv \widetilde{x}_{x_t,\phi_t}(m,M) := \frac{1}{2}1_K - \frac{1}{M-m}\int_T \phi_t\left(\left|x_t - \frac{m+M}{2}1_H\right|\right) d\mu(t), \tag{8}$$

where m, M, $m < M$, are some scalars such that the spectra of x_t, $t \in T$, are in $[m,M]$;

$$\delta_f \equiv \delta_f(m,M) := f(m) + f(M) - 2f\left(\frac{m+M}{2}\right), \tag{9}$$

where $f : [m,M] \to \mathbb{R}$ is a continuous function.

Obviously, $m1_H \le x_t \le M1_H$ implies $-\frac{M-m}{2}1_H \le x_t - \frac{m+M}{2}1_H \le \frac{M-m}{2}1_H$ for $t \in T$ and $\int_T \phi_t(|x_t - \frac{m+M}{2}1_H|)\, d\mu(t) \le \frac{M-m}{2}\int_T \phi_t(1_H)\, d\mu(t) = \frac{M-m}{2}1_K$. It follows $\widetilde{x} \ge 0$. Also, if f is *convex* (resp. *concave*), then $\delta_f \ge 0$ (resp. $\delta_f \le 0$).

To prove our main result related to converse Jensen's inequality, we need the following lemma.

Lemma 3 *Let* f *be a convex function on an interval* I, $m, M \in I$ *and* $p_1, p_2 \in [0,1]$ *such that* $p_1 + p_2 = 1$. *Then*

$$\begin{aligned} &\min\{p_1,p_2\}\left[f(m) + f(M) - 2f\left(\frac{m+M}{2}\right)\right] \\ &\quad\le p_1 f(m) + p_2 f(M) - f(p_1 m + p_2 M) \\ &\quad\le \max\{p_1,p_2\}\left[f(m) + f(M) - 2f\left(\frac{m+M}{2}\right)\right]. \end{aligned} \tag{10}$$

Proof These results follow from [30, Theorem 1, p.717] for $n = 2$. For the reader's convenience, we give an elementary proof of (10).

Let $a_i \le b_i$, $i = 1, 2$, be positive real numbers such that $A = a_1 + a_2 < B = b_1 + b_2$. Using Jensen's inequality and its reverse, we get

$$\begin{aligned} &Bf\left(\frac{b_1 m + b_2 M}{B}\right) - Af\left(\frac{a_1 m + a_2 M}{A}\right) \\ &\quad\le (B-A)f\left(\frac{(b_1 - a_1)m + (b_2 - a_2)M}{B-A}\right) \\ &\quad\le (b_1 - a_1)f(m) + (b_2 - a_2)f(M) \\ &\quad= b_1 f(m) + b_2 f_2(M) - \bigl(a_1 f(m) + a_2 f_2(M)\bigr). \end{aligned} \tag{11}$$

Suppose that $0 < p_1 < p_2 < 1$, $p_1 + p_2 = 1$. Replacing a_1 and a_2 by p_1 and p_2, respectively, and putting $b_1 = b_2 = p_2$, $A = 1$ and $B = 2p_2$ in (11), we get

$$2p_2 f\left(\frac{m+M}{2}\right) - f\bigl(p_1 f(m) + p_2 f(M)\bigr) \le p_2 f(m) + p_2 f_2(M) - \bigl(p_1 f(m) + p_2 f_2(M)\bigr),$$

which gives the right-hand side of (10). Similarly, replacing b_1 and b_2 by p_1 and p_2, respectively, and putting $a_1 = a_2 = p_1$, $A = 2p_1$ and $B = 1$ in (11), we obtain the left-hand side of (10).

If $p_1 = 0$, $p_2 = 1$ or $p_1 = 1$, $p_2 = 0$, then inequality (10) holds, since f is convex. If $p_1 = p_2 = 1/2$, then we have an equality in (10). □

The main result of an improvement of the Mond-Pečarić method follows.

Lemma 4 *Let* $(x_t)_{t\in T}$, $(\phi_t)_{t\in T}$, *m and M be as above. Then*

$$\int_T \phi_t\bigl(f(x_t)\bigr)\,d\mu(t) \le k_f \int_T \phi_t(x_t)\,d\mu(t) + l_f 1_K - \delta_f \widetilde{x} \le k_f \int_T \phi_t(x_t)\,d\mu(t) + l_f 1_K \tag{12}$$

for every continuous convex function $f : [m, M] \to \mathbb{R}$, *where* $\widetilde{x}$ *and* δ_f *are defined by* (8) *and* (9), *respectively.*

If f *is concave, then the reverse inequality is valid in* (12).

Proof We prove only the convex case. By using (10) we get

$$f(p_1 m + p_2 M) \le p_1 f(m) + p_2 f(M) - \min\{p_1, p_2\}\left[f(m) + f(M) - 2f\left(\frac{m+M}{2}\right)\right] \tag{13}$$

for every $p_1, p_2 \in [0, 1]$ such that $p_1 + p_2 = 1$. Let functions $p_1, p_2 : [m, M] \to [0, 1]$ be defined by

$$p_1(z) = \frac{M - z}{M - m}, \qquad p_2(z) = \frac{z - m}{M - m}.$$

Then, for any $z \in [m, M]$, we can write

$$f(z) = f\left(\frac{M-z}{M-m} m + \frac{z-m}{M-m} M\right) = f\bigl(p_1(z)m + p_2(z)M\bigr).$$

By using (13) we get

$$f(z) \le \frac{M-z}{M-m} f(m) + \frac{z-m}{M-m} f(M) - \tilde{z}\left[f(m) + f(M) - 2f\left(\frac{m+M}{2}\right)\right], \tag{14}$$

where

$$\tilde{z} = \frac{1}{2} - \frac{1}{M-m}\left|z - \frac{m+M}{2}\right|,$$

since

$$\min\left\{\frac{M-z}{M-m}, \frac{z-m}{M-m}\right\} = \frac{1}{2} - \frac{1}{M-m}\left|z - \frac{m+M}{2}\right|.$$

Now since $\mathsf{Sp}(x_t) \subseteq [m, M]$, by utilizing the functional calculus to (14), we obtain

$$f(x_t) \le \frac{M-x_t}{M-m} f(m) + \frac{x_t-m}{M-m} f(M) - \widetilde{x}_t\left[f(m) + f(M) - 2f\left(\frac{m+M}{2}\right)\right],$$

where

$$\widetilde{x}_t = \frac{1}{2}1_H - \frac{1}{M-m}\left|x_t - \frac{m+M}{2}1_H\right|.$$

Applying a positive linear mapping ϕ_t, integrating and using $\int_T \phi_t(1_H)\,d\mu(t) = 1_K$, we get the first inequality in (12) since

$$\widetilde{x} = \int_T \phi_t(\widetilde{x}_t)\,d\mu(t) = \frac{1}{2}1_K - \frac{1}{M-m}\int_T \phi_t\left(\left|x_t - \frac{m+M}{2}1_H\right|\right)d\mu(t).$$

By using that $\delta_f \widetilde{x} \ge 0$, the second inequality in (12) holds. □

We can use Lemma 4 to obtain refinements of some other inequalities mentioned in the introduction. First, we present a refinement of Theorem 2.

