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On the Harary index of graph operations

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Abstract

The Harary index is defined as the sum of reciprocals of distances between all pairs of vertices of a connected graph. In this paper, expressions for the Harary indices of the join, corona product, Cartesian product, composition and disjunction of graphs are derived and the indices for some well-known graphs are evaluated. In derivations some terms appear which are similar to the Harary index and we name them the second and third Harary index.

MSC: 05C05; 05C07; 05C90

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1 Introduction and preliminaries

Throughout this paper we consider simple connected graphs without loops and multiple edges. Suppose that G is a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. The distance between the vertices v_i and v_j of $V(G)$ is denoted by $d_G(v_i, v_j)$ and it is defined as the number of edges in a minimal path connecting the vertices v_i and v_j. The Harary index is one of very much studied topological indices and is defined as follows [1, 2]:

$$H(G) = \sum_{v_i, v_j \in V(G), i \neq j} \frac{1}{d_G(v_i, v_j)},$$

where the summation goes over all unordered pairs of vertices of G. Mathematical properties and applications of H are reported in [3–11]. We now propose two more members of the class of Harary indices, the second Harary index and the third Harary index, which are as follows:

$$H_1(G) = \sum_{v_i, v_j \in V(G), i \neq j} \frac{1}{d_G(v_i, v_j) + 1}$$

and

$$H_2(G) = \sum_{v_i, v_j \in V(G), i \neq j} \frac{1}{d_G(v_i, v_j) + 2},$$

where the summation goes over all unordered pairs of the vertices of G. The Wiener index of G is defined as [12]

$$W = W(G) = \sum_{\{v_i, v_j\} \subseteq V} d_G(v_i, v_j).$$

If we denote by $d(G,k)$ the number of vertex pairs of G, the distance of which is equal to k, then the Wiener index of G can be expressed as

$$W(G) = \sum_{k\geq 1} kd(G,k).$$

The maximum value of k, for which $d(G,k)$ is non-zero, is the diameter of the graph G, and it will be denoted by $D(G)$. The Wiener index is of certain importance in chemistry [13]. It is one of the oldest and most thoroughly studied graph-based molecular structure-descriptors (the so-called topological indices) [12–14]. Numerous chemical applications of it have been reported (see, for instance, [15, 16]), and its mathematical properties are reasonably well understood [17–20]. The *degree* of $v_i \in V(G)$, denoted by $d_G(v_i)$, is the number of vertices in G adjacent to v_i. For other undefined notations and terminology from graph theory, the readers are referred to [21].

In [22], Khalifeh *et al.* computed some exact formulae for the hyper-Wiener index of the join, Cartesian product, composition, disjunction and symmetric difference of graphs. Some more properties and applications of graph products can be seen in the classical book [23].

The paper is organized as follows. In Section 2, we obtain lower and upper bounds on the Harary index of graphs. In Section 3, we give some exact expressions for the Harary index of various graph operations, such as join, corona product, Cartesian product, composition, disjunction, *etc.* Moreover, computations are done for some well-known graphs.

2 Bounds on the Harary index

We define

$$H_t(G) = \sum_{u_i,u_k\in V(G),i\neq k} \frac{1}{d_G(u_i,u_k)+t},$$

where t is any non-negative real number. In this section we obtain lower and upper bounds on $H_t(G)$ of the graph G. From that we can find lower and upper bounds on the Harary index of graphs. These results are useful in the next section. We begin with the following lower and upper bounds on $H_t(G)$.

Theorem 1 *Let G ($\ncong K_{|G|}$) be a connected graph of order $|G|$, $\|G\|$ edges and diameter $D(G)$. Then*

$$H_t(G) \geq \frac{\|G\|}{t+1} + \frac{(\frac{|G|(|G|-1)}{2} - \|G\|)^2}{W(G) + \frac{|G|(|G|-1)}{2}t - (t+1)\|G\|} \tag{1}$$

with equality holding if and only if G is isomorphic to a graph of diameter 2. *Moreover,*

$$H_t(G) \leq \frac{\|G\|}{t+1} + \frac{(\frac{|G|(|G|-1)}{2} - \|G\|)[2 + (\frac{|G|(|G|-1)}{2} - \|G\| - 1) \times (\frac{D(G)+t}{t+2} + \frac{t+2}{D(G)+t})]}{2(W(G) + \frac{|G|(|G|-1)}{2}t - (t+1)\|G\|)} \tag{2}$$

with equality holding if and only if G is isomorphic to a graph of diameter 2.