Theorem 5 *Let m_x and M_x, $m_x \le M_x$, be the bounds of the operator $x = \int_T \phi_t(x_t)\,d\mu(t)$ and let $m_{\widetilde{x}}$ be the lower bound of the operator $\widetilde{x}$. Let $f : [m, M] \to \mathbb{R}$, $g : [m_x, M_x] \to \mathbb{R}$, $F : U \times V \to \mathbb{R}$, where $f([m, M]) \subseteq U$, $g([m_x, M_x]) \subseteq V$ and F is bounded.*

If f is convex and F is operator monotone in the first variable, then

$$\begin{aligned} &F\left[\int_T \phi_t\big(f(x_t)\big)\,d\mu(t), g\left(\int_T \phi_t(x_t)\,d\mu(t)\right)\right] \\ &\quad\le F\left[k_f x + l_f - \delta_f \widetilde{x}, g\left(\int_T \phi_t(x_t)\,d\mu(t)\right)\right] \\ &\quad\le \sup_{m_x \le z \le M_x} F\big[k_f z + l_f - \delta_f m_{\widetilde{x}}, g(z)\big]1_K \le \sup_{m_x \le z \le M_x} F\big[k_f z + l_f, g(z)\big]1_K. \end{aligned} \tag{15}$$

If f is concave, then the reverse inequality is valid in (15) *with* inf *instead of* sup.

Proof We prove only the convex case. Then $\delta_f \geq 0$ implies $0 \leq \delta_f m_{\widetilde{x}} 1_K \leq \delta_f \widetilde{x}$. By using (12) it follows that

$$\int_T \phi_t(f(x_t))\, d\mu(t) \leq k_f x + l_f - \delta_f \widetilde{x} \leq k_f x + l_f - \delta_f m_{\widetilde{x}} 1_K \leq k_f x + l_f.$$

Using operator monotonicity of $F(\cdot, v)$ in the first variable, we obtain (15). □

3 Difference-type converse inequalities

By using Jensen's operator inequality, we obtain that

$$\alpha g\left(\int_T \phi_t(x_t)\, d\mu(t)\right) \leq \int_T \phi_t(f(x_t))\, d\mu(t) \tag{16}$$

holds for every operator convex function f on $[m, M]$, every function g and real number α such that $\alpha g \leq f$ on $[m, M]$. Now, applying Theorem 5 to the function $F(u, v) = u - \alpha v$, $\alpha \in \mathbb{R}$, we obtain the following converse of (16). It is also a refinement of [29, Theorem 3.1].

Theorem 6 *Let m_x and M_x, $m_x \leq M_x$, be the bounds of the operator $x = \int_T \phi_t(x_t)\, d\mu(t)$ and $f : [m, M] \to \mathbb{R}$, $g : [m_x, M_x] \to \mathbb{R}$ be continuous functions.*

If f is convex and $\alpha \in \mathbb{R}$, then

$$\int_T \phi_t(f(x_t))\, d\mu(t) - \alpha g\left(\int_T \phi_t(x_t)\, d\mu(t)\right) \leq \max_{m_x \leq z \leq M_x} \{k_f z + l_f - \alpha g(z)\} 1_K - \delta_f \widetilde{x}. \tag{17}$$

If f is concave, then the reverse inequality is valid in (17) *with* min *instead of* max.

Remark 1 (1) Obviously,

$$\begin{aligned} &\int_T \phi_t(f(x_t))\, d\mu(t) - \alpha g\left(\int_T \phi_t(x_t)\, d\mu(t)\right) \\ &\quad \leq \max_{m_x \leq z \leq M_x} \{k_f z + l_f - \alpha g(z)\} 1_K - \delta_f \widetilde{y} \leq \max_{m_x \leq z \leq M_x} \{k_f z + l_f - \alpha g(z)\} 1_K \end{aligned}$$

for every convex function f, every $\alpha \in \mathbb{R}$, and $m_{\widetilde{x}} 1_K \leq \widetilde{y} \leq \widetilde{x}$, where $m_{\widetilde{x}}$ is the lower bound of $\widetilde{x}$.

(2) According to [29, Corollary 3.2], we can determine the constant in the RHS of (17).

(i) Let f be convex. We can determine the value C_α in

$$\int_T \phi_t(f(x_t))\, d\mu(t) - \alpha g\left(\int_T \phi_t(x_t)\, d\mu(t)\right) \leq C_\alpha 1_K - \delta_f \widetilde{x}$$

as follows:

- if $\alpha \leq 0$, g is convex or $\alpha \geq 0$, g is concave, then

$$C_\alpha = \max\{k_f m_x + l_f - \alpha g(m_x), k_f M_x + l_f - \alpha g(M_x)\}; \tag{18}$$

- if $\alpha \le 0$, g is concave or $\alpha \ge 0$, g is convex, then

$$C_\alpha = \begin{cases} k_f m_x + l_f - \alpha g(m_x) & \text{if } \alpha g'_-(z) \ge k_f \text{ for every } z \in (m_x, M_x), \\ k_f z_0 + l_f - \alpha g(z_0) & \text{if } \alpha g'_-(z_0) \le k_f \le \alpha g'_+(z_0) \\ & \text{for some } z_0 \in (m_x, M_x), \\ k_f M_x + l_f - \alpha g(M_x) & \text{if } \alpha g'_+(z) \le k_f \text{ for every } z \in (m_x, M_x). \end{cases} \tag{19}$$

(ii) Let f be concave. We can determine the value c_α in

$$c_\alpha 1_K - \delta_f \widetilde{x} \le \int_T \phi_t\big(f(x_t)\big)\, d\mu(t) - \alpha g\left(\int_T \phi_t(x_t)\, d\mu(t)\right)$$

as follows:

- if $\alpha \le 0$, g is convex or $\alpha \ge 0$, g is concave, then c_α is equal to the right-hand side in (19) with reverse inequality signs;
- if $\alpha \le 0$, g is concave or $\alpha \ge 0$, g is convex, then c_α is equal to the right-hand side in (18) with min instead of max.

Theorem 6 and Remark 1(2) applied to functions $f(z) = z^p$ and $g(z) = z^q$ give the following corollary, which is a refinement of [29, Corollary 3.3].

Corollary 7 *Let $(x_t)_{t\in T}$ be a field of strictly positive operators, let m_x and M_x, $m_x \le M_x$, be the bounds of the operator $x = \int_T \phi_t(x_t)\, d\mu(t)$. Let $\widetilde{x}$ be defined by* (8).