Proof For $(u_i,u_k)\neq(u_j,u_\ell)$, $i\neq k$, $j\neq\ell$, $2\le d_G(u_i,u_k)\le D(G)$, $2\le d_G(u_j,u_\ell)\le D(G)$, we have

$$2\le\left(\frac{d_G(u_i,u_k)+t}{d_G(u_j,u_\ell)+t}+\frac{d_G(u_j,u_\ell)+t}{d_G(u_i,u_k)+t}\right)\le\left(\frac{D(G)+t}{t+2}+\frac{t+2}{D(G)+t}\right). \tag{3}$$

From the definition of the Wiener index, we have

$$W(G)=\sum_{u_i,u_k\in V(G),i\neq k}d_G(u_i,u_k).$$

Using the above, we get

$$\sum_{u_i,u_k\in V(G),i\neq k,d_G(u_i,u_k)\ge 2}\bigl(d_G(u_i,u_k)+t\bigr)=W(G)+\frac{|G|(|G|-1)}{2}t-(t+1)\|G\|. \tag{4}$$

Now,

$$\begin{aligned}&\sum_{u_i,u_k\in V(G),i\neq k,d_G(u_i,u_k)\ge 2}\bigl(d_G(u_i,u_k)+t\bigr)\sum_{u_i,u_k\in V(G),i\neq k,d_G(u_i,u_k)\ge 2}\frac{1}{d_G(u_i,u_k)+t}\\&=\left(\frac{|G|(|G|-1)}{2}-\|G\|\right)\\&\quad+\sum_{(u_i,u_k)\neq(u_j,u_\ell),i\neq k,j\neq\ell,d_G(u_i,u_k)\ge 2,d_G(u_j,u_\ell)\ge 2}\left(\frac{d_G(u_i,u_k)+t}{d_G(u_j,u_\ell)+t}+\frac{d_G(u_j,u_\ell)+t}{d_G(u_i,u_k)+t}\right).\end{aligned} \tag{5}$$

Using (3) in the above, we get

$$\begin{aligned}&\sum_{u_i,u_k\in V(G),i\neq k,d_G(u_i,u_k)\ge 2}\bigl(d_G(u_i,u_k)+t\bigr)\sum_{u_i,u_k\in V(G),i\neq k,d_G(u_i,u_k)\ge 2}\frac{1}{d_G(u_i,u_k)+t}\\&\ge\left(\frac{|G|(|G|-1)}{2}-\|G\|\right)+\left(\frac{|G|(|G|-1)}{2}-\|G\|\right)\left(\frac{|G|(|G|-1)}{2}-\|G\|-1\right)\quad\text{by (3)}\\&=\left(\frac{|G|(|G|-1)}{2}-\|G\|\right)^2,\end{aligned} \tag{6}$$

and

$$\begin{aligned}&\sum_{u_i,u_k\in V(G),i\neq k,d_G(u_i,u_k)\ge 2}\bigl(d_G(u_i,u_k)+t\bigr)\sum_{u_i,u_k\in V(G),i\neq k,d_G(u_i,u_k)\ge 2}\frac{1}{d_G(u_i,u_k)+t}\\&\le\left(\frac{|G|(|G|-1)}{2}-\|G\|\right)+\frac{(\frac{|G|(|G|-1)}{2}-\|G\|)(\frac{|G|(|G|-1)}{2}-\|G\|-1)}{2}\\&\quad\times\left(\frac{D(G)+t}{t+2}+\frac{t+2}{D(G)+t}\right)\quad\text{by (3)}.\end{aligned} \tag{7}$$

Now,

$$\begin{aligned}H_t(G)&=\sum_{u_i,u_k\in V(G),i\neq k}\frac{1}{d_G(u_i,u_k)+t}\\&=\frac{\|G\|}{t+1}+\sum_{u_i,u_k\in V(G),i\neq k,d_G(u_i,u_k)\ge 2}\frac{1}{d_G(u_i,u_k)+t}.\end{aligned}$$

Using (4), (6) and (7) in the above, we get the lower bound in (1) and upper bound in (2) on $H_t(G)$ of the graph G.