(i) *Let $p \in (-\infty, 0] \cup [1, \infty)$. Then*

$$\int_T \phi_t\big(x_t^p\big)\, d\mu(t) - \alpha\left(\int_T \phi_t(x_t)\, d\mu(t)\right)^q \le C_\alpha^\star 1_K - \big(m^p + M^p - 2^{1-p}(m+M)^p\big)\widetilde{x},$$

where the constant $C_\alpha^\star$ is determined as follows:

- *if $\alpha \le 0$, $q \in (-\infty, 0] \cup [1, \infty)$ or $\alpha \ge 0$, $q \in (0,1)$, then*

$$C_\alpha^\star = \max\big\{k_{t^p} m_x + l_{t^p} - \alpha m_x^q, k_{t^p} M_x + l_{t^p} - \alpha M_x^q\big\}; \tag{20}$$

- *if $\alpha \le 0$, $q \in (0,1)$ or $\alpha \ge 0$, $q \in (-\infty, 0] \cup [1, \infty)$, then*

$$C_\alpha^\star = \begin{cases} k_{t^p} m_x + l_{t^p} - \alpha m_x^q & \text{if } (\alpha q/k_{t^p})^{1/(1-q)} \le m_x, \\ l_{t^p} + \alpha(q-1)(\alpha q/k_{t^p})^{q/(1-q)} & \text{if } m_x \le (\alpha q/k_{t^p})^{1/(1-q)} \le M_x, \\ k_{t^p} M_x + l_{t^p} - \alpha M_x^q & \text{if } (\alpha q/k_{t^p})^{1/(1-q)} \ge M_x, \end{cases} \tag{21}$$

where $k_{t^p} := (M^p - m^p)/(M - m)$ and $l_{t^p} := (Mm^p - mM^p)/(M - m)$.

(ii) *Let $p \in (0,1)$. Then*

$$c_\alpha^\star 1_K + \big(2^{1-p}(m+M)^p - m^p - M^p\big)\widetilde{x} \le \int_T \phi_t\big(x_t^p\big)\, d\mu(t) - \alpha\left(\int_T \phi_t(x_t)\, d\mu(t)\right)^q,$$

where the constant $c_\alpha^\star$ is determined as follows:

- *if* $\alpha \le 0$, $q \in (-\infty, 0] \cup [1, \infty)$ *or* $\alpha \ge 0$, $q \in (0,1)$, *then* $c_\alpha^\star$ *is equal to the right-hand side in* (21);
- *if* $\alpha \le 0$, $q \in (0,1)$ *or* $\alpha \ge 0$, $q \in (-\infty, 0] \cup [1, \infty)$, *then* $c_\alpha^\star$ *is equal to the right-hand side in* (20) *with* min *instead of* max.

Using Theorem 6 and Remark 1 for $g \equiv f$ and $\alpha = 1$ and utilizing elementary calculations, we obtain the following converse of Jensen's inequality.

Theorem 8 *Let* m_x *and* M_x, $m_x \le M_x$, *be the bounds of the operator* $x = \int_T \phi_t(x_t)\, d\mu(t)$ *and let* $f : [m, M] \to \mathbb{R}$ *be a continuous function.*

If f *is convex, then*

$$0 \le \int_T \phi_t\big(f(x_t)\big)\, d\mu(t) - f\left(\int_T \phi_t(x_t)\, d\mu(t)\right) \le \bar{C} 1_K - \delta_f \widetilde{x}, \tag{22}$$

where $\widetilde{x}$ *and* δ_f *are defined by* (8) *and* (9), *respectively, and*

$$\bar{C} = \max_{m_x \le z \le M_x} \big\{ k_f z + l_f - f(z) \big\}. \tag{23}$$

Furthermore, if f *is strictly convex differentiable, then the bound* $\bar{C}1_K - \delta_f \widetilde{x}$ *satisfies the following condition*:

$$0 \le \bar{C} 1_K - \delta_f \widetilde{x} \le \big\{ f(M) - f(m) - f'(m)(M - m) - \delta_f m_{\widetilde{x}} \big\} 1_K,$$

where $m_{\widetilde{x}}$ *is the lower bound of the operator* $\widetilde{x}$. *We can determine the value* $\bar{C}$ *in* (23) *as follows*:

$$\bar{C} = k_f z_0 + l_f - f(z_0), \tag{24}$$

where

$$z_0 = \begin{cases} m_x & \text{if } f'(m_x) \ge k_f, \\ f'^{-1}(k_f) & \text{if } f'(m_x) \le k_f \le f'(M_x), \\ M_x & \text{if } f'(M_x) \le k_f. \end{cases} \tag{25}$$

In the dual case, when f *is concave, the reverse inequality is valid in* (22) *with* min *instead of* max *in* (23). *Furthermore, if* f *is strictly concave differentiable, then the bound* $\bar{C}1_K - \delta_f \widetilde{x}$ *satisfies the following condition*:

$$\big\{ f(M) - f(m) - f'(m)(M - m) - \delta_f m_{\widetilde{x}} \big\} 1_K \le \bar{C} 1_K - \delta_f \widetilde{x} \le 0.$$

We can determine the value $\bar{C}$ *in* (24) *with* z_0, *which equals the right-hand side in* (25) *with reverse inequality signs.*

Example 1 We give examples for the matrix cases and $T = \{1, 2\}$. We put $f(t) = t^4$, which is convex, but not operator convex. Also, we define mappings $\Phi_1, \Phi_2 : M_3(\mathbb{C}) \to M_2(\mathbb{C})$ by $\Phi_1((a_{ij})_{1 \le i,j \le 3}) = \frac{1}{2}(a_{ij})_{1 \le i,j \le 2}$, $\Phi_2 = \Phi_1$ and measures by $\mu(\{1\}) = \mu(\{2\}) = 1$.

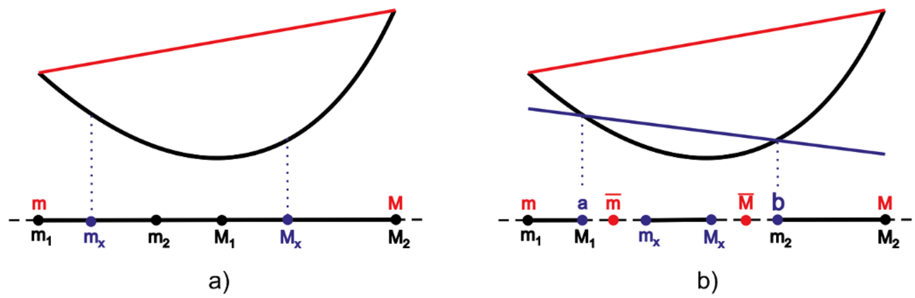


Figure 1 Refinement for two operators and a convex function *f*.

(I) First, we observe an example without the spectra condition (see Figure 1(a)). Then we obtain a refined inequality as in (22), but do not have refined Jensen's inequality.

$$\text{If } X_1 = 2\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad X_2 = 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{then } X = 2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and $m_1 = -1.604$, $M_1 = 4.494$, $m_2 = 0$, $M_2 = 2$, $m = -1.604$, $M = 4.494$ (rounded to three decimal places). We have

$$\left(\Phi_1(X_1) + \Phi_2(X_2)\right)^4 = \begin{pmatrix} 16 & 0 \\ 0 & 0 \end{pmatrix} \ngeq \begin{pmatrix} 80 & 40 \\ 40 & 24 \end{pmatrix} = \Phi_1\left(X_1^4\right) + \Phi_2\left(X_2^4\right)$$

and

$$\begin{aligned} \Phi_1\left(X_1^4\right) + \Phi_2\left(X_2^4\right) &= \begin{pmatrix} 80 & 40 \\ 40 & 24 \end{pmatrix} \\ < \Phi_1\left(X_1^4\right) + \Phi_2\left(X_2^4\right) + \bar{C}I_2 - \delta_f \widetilde{X} &= \begin{pmatrix} 111.742 & 39.327 \\ 39.327 & 142.858 \end{pmatrix} \\ < \left(\Phi_1(X_1) + \Phi_2(X_2)\right)^4 + \bar{C}I_2 &= \begin{pmatrix} 243.758 & 0 \\ 0 & 227.758 \end{pmatrix}, \end{aligned}$$

since $\bar{C} = 227.758$, $\delta_f = 405.762$, $\widetilde{X} = \left(\begin{smallmatrix} 0.325 & -0.097 \\ -0.097 & 0.2092 \end{smallmatrix}\right)$.