Now suppose that the equality holds in (1) and (2). Then all inequalities in the above argument must be equalities. For the lower bound, we must have

$$d_G(u_i, u_k) = 2 \quad \text{for any pair of vertices } (u_i, u_k) \text{ in } V(G).$$

For the upper bound, we must have

$$d_G(u_i, u_k) = 2 = D(G) \quad \text{for any pair of vertices } (u_i, u_k) \text{ in } V(G).$$

Thus G is isomorphic to a graph of diameter 2.

Conversely, one can see easily that both equalities hold in (1) and (2) for graphs of diameter 2. □

Corollary 1 *Let G ($\ncong K_{|G|}$) be a connected graph of order $|G|$, $\|G\|$ edges and diameter $D(G)$. Then*

$$H_1(G) \geq \frac{\|G\|}{2} + \frac{(\frac{|G|(|G|-1)}{2} - \|G\|)^2}{W(G) - 2\|G\| + \frac{|G|(|G|-1)}{2}}$$

with equality holding if and only if G is isomorphic to a graph of diameter 2. *Moreover,*

$$H_1(G) \leq \frac{\|G\|}{2} + \frac{(\frac{|G|(|G|-1)}{2} - \|G\|)[2 + (\frac{|G|(|G|-1)}{2} - \|G\| - 1) \times (\frac{D(G)+1}{3} + \frac{3}{D(G)+1})]}{2(W(G) + \frac{|G|(|G|-1)}{2} - 2\|G\|)}$$

with equality holding if and only if G is isomorphic to a graph of diameter 2.

Proof Putting $t = 1$ in (1) and (2), we get the required result. By Theorem 1, we have that both equalities hold if and only if G is isomorphic to a graph of diameter 2. □

Corollary 2 *Let G ($\ncong K_{|G|}$) be a connected graph of order $|G|$, $\|G\|$ edges and diameter $D(G)$. Then*

$$H_2(G) \geq \frac{\|G\|}{3} + \frac{(\frac{|G|(|G|-1)}{2} - \|G\|)^2}{W(G) - 3\|G\| + |G|(|G|-1)}$$

with equality holding if and only if G is isomorphic to a graph of diameter 2. *Moreover,*

$$H_2(G) \leq \frac{\|G\|}{3} + \frac{(\frac{|G|(|G|-1)}{2} - \|G\|)[2 + (\frac{|G|(|G|-1)}{2} - \|G\| - 1) \times (\frac{D(G)+2}{4} + \frac{4}{D(G)+2})]}{2(W(G) + |G|(|G|-1) - 3\|G\|)}$$

with equality holding if and only if G is isomorphic to a graph of diameter 2.

Proof Putting $t = 2$ in (1) and (2), we get the required result. By Theorem 1, we have that both equalities hold if and only if G is isomorphic to a graph of diameter 2. □

From the above Theorem 1, we get the lower and upper bounds on the Harary index of graphs.

Theorem 2 *Let G ($\ncong K_{|G|}$) be a connected graph of order $|G|$, $\|G\|$ edges and diameter $D(G)$. Then*

$$H(G) \geq \|G\| + \frac{(\frac{|G|(|G|-1)}{2} - \|G\|)^2}{W(G) - \|G\|}$$

with equality holding if and only if G is isomorphic to a graph of diameter 2. *Moreover,*

$$H(G) \leq \|G\| + \frac{(\frac{|G|(|G|-1)}{2} - \|G\|)[2 + (\frac{|G|(|G|-1)}{2} - \|G\| - 1) \times (\frac{D(G)}{2} + \frac{2}{D(G)})]}{2(W(G) - \|G\|)}$$

with equality holding if and only if G is isomorphic to a graph of diameter 2.

Proof Putting $t = 0$ in (1) and (2), we get the required result. By Theorem 1, we have that both equalities hold if and only if G is isomorphic to a graph of diameter 2. □

3 Harary index of graph operations

In this section, some exact formulae for the Harary index of some graph operations are presented.

Let G_1 and G_2 be two graphs with $|G_1|$ and $|G_2|$ vertices, and $\|G_1\|$ and $\|G_1\|$ edges, respectively. The join $G_1 \vee G_2$ of graphs G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$, and edge sets $E(G_1)$ and $E(G_2)$ is the graph union $G_1 \cup G_2$ together with all the edges joining $V(G_1)$ and $V(G_2)$. Thus, for example, $\overline{K}_p \vee \overline{K}_q = K_{p,q}$, the complete bipartite graph.

Theorem 3 *Let G_1 and G_2 be two graphs. Then*

$$H(G_1 \vee G_2) = \frac{1}{2}\big(|G_1||G_2| + \|G_1\| + \|G_2\|\big) + \frac{1}{4}\big(|G_1| + |G_2|\big)\big(|G_1| + |G_2| - 1\big).$$

Proof By the definition of the Harary index, we have

$$\begin{aligned} H(G_1 \vee G_2) &= \sum_{u_i,v_j \in V(G_1 \vee G_2), u_i \neq v_j} \frac{1}{d_{G_1 \vee G_2}(u_i, v_j)} \\ &= \sum_{u_i,v_j \in V(G_1), u_i \neq v_j} \frac{1}{d_{G_1 \vee G_2}(u_i, v_j)} + \sum_{u_i,v_j \in V(G_2), u_i \neq v_j} \frac{1}{d_{G_1 \vee G_2}(u_i, v_j)} \\ &\quad + \sum_{u_i \in V(G_1)} \sum_{v_j \in V(G_2)} \frac{1}{d_{G_1 \vee G_2}(u_i, v_j)} \\ &= \frac{1}{2} \sum_{u_i \in V(G_1)} \left(d_{G_1}(u_i) + \frac{|G_1| - d_{G_1}(u_i) - 1}{2} \right) \\ &\quad + \frac{1}{2} \sum_{v_j \in V(G_2)} \left(d_{G_2}(v_j) + \frac{|G_2| - d_{G_2}(v_j) - 1}{2} \right) + |G_1||G_2| \end{aligned}$$

$$\text{as } d_{G_1 \vee G_2}(u_i, v_j) = \begin{cases} 0 & \text{if } u_i = v_j, \\ 1 & \text{if } u_i v_j \in E(G_1 \vee G_2), \\ 2 & \text{otherwise} \end{cases}$$

$$= \frac{1}{2}\big(|G_1||G_2| + \|G_1\| + \|G_2\|\big) + \frac{1}{4}\big(|G_1| + |G_2|\big)\big(|G_1| + |G_2| - 1\big)$$

$$\text{as} \sum_{u_i \in V(G_1)} d_{G_1}(u_i) = 2\|G_1\| \text{ and } \sum_{v_j \in V(G_2)} d_{G_2}(v_j) = 2\|G_2\|. \qquad \square$$

For two cycles C_m and C_n, we have the following.

Example 1

$$\begin{aligned} H(C_m \vee C_n) &= \frac{1}{2}(mn + m + n) + \frac{1}{4}(m + n)(m + n - 1) \\ &= \frac{1}{4}\big(m^2 + n^2\big) + mn + \frac{1}{4}(m + n). \end{aligned}$$

The corona product $G_1 \circ G_2$ of two graphs G_1 and G_2 is defined to be the graph Γ obtained by taking one copy of G_1 (which has $|G_1|$ vertices) and $|G_1|$ copies of G_2, and then joining the ith vertex of G_1 to every vertex in the ith copy of G_2, $i = 1, 2, \ldots, |G_1|$. Let $G_1 = (V, E)$ and $G_2 = (V, E)$ be two graphs such that $V(G_1) = \{u_1, u_2, \ldots, u_{p_1}\}$, $p_1 = |G_1|$, $|E(G_1)| = \|G_1\|$ and $V(G_2) = \{v_1, v_2, \ldots, v_{p_2}\}$, $p_2 = |G_2|$, $|E(G_2)| = \|G_2\|$. Then it follows from the definition of the corona that $G_1 \circ G_2$ has $|G_1|(1 + |G_2|)$ vertices and $\|G_1\| + |G_1|\|G_2\| + |G_1||G_2|$ edges, where