(II) Next, we observe an example with the spectra condition (see Figure 1(b)). Then we obtain a series of inequalities involving refined Jensen's inequality and its converses.

$$\text{If } X_1 = \begin{pmatrix} -4 & 1 & 1 \\ 1 & -2 & -1 \\ 1 & -1 & -1 \end{pmatrix} \quad \text{and} \quad X_2 = \begin{pmatrix} 5 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 3 \end{pmatrix}, \quad \text{then } X = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and $m_1 = -4.866$, $M_1 = -0.345$, $m_2 = 1.345$, $M_2 = 5.866$, $m = -4.866$, $M = 5.866$, $a = -0.345$, $b = 1.345$ and we put $\bar{m} = a$, $\bar{M} = b$ (rounded to three decimal places). We have

$$\begin{aligned}
\left(\Phi_1(X_1)+\Phi_2(X_2)\right)^4 &= \begin{pmatrix} 0.0625 & 0 \\ 0 & 0 \end{pmatrix} \\
< \Phi_1\left(X_1^4\right)+\Phi_2\left(X_2^4\right)-\delta_f(a,b)\bar{X} &= \begin{pmatrix} 639.921 & -255 \\ -255 & 117.856 \end{pmatrix} \\
< \Phi_1\left(X_1^4\right)+\Phi_2\left(X_2^4\right) &= \begin{pmatrix} 641.5 & -255 \\ -255 & 118.5 \end{pmatrix} \\
< \left(\Phi_1(X_1)+\Phi_2(X_2)\right)^4+\bar{C}I_2-\delta_f(m,M)\widetilde{X} &= \begin{pmatrix} 731.649 & -162.575 \\ -162.575 & 325.15 \end{pmatrix} \\
< \left(\Phi_1(X_1)+\Phi_2(X_2)\right)^4+\bar{C}I_2 &= \begin{pmatrix} 872.471 & 0 \\ 0 & 872.409 \end{pmatrix},
\end{aligned}$$

since $\delta_f(a,b) = 3.158$, $\bar{X} = \left(\begin{smallmatrix} 0.5 & 0 \\ 0 & 0.204 \end{smallmatrix}\right)$, $\delta_f(m,M) = 1744.82$, $\widetilde{X} = \left(\begin{smallmatrix} 0.325 & -0.097 \\ -0.097 & 0.2092 \end{smallmatrix}\right)$ and $\bar{C} = 872.409$.

Applying Theorem 8 to $f(t) = t^p$, we obtain the following refinement of [29, Corollary 3.6].

Corollary 9 *Let $(x_t)_{t\in T}$ be a field of strictly positive operators, let m_x and M_x, $m_x \leq M_x$, be the bounds of the operator $x = \int_T \phi_t(x_t)\,d\mu(t)$. Let $\widetilde{x}$ be defined by* (8). *Then*

$$\begin{aligned}
0 &\leq \int_T \phi_t\left(x_t^p\right)d\mu(t) - \left(\int_T \phi_t(x_t)\,d\mu(t)\right)^p \\
&\leq \bar{C}(m_x,M_x,m,M,p)1_K - \left(m^p+M^p-2^{1-p}(m+M)^p\right)\widetilde{x} \\
&\leq \bar{C}(m_x,M_x,m,M,p)1_K \leq C(m,M,p)1_K
\end{aligned}$$

for $p \notin (0,1)$, and

$$\begin{aligned}
C(m,M,p)1_K &\leq \bar{c}(m_x,M_x,m,M,p)1_K \\
&\leq \bar{c}(m_x,M_x,m,M,p)1_K + \left(2^{1-p}(m+M)^p-m^p-M^p\right)\widetilde{x} \\
&\leq \int_T \phi_t\left(x_t^p\right)d\mu(t) - \left(\int_T \phi_t(x_t)\,d\mu(t)\right)^p \leq 0
\end{aligned}$$

for $p \in (0,1)$, where

$$\bar{C}(m_x,M_x,m,M,p) = \begin{cases} k_{t^p}m_x + l_{t^p} - m_x^p & \text{if } pm_x^{p-1} \geq k_{t^p}, \\ C(m,M,p) & \text{if } pm_x^{p-1} \leq k_{t^p} \leq pM_x^{p-1}, \\ k_{t^p}M_x + l_{t^p} - M_x^p & \text{if } pM_x^{p-1} \leq k_{t^p}, \end{cases} \tag{26}$$

and $\bar{c}(m_x,M_x,m,M,p)$ equals the right-hand side in (26) *with reverse inequality signs. $C(m,M,p)$ is the known Kantorovich-type constant for difference* (*see, i.e.*, [6, §2.7]):

$$C(m,M,p) = (p-1)\left(\frac{M^p-m^p}{p(M-m)}\right)^{1/(p-1)} + \frac{Mm^p-mM^p}{M-m} \quad \textit{for } p \in \mathbb{R}.$$

4 Ratio-type converse inequalities

In [29, Theorem 4.1] the following ratio-type converse of (16) is given:

$$\int_T \phi_t\big(f(x_t)\big)\,d\mu(t) \le \max_{m_x\le z\le M_x}\left\{\frac{k_f z + l_f}{g(z)}\right\} g\left(\int_T \phi_t(x_t)\,d\mu(t)\right), \tag{27}$$

where f is convex and $g > 0$. Applying Theorem 5 and Theorem 6, we obtain the following two refinements of (27).

Theorem 10 *Let m_x and M_x, $m_x \le M_x$, be the bounds of the operator $x = \int_T \phi_t(x_t)\,d\mu(t)$ and let $f : [m, M] \to \mathbb{R}$, $g : [m_x, M_x] \to \mathbb{R}$ be continuous functions.*

If f is convex and $g > 0$, then

$$\int_T \phi_t\big(f(x_t)\big)\,d\mu(t) \le \max_{m_x\le z\le M_x}\left\{\frac{k_f z + l_f}{g(z)}\right\} g\left(\int_T \phi_t(x_t)\,d\mu(t)\right) - \delta_f \widetilde{x} \tag{28}$$

and

$$\int_T \phi_t\big(f(x_t)\big)\,d\mu(t) \le \max_{m_x\le z\le M_x}\left\{\frac{k_f z + l_f - \delta_f m_{\widetilde{x}}}{g(z)}\right\} g\left(\int_T \phi_t(x_t)\,d\mu(t)\right), \tag{29}$$

where $\widetilde{x}$ and δ_f are defined by (8) *and* (9), *respectively, and $m_{\widetilde{x}}$ is the lower bound of the operator $\widetilde{x}$. If f is concave, then reverse inequalities are valid in* (28) *and* (29) *with* min *instead of* max.

Proof We prove only the convex case. Let $\alpha_1 = \max_{m_x\le z\le M_x}\{\frac{k_f z + l_f}{g(z)}\}$. Then there is $z_0 \in [m_x, M_x]$ such that $\alpha_1 = \frac{k_f z_0 + l_f}{g(z_0)}$ and $\frac{k_f z + l_f}{g(z)} \le \alpha_1$ for all $z \in [m_x, M_x]$. It follows that $k_f z_0 + l_f - \alpha_1 g(z_0) = 0$ and $k_f z + l_f - \alpha_1 g(z) \le 0$ for all $z \in [m_x, M_x]$. So,

$$\max_{m_x\le z\le M_x}\big\{k_f z + l_f - \alpha_1 g(z)\big\} = 0.$$

By using (17), we obtain (28). Inequality (29) follows directly from Theorem 5 by putting $F(u, v) = v^{-1/2} u v^{-1/2}$. □

Remark 2 (1) Inequality (28) is a refinement of (27) since $\delta_f \widetilde{x} \ge 0$. Also, (29) is a refinement of (27) since $m_{\widetilde{x}} \ge 0$ and $g > 0$ implies

$$\max_{m_x\le z\le M_x}\left\{\frac{k_f z + l_f - \delta_f m_{\widetilde{x}}}{g(z)}\right\} \le \max_{m_x\le z\le M_x}\left\{\frac{k_f z + l_f}{g(z)}\right\}.$$

(2) Let the assumptions of Theorem 10 hold. Generally, there is no relation between the right-hand sides of inequalities (28) and (29) under the operator order (see Example 2). But, for example, if $g(\int_T \phi_t(x_t)\,d\mu(t)) \le g(z_0)1_K$, where $z_0 \in [m_x, M_x]$ is the point where it achieves $\max_{m_x\le z\le M_x}\{\frac{k_f z + l_f}{g(z)}\}$, then the following order holds:

$$\begin{aligned}\int_T \phi_t\big(f(x_t)\big)\,d\mu(t) &\le \max_{m_x\le z\le M_x}\left\{\frac{k_f z + l_f}{g(z)}\right\} g\left(\int_T \phi_t(x_t)\,d\mu(t)\right) - \delta_f \widetilde{x} \\ &\le \max_{m_x\le z\le M_x}\left\{\frac{k_f z + l_f - \delta_f m_{\widetilde{x}}}{g(z)}\right\} g\left(\int_T \phi_t(x_t)\,d\mu(t)\right).\end{aligned}$$

Example 2 Let $f(t) = g(t) = t^4$, $\Phi_k((a_{ij})_{1\le i,j\le 3}) = \frac{1}{2}(a_{ij})_{1\le i,j\le 2}$ and $\mu(\{k\}) = 1$, $k = 1, 2$.

$$\text{If } X_1 = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad X_2 = \begin{pmatrix} 5 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 3 \end{pmatrix}, \quad \text{then } X = \begin{pmatrix} 4.5 & 0 \\ 0 & 2 \end{pmatrix}$$

and $m_1 = 0.623$, $M_1 = 4.651$, $m_2 = 1.345$, $M_2 = 5.866$, $m = 0.623$, $M = 5.866$ (rounded to three decimal places). We have

$$\begin{aligned} \Phi_1\bigl(X_1^4\bigr) + \Phi_2\bigl(X_2^4\bigr) &= \begin{pmatrix} 629.5 & -87.5 \\ -87.5 & 99 \end{pmatrix} \\ < \alpha_1\bigl(\Phi_1(X_1) + \Phi_2(X_2)\bigr)^4 - \delta_f \widetilde{x} &= \begin{pmatrix} 7823.449 & -53.737 \\ -53.737 & 139.768 \end{pmatrix} \\ < \alpha_1\bigl(\Phi_1(X_1) + \Phi_2(X_2)\bigr)^4 &= \begin{pmatrix} 7974.38 & 0 \\ 0 & 311.148 \end{pmatrix}, \end{aligned} \tag{30}$$

since $\alpha_1 = \max_{m_x \le z \le M_x}\{\frac{k_f z + l_f}{g(z)}\} = 19.447$, $\delta_f = 962.73$, $\widetilde{x} = \bigl(\begin{smallmatrix} 0.157 & 0.056 \\ 0.056 & 0.178 \end{smallmatrix}\bigr)$. Further,

$$\begin{aligned} \Phi_1\bigl(X_1^4\bigr) + \Phi_2\bigl(X_2^4\bigr) &= \begin{pmatrix} 629.5 & -87.5 \\ -87.5 & 99 \end{pmatrix} \\ < \alpha_2\bigl(\Phi_1(X_1) + \Phi_2(X_2)\bigr)^4 &= \begin{pmatrix} 5246.13 & 0 \\ 0 & 204.696 \end{pmatrix} \\ < \alpha_1\bigl(\Phi_1(X_1) + \Phi_2(X_2)\bigr)^4 &= \begin{pmatrix} 7974.38 & 0 \\ 0 & 311.148 \end{pmatrix}, \end{aligned} \tag{31}$$

since $\alpha_2 = \max_{m_x \le z \le M_x}\{\frac{k_f z + l_f - \delta_f m_{\widetilde{x}}}{g(z)}\} = 12.794$. We remark that there is no relation between matrices in the right-hand sides of equalities (30) and (31).

Remark 3 Similar to [29, Corollary 4.2], we can determine the constant in the RHS of (29).

(i) Let f be convex. We can determine the value C in

$$\int_T \phi_t\bigl(f(x_t)\bigr)\,d\mu(t) \le Cg\biggl(\int_T \phi_t(x_t)\,d\mu(t)\biggr)$$

as follows:

- if g is convex, then

$$C_\alpha = \begin{cases} \frac{k_f m_x + l_f - \delta_f m_{\widetilde{x}}}{g(m_x)} & \text{if } g'_-(z) \ge \frac{k_f g(z)}{k_f z + l_f - \delta_f m_{\widetilde{x}}} \text{ for every } z \in (m_x, M_x), \\ \frac{k_f z_0 + l_f - \delta_f m_{\widetilde{x}}}{g(z_0)} & \text{if } g'_-(z_0) \le \frac{k_f g(z_0)}{k_f z_0 + l_f - \delta_f m_{\widetilde{x}}} \le g'_+(z_0) \\ & \text{for some } z_0 \in (m_x, M_x), \\ \frac{k_f M_x + l_f - \delta_f m_{\widetilde{x}}}{g(M_x)} & \text{if } g'_+(z) \le \frac{k_f g(z)}{k_f z + l_f - \delta_f m_{\widetilde{x}}} \text{ for every } z \in (m_x, M_x); \end{cases} \tag{32}$$

- if g is concave, then

$$C = \max\biggl\{\frac{k_f m_x + l_f - \delta_f m_{\widetilde{x}}}{g(m_x)}, \frac{k_f M_x + l_f - \delta_f m_{\widetilde{x}}}{g(M_x)}\biggr\}. \tag{33}$$

Also, we can determine the constant D in

$$\int_T \phi_t\big(f(x_t)\big)\,d\mu(t) \le Dg\biggl(\int_T \phi_t(x_t)\,d\mu(t)\biggr) - \delta_f \widetilde{x}$$

in the same way as the above constant C but without $m_{\widetilde{x}}$.