$$\begin{aligned} V(G_1 \circ G_2) &= \big\{(u_i, v_j), i = 1, 2, \ldots, |G_1|; j = 0, 1, 2 \ldots, |G_2|\big\} \quad \text{and} \\ E(G_1 \circ G_2) &= \big\{\big((u_i, v_0), (u_k, v_0)\big), (u_i, u_k) \in E(G_1)\big\} \\ &\quad \cup \big\{\big((u_i, v_j), (u_i, v_\ell)\big), (v_j, v_\ell) \in E(G_2), i = 1, 2, \ldots, |G_1|\big\} \\ &\quad \cup \big\{\big((u_i, v_0), (u_i, v_\ell)\big), \ell = 1, 2, \ldots, |G_2|, i = 1, 2, \ldots, |G_1|\big\}. \end{aligned}$$

It is clear that if G_1 is connected, then $G_1 \circ G_2$ is connected, and in general $G_1 \circ G_2$ is not isomorphic to $G_2 \circ G_1$.

Theorem 4 *The Harary index of the corona product is computed as follows*:

$$H(G_1 \circ G_2) = H(G_1) + |G_2|H_1(G_1) + |G_2|^2 H_2(G_1) + \frac{1}{4}\big(|G_2| + 3\big)|G_1||G_2| + \frac{1}{2}|G_1|\|G_2\|,$$

where $H_1(G_1) = \sum_{u_i, u_k \in V(G_1), u_i \neq u_k} \frac{1}{d_{G_1}(u_i, u_k) + 1}$ *and* $H_2(G_1) = \sum_{u_i, u_k \in V(G_1), u_i \neq u_k} \frac{1}{d_{G_1}(u_i, u_k) + 2}$.

Proof Note that

$$H_k(G_1) = \sum_{u_i, u_j \in V(G_1), u_i \neq u_j} \frac{1}{d_{G_1}(u_i, u_j) + k} \quad \text{for } k = 1, 2.$$

By the definition of the Harary index, we have

$$\begin{aligned} H(G_1 \circ G_2) &= \sum_{(u_i, v_j), (u_k, v_\ell) \in V(G_1 \circ G_2), (u_i, v_j) \neq (u_k, v_\ell)} \frac{1}{d_{G_1 \circ G_2}((u_i, v_j), (u_k, v_\ell))} \\ &= \sum_{(u_i, v_0), (u_k, v_0) \in V(G_1 \circ G_2), i \neq k} \frac{1}{d_{G_1 \circ G_2}((u_i, v_0), (u_k, v_0))} \end{aligned}$$

$$\begin{aligned}
&+ \sum_{(u_i,v_0),(u_k,v_\ell)\in V(G_1\circ G_2),\ell\neq 0} \frac{1}{d_{G_1\circ G_2}((u_i,v_0),(u_k,v_\ell))} \\
&+ \sum_{(u_i,v_j),(u_k,v_\ell)\in V(G_1\circ G_2),(u_i,v_j)\neq(u_k,v_\ell),j\neq 0\neq \ell} \frac{1}{d_{G_1\circ G_2}((u_i,v_j),(u_k,v_\ell))} \\
&= \sum_{u_i,u_k\in V(G_1),i\neq k} \frac{1}{d_{G_1}(u_i,u_k)} \\
&+ \sum_{(u_i,v_0),(u_i,v_\ell)\in V(G_1\circ G_2),\ell\neq 0} \frac{1}{d_{G_1\circ G_2}((u_i,v_0),(u_i,v_\ell))} \\
&+ \sum_{(u_i,v_0),(u_k,v_\ell)\in V(G_1\circ G_2),\ell\neq 0,i\neq k} \frac{1}{d_{G_1\circ G_2}((u_i,v_0),(u_k,v_\ell))} \\
&+ \sum_{(u_i,v_j),(u_i,v_\ell)\in V(G_1\circ G_2),0\neq j\neq \ell\neq 0} \frac{1}{d_{G_1\circ G_2}((u_i,v_j),(u_i,v_\ell))} \\
&+ \sum_{(u_i,v_j),(u_k,v_\ell)\in V(G_1\circ G_2),j\neq 0\neq \ell,i\neq k} \frac{1}{d_{G_1\circ G_2}((u_i,v_j),(u_k,v_\ell))} \\
&= H(G_1) + \sum_{u_i\in V(G_1)} |G_2| + \sum_{v_j\in V(G_2)} \sum_{u_i,u_k\in V(G_1),i\neq k} \frac{1}{d_{G_1}(u_i,u_k)+1} \\
&+ \sum_{u_i\in V(G_1)} \frac{1}{2} \sum_{j=1}^{|G_2|} \left[d_{G_2}(v_j) + \frac{1}{2}\bigl(|G_2| - d_{G_2}(v_j) - 1\bigr) \right] \\
&+ \sum_{v_j\in V(G_2)} \sum_{u_i,u_k\in V(G_1),i\neq k} \frac{|G_2|}{d_{G_1}(u_i,u_k)+2} \\
&= H(G_1) + |G_1||G_2| + |G_2| \sum_{u_i,u_k\in V(G_1),i\neq k} \frac{1}{d_{G_1}(u_i,u_k)+1} \\
&+ \frac{1}{4}|G_1| \sum_{j=1}^{|G_2|} \bigl(|G_2| + d_{G_2}(v_j) - 1\bigr) + |G_2|^2 \sum_{u_i,u_k\in V(G_1),i\neq k} \frac{1}{d_{G_1}(u_i,u_k)+2} \\
&= H(G_1) + |G_2|H_1(G_1) + |G_2|^2 H_2(G_1) \\
&+ \frac{1}{4}\bigl(|G_2|+3\bigr)|G_1||G_2| + \frac{1}{2}|G_1|\|G_2\|. \qquad \square
\end{aligned}$$