(ii) Let f be concave. We can determine the value c in

$$cg\biggl(\int_T \phi_t(x_t)\,d\mu(t)\biggr) \le \int_T \phi_t\big(f(x_t)\big)\,d\mu(t)$$

as follows:

- if g is convex, then c is equal to the right-hand side in (33) with min instead of max;
- if g is concave, then c is equal to the right-hand side in (32) with reverse inequality signs.

Also, we can determine the constant d in

$$dg\biggl(\int_T \phi_t(x_t)\,d\mu(t)\biggr) - \delta_f \widetilde{x} \le \int_T \phi_t\big(f(x_t)\big)\,d\mu(t)$$

in the same way as the above constant c but without $m_{\widetilde{x}}$.

Theorem 10 and Remark 3 applied to functions $f(z) = z^p$ and $g(z) = z^q$ give the following corollary, which is a refinement of [29, Corollary 4.4].

Corollary 11 *Let $(x_t)_{t\in T}$ be a field of strictly positive operators, let m_x and M_x, $m_x \le M_x$, be the bounds of the operator $x = \int_T \phi_t(x_t)\,d\mu(t)$. Let $\widetilde{x}$ be defined by* (8), *$m_{\widetilde{x}}$ be the lower bound of the operator $\widetilde{x}$ and $\delta_p := m^p + M^p - 2^{1-p}(m+M)^p$.*

(i) *Let $p \in (-\infty, 0] \cup [1, \infty)$. Then*

$$\int_T \phi_t\big(x_t^p\big)\,d\mu(t) \le C^\star \biggl(\int_T \phi_t(x_t)\,d\mu(t)\biggr)^q,$$

where the constant $C^\star$ is determined as follows:

- *if $q \in (-\infty, 0] \cup [1, \infty)$, then*

$$C^\star = \begin{cases} \frac{k_{t^p} m_x + l_{t^p} - \delta_p m_{\widetilde{x}}}{m_x^q} & \textit{if } \frac{q}{1-q}\frac{l_{t^p} - \delta_p m_{\widetilde{x}}}{k_{t^p}} \le m_x, \\ \frac{l_{t^p} - \delta_p m_{\widetilde{x}}}{1-q}\bigl(\frac{1-q}{q}\frac{k_{t^p}}{l_{t^p} - \delta_p m_{\widetilde{x}}}\bigr)^q & \textit{if } m_x \le \frac{q}{1-q}\frac{l_{t^p} - \delta_p m_{\widetilde{x}}}{k_{t^p}} \le M_x, \\ \frac{k_{t^p} M_x + l_{t^p} - \delta_p m_{\widetilde{x}}}{M_x^q} & \textit{if } \frac{q}{1-q}\frac{l_{t^p} - \delta_p m_{\widetilde{x}}}{k_{t^p}} \ge M_x; \end{cases} \tag{34}$$

- *if $q \in (0,1)$, then*

$$C^\star = \max\biggl\{\frac{k_{t^p} m_x + l_{t^p} - \delta_p m_{\widetilde{x}}}{m_x^q}, \frac{k_{t^p} q, M_x + l_{t^p} - \delta_p m_{\widetilde{x}}}{M_x^q}\biggr\}. \tag{35}$$

Also,

$$\int_T \phi_t\big(x_t^p\big)\,d\mu(t) \le D^\star \biggl(\int_T \phi_t(x_t)\,d\mu(t)\biggr)^q - \delta_p \widetilde{x}$$

holds, where $D^\star$ is determined in the same way as the above constant $C^\star$ but without $m_{\widetilde{x}}$.

(ii) *Let $p \in (0,1)$. Then*

$$c^\star \left(\int_T \phi_t(x_t)\, d\mu(t) \right)^q \leq \int_T \phi_t\big(x_t^p\big)\, d\mu(t),$$

where the constant $c^\star$ is determined as follows:

- *if $q \in (-\infty, 0] \cup [1, \infty)$, then $c^\star$ is equal to the right-hand side in* (35) *with* min *instead of* max;
- *if $q \in (0,1)$, then $c_\alpha^\star$ is equal to the right-hand side in* (34).

Also,

$$d^\star \left(\int_T \phi_t(x_t)\, d\mu(t) \right)^q - \delta_p \widetilde{x} \leq \int_T \phi_t\big(x_t^p\big)\, d\mu(t)$$

holds, where $\delta_p \leq 0$, $\widetilde{x} \geq 0$ and $d^\star$ is determined in the same way as the above constant $d^\star$ but without $m_{\widetilde{x}}$.

Using Theorem 10 and Remark 3 for $g \equiv f$ and utilizing elementary calculations, we obtain the following converse of Jensen's operator inequality.

Theorem 12 *Let m_x and M_x, $m_x \leq M_x$, be the bounds of the operator $x = \int_T \phi_t(x_t)\, d\mu(t)$. If $f : [m, M] \to \mathbb{R}$ is a continuous convex function and strictly positive on $[m_x, M_x]$, then*

$$\int_T \phi_t\big(f(x_t)\big)\, d\mu(t) \leq \max_{m_x \leq z \leq M_x} \left\{ \frac{k_f z + l_f - \delta_f m_{\widetilde{x}}}{f(z)} \right\} f\left(\int_T \phi_t(x_t)\, d\mu(t) \right) \tag{36}$$

and

$$\int_T \phi_t\big(f(x_t)\big)\, d\mu(t) \leq \max_{m_x \leq z \leq M_x} \left\{ \frac{k_f z + l_f}{f(z)} \right\} f\left(\int_T \phi_t(x_t)\, d\mu(t) \right) - \delta_f \widetilde{x}, \tag{37}$$

where $\widetilde{x}$ and δ_f are defined by (8) *and* (9), *respectively, and $m_{\widetilde{x}}$ is the lower bound of the operator $\widetilde{x}$.*

In the dual case, if f is concave, then the reverse inequalities are valid in (36) *and* (37) *with* min *instead of* max.

Furthermore, if f is convex differentiable on $[m_x, M_x]$, we can determine the constant

$$\alpha_1 \equiv \alpha_1(m, M, m_x, M_x, f) = \max_{m_x \leq z \leq M_x} \left\{ \frac{k_f z + l_f - \delta_f m_{\widetilde{x}}}{f(z)} \right\}$$

in (36) *as follows*:

$$\alpha_1 = \begin{cases} \frac{k_f m_x + l_f - \delta_f m_{\widetilde{x}}}{f(m_x)} & \textit{if } f'(z) \geq \frac{k_f f(z)}{k_f z + l_f - \delta_f m_{\widetilde{x}}} \textit{ for every } z \in (m_x, M_x), \\ \frac{k_f z_0 + l_f - \delta_f m_{\widetilde{x}}}{f(z_0)} & \textit{if } f'(z_0) = \frac{k_f f(z_0)}{k_f z_0 + l_f - \delta_f m_{\widetilde{x}}} \textit{ for some } z_0 \in (m_x, M_x), \\ \frac{k_f M_x + l_f - \delta_f m_{\widetilde{x}}}{f(M_x)} & \textit{if } f'(z) \leq \frac{k_f f(z)}{k_f z + l_f - \delta_f m_{\widetilde{x}}} \textit{ for every } z \in (m_x, M_x). \end{cases} \tag{38}$$

Also, if f is strictly convex twice differentiable on $[m_x, M_x]$, then we can determine the constant

$$\alpha_2 \equiv \alpha_2(m, M, m_x, M_x, f) = \max_{m_x \le z \le M_x} \left\{ \frac{k_f z + l_f}{f(z)} \right\}$$

in (37) *as follows*:

$$\alpha_2 = \frac{k_f z_0 + l_f}{f(z_0)}, \tag{39}$$

where $z_0 \in (m_x, M_x)$ is defined as the unique solution of the equation $k_f f(z) = (k_f z + l_f) f'(z)$ provided $(k_f m_x + l_f) f'(m_x)/f(m_x) \le k_f \le (k_f M_x + l_f) f'(M_x)/f(M_x)$. Otherwise, z_0 is defined as m_x or M_x provided $k_f \le (k_f m_x + l_f) f'(m_x)/f(m_x)$ or $k_f \ge (k_f M_x + l_f) f'(M_x)/f(M_x)$, respectively.

In the dual case, if f is concave differentiable, then the value α_1 is equal to the right-hand side in (38) *with reverse inequality signs. Also, if f is strictly concave twice differentiable, then we can determine the value α_2 in* (39) *with z_0, which equals the right-hand side in* (39) *with reverse inequality signs.*

Remark 4 If f is convex and strictly negative on $[m_x, M_x]$, then (36) and (37) are valid with min instead of max. If f is concave and strictly negative, then reverse inequalities are valid in (36) and (37).

Applying Theorem 12 to $f(t) = t^p$, we obtain the following refinement of [29, Corollary 4.8].

Corollary 13 *Let $(x_t)_{t \in T}$ be a field of strictly positive operators, let m_x and M_x, $m_x \le M_x$, be the bounds of the operator $x = \int_T \phi_t(x_t)\, d\mu(t)$. Let $\widetilde{x}$ be defined by* (8), *$m_{\widetilde{x}}$ be the lower bound of the operator $\widetilde{x}$ and $\delta_p := m^p + M^p - 2^{1-p}(m+M)^p$.*

If $p \notin (0,1)$, then

$$\begin{aligned} 0 &\le \int_T \phi_t(x_t^p)\, d\mu(t) \le \bar{K}(m_x, M_x, m, M, p, 0) \left(\int_T \phi_t(x_t)\, d\mu(t) \right)^p - \delta_p \\ &\le \bar{K}(m_x, M_x, m, M, p, 0) \left(\int_T \phi_t(x_t)\, d\mu(t) \right)^p \\ &\le K(m, M, p) \left(\int_T \phi_t(x_t)\, d\mu(t) \right)^p \end{aligned} \tag{40}$$

and

$$\begin{aligned} 0 &\le \int_T \phi_t(x_t^p)\, d\mu(t) \le \bar{K}(m_x, M_x, m, M, p, m_{\widetilde{x}}) \left(\int_T \phi_t(x_t)\, d\mu(t) \right)^p \\ &\le \bar{K}(m_x, M_x, m, M, p, 0) \left(\int_T \phi_t(x_t)\, d\mu(t) \right)^p \\ &\le K(m, M, p) \left(\int_T \phi_t(x_t)\, d\mu(t) \right)^p, \end{aligned} \tag{41}$$

where

$$\bar{K}(m_x, M_x, m, M, p, c) = \begin{cases} \frac{k_{t^p} m_x + l_{t^p} - c\delta_p}{m_x^p} & \text{if } \frac{p(l_{t^p} - c\delta_p)}{m_x} \geq (1-p)k_{t^p}, \\ K(m, M, p, c) & \text{if } \frac{p(l_{t^p} - c\delta_p)}{m_x} < (1-p)k_{t^p} < \frac{p(l_{t^p} - c\delta_p)}{M_x}, \\ \frac{k_{t^p} M_x + l_{t^p} - c\delta_p}{M_x^p} & \text{if } \frac{p(l_{t^p} - c\delta_p)}{M_x} \leq (1-p)k_{t^p}. \end{cases} \tag{42}$$

$K(m, M, p, c)$ is a generalization of the known Kantorovich constant $K(m, M, p) \equiv K(m, M, p, 0)$ (defined in [6, §2.7]) *as follows*:

$$\begin{aligned} &K(m, M, p, c) \\ &\quad := \frac{mM^p - Mm^p + c\delta_p(M-m)}{(p-1)(M-m)} \left(\frac{p-1}{p} \frac{M^p - m^p}{mM^p - Mm^p + c\delta_p(M-m)} \right)^p, \end{aligned} \tag{43}$$

for $p \in \mathbb{R}$ and $0 \leq c \leq 0.5$.

If $p \in (0,1)$, then

$$\begin{aligned} \int_T \phi_t(x_t^p)\, d\mu(t) &\geq \bar{k}(m_x, M_x, m, M, p, 0) \left(\int_T \phi_t(x_t)\, d\mu(t) \right)^p - \delta_p \widetilde{x} \\ &\geq \bar{k}(m_x, M_x, m, M, p, 0) \left(\int_T \phi_t(x_t)\, d\mu(t) \right)^p \\ &\geq K(m, M, p) \left(\int_T \phi_t(x_t)\, d\mu(t) \right)^p \geq 0 \end{aligned}$$

and

$$\begin{aligned} \int_T \phi_t(x_t^p)\, d\mu(t) &\geq \bar{k}(m_x, M_x, m, M, p, m_{\widetilde{x}}) \left(\int_T \phi_t(x_t)\, d\mu(t) \right)^p \\ &\geq \bar{k}(m_x, M_x, m, M, p, 0) \left(\int_T \phi_t(x_t)\, d\mu(t) \right)^p \\ &\geq K(m, M, p) \left(\int_T \phi_t(x_t)\, d\mu(t) \right)^p \geq 0, \end{aligned}$$

where $\bar{k}(m_x, M_x, m, M, p, c)$ equals the right-hand side in (42) *with reverse inequality signs.*

Proof The second inequalities in (40) and (41) follow directly from (37) and (36) by using (39) and (38), respectively. The last inequality in (40) follows from

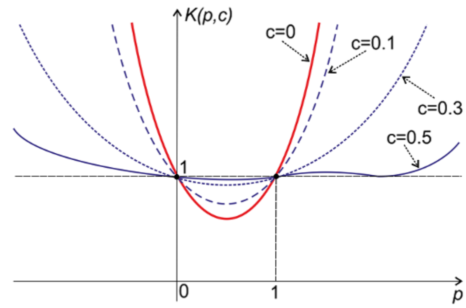
$$\begin{aligned} \bar{K}(m_x, M_x, m, M, p, 0) &= \max_{m_x \leq z \leq M_x} \left\{ \frac{k_{t^p} z + l_{t^p}}{z^p} \right\} \\ &\leq \max_{m \leq z \leq M} \left\{ \frac{k_{t^p} z + l_{t^p}}{z^p} \right\} = K(m, M, p). \end{aligned}$$

The third inequality in (41) follows from

$$\bar{K}(m_x, M_x, m, M, p, m_{\widetilde{x}}) = \max_{m_x \leq z \leq M_x} \left\{ \frac{k_{t^p} z + l_{t^p} - \delta_p m_{\widetilde{x}}}{z^p} \right\} \leq \bar{K}(m_x, M_x, m, M, p, 0),$$

since $\delta_p m_{\widetilde{x}} \geq 0$ for $p \notin (0,1)$ and $M_x \geq m_x \geq 0$. $\square$

Figure 2 Relation between $K(p,c)$ for $p \in \mathbb{R}$ and $0 \leq c \leq 0.5$.