Theorem 5 *Let G_1 ($\ncong K_{|G_1|}$) and G_2 be two connected graphs with diameter of G_1, $D(G_1)$. Then the lower and upper bounds on the Harary index of the corona product are as follows*:

$$\begin{aligned}
&H(G_1\circ G_2) \\
&\geq \left[\frac{1}{W(G_1) - \|G_1\|} + \frac{|G_2|}{W(G_1) - 2\|G_1\| + \frac{|G_1|(|G_1|-1)}{2}} \right. \\
&\left. + \frac{|G_2|^2}{W(G_1) - 3\|G_1\| + |G_1|(|G_1|-1)} \right] \\
&\times \left(\frac{|G_1|(|G_1|-1)}{2} - \|G_1\| \right)^2 + \|G_1\| \left(1 + \frac{|G_2|}{2} + \frac{|G_2|^2}{3} \right) \\
&+ \frac{1}{4}\bigl(|G_2|+3\bigr)|G_1||G_2| + \frac{1}{2}|G_1|\|G_2\|
\end{aligned}$$

with equality holding if and only if G_1 *is isomorphic to a graph of diameter* 2. *Moreover,*

$$\begin{aligned}
&H(G_1 \circ G_2)\\
&\quad\le \left(\frac{|G_1|(|G_1|-1)}{2} - \|G_1\|\right) \times \left[\frac{2 + (\frac{|G_1|(|G_1|-1)}{2} - \|G_1\| - 1)(\frac{D(G_1)}{2} + \frac{2}{D(G_1)})}{2(W(G_1) - \|G_1\|)}\right.\\
&\qquad + \frac{|G_2|(2 + (\frac{|G_1|(|G_1|-1)}{2} - \|G_1\| - 1)(\frac{D(G_1)+1}{3} + \frac{3}{D(G_1)+1}))}{2(W(G_1) - 2\|G_1\| + \frac{|G_1|(|G_1|-1)}{2})}\\
&\qquad \left. + \frac{|G_2|^2(2 + (\frac{|G_1|(|G_1|-1)}{2} - \|G_1\| - 1)(\frac{D(G_1)+2}{4} + \frac{4}{D(G_1)+2}))}{2(W(G_1) - 3\|G_1\| + |G_1|(|G_1|-1))}\right]\\
&\qquad + \|G_1\|\left(1 + \frac{|G_2|}{2} + \frac{|G_2|^2}{3}\right) + \frac{1}{4}\big(|G_2| + 3\big)|G_1||G_2| + \frac{1}{2}|G_1|\|G_2\|
\end{aligned}$$

with equality holding if and only if G_1 *is isomorphic to a graph of diameter* 2.

Proof Using Corollaries 1 and 2 with Theorem 2, from Theorem 4, we get the lower and upper bounds on the Harary index of the corona product. Moreover, the equality holds if and only if G_1 is isomorphic to a graph of diameter 2. □

Given a graph G with vertex set $V = \{v_1, v_2, \dots, v_n\}$, the thorn graph $G(p_1, p_2, \dots, p_n)$ first introduced by Gutman [24], is a graph obtained by attaching p_i pendent vertices to vertex v_i for $i = 1, 2, \dots, n$. In particular, if $p_1 = p_2 = \cdots = p_n = p$, we denote by $G^{(p)}$ the thorn graph $G(p_1, p_2, \dots, p_n)$ for short. Recall the definition of the corona product, the graph $G^{(p)} \cong G \circ \overline{K_p}$, where $\overline{K_p}$ denotes the complement of a complete graph K_p. Therefore, for a connected graph G of order n, we have the following.

Example 2

$$H\big(G^{(p)}\big) = H(G \circ \overline{K_p}) = H(G) + pH_1(G) + p^2H_2(G_1) + \frac{1}{4}(p+3)np.$$

The Cartesian product $G_1 \times G_2$ of graphs G_1 and G_2 has the vertex set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and $(u_i, v_j)(u_k, v_\ell)$ is an edge of $G_1 \times G_2$ if $u_i = u_k$ and $v_j v_\ell \in E(G_2)$, or $u_i u_k \in E(G_1)$ and $v_j = v_\ell$.

Theorem 6 *Let* G_1 *and* G_2 *be two connected graphs with diameter* $D(G_2)$ *of the graph* G_2. *Then*

$$\begin{aligned}
&|G_1|H(G_2) + |G_2|H(G_1) + |G_2|\big(|G_2| - 1\big)H_D(G_1)\\
&\quad\le H(G_1 \times G_2)\\
&\quad\le |G_1|H(G_2) + |G_2|H(G_1) + |G_2|\big(|G_2| - 1\big)H_1(G_1),
\end{aligned} \tag{8}$$

where

$$H_D(G_1) = \sum_{u_i, u_k \in V(G_1), i \ne k} \frac{1}{d_{G_1}(u_i, u_k) + D(G_2)} \quad \textit{and}$$

$$H_1(G_1) = \sum_{u_i, u_k \in V(G_1), i \ne k} \frac{1}{d_{G_1}(u_i, u_k) + 1}.$$

Moreover, both sides of the equality hold in (8) *if and only if* G_2 *is isomorphic to a complete graph of order* $|G_2|$.

Proof By the definition of the Harary index, we have

$$\begin{aligned}
H(G_1 \times G_2) &= \sum_{(u_i,v_j),(u_k,v_\ell)\in V(G_1\times G_2),(u_i,v_j)\neq(u_k,v_\ell)} \frac{1}{d_{G_1\times G_2}((u_i,v_j),(u_k,v_\ell))} \\
&= \sum_{(u_i,v_j),(u_k,v_\ell)\in V(G_1\times G_2),i=k,j\neq\ell} \frac{1}{d_{G_1}(u_i,u_k)+d_{G_2}(v_j,v_\ell)} \\
&\quad + \sum_{(u_i,v_j),(u_k,v_\ell)\in V(G_1\times G_2),i\neq k,j=\ell} \frac{1}{d_{G_1}(u_i,u_k)+d_{G_2}(v_j,v_\ell)} \\
&\quad + \sum_{(u_i,v_j),(u_k,v_\ell)\in V(G_1\times G_2),i\neq k,j\neq\ell} \frac{1}{d_{G_1}(u_i,u_k)+d_{G_2}(v_j,v_\ell)} \\
&= \sum_{u_i\in V(G_1)} \sum_{v_j,v_\ell\in V(G_2),j\neq\ell} \frac{1}{d_{G_2}(v_j,v_\ell)} \\
&\quad + \sum_{u_i,u_k\in V(G_1),i\neq k} \sum_{v_j\in V(G_2)} \frac{1}{d_{G_1}(u_i,u_k)} \\
&\quad + \sum_{u_i,u_k\in V(G_1),i\neq k} \sum_{v_j,v_\ell\in V(G_2),j\neq\ell} \frac{1}{d_{G_1}(u_i,u_k)+d_{G_2}(v_j,v_\ell)} \\
&\quad \text{as } d_{G_1}(u_i,u_i)=0 \text{ and } d_{G_2}(v_j,v_j)=0 \\
&\leq |G_1|H(G_2)+|G_2|H(G_1) \\
&\quad + \sum_{(u_i,u_k)\in V(G_1)} \sum_{(v_j,v_\ell)\in V(G_2)} \frac{1}{d_{G_1}(u_i,u_k)+1} \\
&\quad \text{as } d_{G_2}(v_j,v_\ell)\geq 1, j\neq\ell \\
&= |G_1|H(G_2)+|G_2|H(G_1)+|G_2|\big(|G_2|-1\big)H_1(G_1),
\end{aligned} \tag{9}$$