Appendix A: A new generalization of the Kantorovich constant

Definition 1 Let $h > 0$. Further generalization of Kantorovich constant $K(h,p)$ (given in [6, Definition 2.2]) is defined by

$$K(h,p,c) := \frac{h^p - h + c(h^p + 1 - 2^{1-p}(h+1)^p)(h-1)}{(p-1)(h-1)} \times \left(\frac{p-1}{p}\frac{h^p - 1}{h^p - h + c(h^p + 1 - 2^{1-p}(h+1)^p)(h-1)}\right)^p$$

for any real number $p \in \mathbb{R}$ and any c, $0 \leq c \leq 0.5$. The constant $K(h,p,c)$ is sometimes denoted by $K(p,c)$ briefly. Some of those constants are depicted in Figure 2.

By inserting $c = 0$ in $K(h,p,c)$, we obtain the Kantorovich constant $K(h,p)$. The constant $K(m,M,p,c)$ defined by (43) coincides with $K(h,p,c)$ by putting $h = M/m > 1$.

Lemma 14 *Let $h > 0$. The generalized Kantorovich constant $K(h,p,c)$ has the following properties*:

(i) $K(h,p,c) = K(\frac{1}{h},p,c)$ *for all* $p \in \mathbb{R}$,
(ii) $K(h,0,c) = K(h,1,c) = 1$ *for all* $0 \leq c \leq 0.5$ *and* $K(1,p,c) = 1$ *for all* $p \in \mathbb{R}$,
(iii) $K(h,p,c)$ *is decreasing of c for* $p \notin (0,1)$ *and increasing of c for* $p \in (0,1)$,
(iv) $K(h,p,c) \geq 1$ *for all* $p \notin (0,1)$ *and* $0 < K(h,0.5,0) \leq K(h,p,c) \leq 1$ *for all* $p \in (0,1)$,
(v) $K(h,p,c) \leq h^{p-1}$ *for all* $p \geq 1$.

Proof (i) We use an easy calculation:

$$\begin{aligned} K\left(\frac{1}{h},p,c\right) &= \frac{h^{-p} - h^{-1} + c(h^{-p} + 1 - 2^{1-p}(h^{-1}+1)^p)(h^{-1}-1)}{(p-1)(h^{-1}-1)} \\ &\quad \times \left(\frac{p-1}{p}\frac{h^{-p} - 1}{h^{-p} - h^{-1} + c(h^{-p} + 1 - 2^{1-p}(h^{-1}+1)^p)(h^{-1}-1)}\right)^p \\ &= \frac{h - h^p + c(1 + h^p - 2^{1-p}(h+1)^p)(1-h)}{(p-1)(1-h)} \\ &\quad \times \left(\frac{p-1}{p}\frac{1 - h^p}{h - h^p + c(1 + h^p - 2^{1-p}(h+1)^p)(1-h)}\right)^p \\ &= K(h,p,c). \end{aligned}$$

(ii) Let $h > 1$. The logarithms calculation and l'Hospital's theorem give $K(h,p,b) \to 1$ as $p \to 1$, $K(h,p,b) \to 1$ as $p \to 0$ and $K(h,p,b) \to 1$ as $h \to 1+$. Now using (i) we obtain (ii).

(iii) Let $h > 0$ and $0 \leq c \leq 0.5$.

$$\begin{aligned}\frac{\mathrm{d}K(h,p,c)}{\mathrm{d}c} &= 2\left(\left(\frac{h+1}{2}\right)^p - \frac{h^p+1}{2}\right)\\ &\quad\times\left(\frac{p-1}{p}\frac{h^p-1}{h-h^p+c(h^p+1-2^{1-p}(h+1)^p)(h-1)}\right)^p.\end{aligned}$$

Since the function $z \to z^p$ is convex (resp. concave) on $(0,\infty)$ if $p \notin (0,1)$ (resp. $p \in (0,1)$), then $(\frac{h+1}{2})^p \leq \frac{h^p+1}{2}$ (resp. $(\frac{h+1}{2})^p \geq \frac{h^p+1}{2}$) for every $h > 0$. Then $\frac{\mathrm{d}K(h,p,c)}{\mathrm{d}c} \leq 0$ if $p \notin (0,1)$ and $\frac{\mathrm{d}K(h,p,c)}{\mathrm{d}c} \geq 0$ if $p \in (0,1)$, which gives that $K(h,p,c)$ is decreasing of c if $p \notin (0,1)$ and increasing of c if $p \in (0,1)$.

(iv) Let $h > 1$ and $0 \leq c \leq 0.5$. If $p > 1$ then

$$\begin{aligned}0 &< \frac{(p-1)(h-1)}{h^p-h+c(h^p+1-2^{1-p}(h+1)^p)(h-1)}\\ &\leq \frac{p-1}{p}\frac{h^p-1}{h^p-h+c(h^p+1-2^{1-p}(h+1)^p)(h-1)}\end{aligned}$$

implies

$$\begin{aligned}&\frac{(p-1)(h-1)}{h^p-h+c(h^p+1-2^{1-p}(h+1)^p)(h-1)}\\ &\quad\leq \left(\frac{p-1}{p}\frac{h^p-1}{h^p-h+c(h^p+1-2^{1-p}(h+1)^p)(h-1)}\right)^p,\end{aligned}$$

which gives $K(h,p,c) \geq 1$. Similarly, $K(h,p,c) \geq 1$ if $p < 0$ and $K(h,p,c) \leq 1$ if $p \in (0,1)$. Next, using (iii) and [6, Theorem 2.54(iv)], $K(h,p,c) \geq K(h,p,0) \geq K(h,0.5,0)$ for $p \in (0,1)$.

(v) Let $p \geq 1$. Using (iii) and [6, Theorem 2.54(vi)], $K(h,p,c) \leq K(h,p,0) \leq h^{p-1}$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Author details
[1]Faculty of Mechanical Engineering and Naval Architecture, University of Zagreb, Ivana Lučića 5, Zagreb, 10000, Croatia. [2]Faculty of Textile Technology, University of Zagreb, Prilaz baruna Filipovića 30, Zagreb, 10000, Croatia. [3]Faculty of Science, Department of Mathematics, University of Split, Teslina 12, Split, 21000, Croatia.

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