where $H_1(G_1)$ is given in the statement of the theorem. Since $d_{G_2}(v_j,v_\ell)\leq D(G_2)$, similarly, we get

$$\begin{aligned}
H(G_1 \times G_2) &\geq |G_1|H(G_2)+|G_2|H(G_1) \\
&\quad + \sum_{(u_i,u_k)\in V(G_1)} \sum_{(v_j,v_\ell)\in V(G_2)} \frac{1}{d_{G_1}(u_i,u_k)+D(G_2)} \\
&= |G_1|H(G_2)+|G_2|H(G_1)+|G_2|\big(|G_2|-1\big)H_D(G_1),
\end{aligned}$$

where $H_D(G_1)$ is given in the statement of the theorem. The first part of the proof is over.

Suppose that both sides of the equality hold in (8). Then we must have $d_{G_2}(v_j,v_\ell)=1$, $v_j\neq v_\ell$, for all $v_j,v_\ell\in V(G_2)$ or $d_{G_2}(v_j,v_\ell)=D(G_2)$, $v_j\neq v_\ell$, for all $v_j,v_\ell\in V(G_2)$. Hence G_2 is isomorphic to a complete graph of order $|G_2|$.

Conversely, one can see easily that (8) holds for G_2, a complete graph of order $|G_2|$. This completes the proof. □

Theorem 7 *Let* G_1 $(\ncong K_{|G_1|})$ *and* G_2 $(\ncong K_{|G_2|})$ *be two connected graphs with diameter* $D(G_1)$ *of the graph* G_1 *and* $D(G_2)$ *of the graph* G_2. *Then*

$$\begin{aligned} H(G_1 \times G_2) &> |G_1|\|G_2\| + |G_2|\|G_1\| + \frac{|G_2|(|G_2|-1)\|G_1\|}{D(G_2)+1} + \frac{(\frac{|G_2|(|G_2|-1)}{2} - \|G_2\|)^2|G_1|}{W(G_2) - \|G_2\|} \\ &\quad + \left[\frac{|G_2|}{W(G_1) - \|G_1\|} + \frac{|G_2|(|G_2|-1)}{W(G_1) + \frac{|G_1|(|G_1|-1)}{2}D(G_2) - (D(G_2)+1)\|G_1\|}\right] \\ &\quad \times \left(\frac{|G_1|(|G_1|-1)}{2} - \|G_1\|\right)^2. \end{aligned}$$

Moreover,

$$\begin{aligned} H(G_1 \times G_2) &< |G_1|\|G_2\| + |G_2|\|G_1\| + \frac{|G_2|(|G_2|-1)\|G_1\|}{2} \\ &\quad + \frac{|G_1|(\frac{|G_2|(|G_2|-1)}{2} - \|G_2\|)[2 + (\frac{|G_2|(|G_2|-1)}{2} - \|G_2\|) \times (\frac{D(G_2)}{2} + \frac{2}{D(G_2)})]}{2(W(G_2) - \|G_2\|)} \\ &\quad + \left[\frac{|G_2|(2 + (\frac{|G_1|(|G_1|-1)}{2} - \|G_1\| - 1)(\frac{D(G_1)}{2} + \frac{2}{D(G_1)}))}{2(W(G_1) - \|G_1\|)}\right. \\ &\quad \left. + \frac{|G_2|(|G_2|-1)(2 + (\frac{|G_1|(|G_1|-1)}{2} - \|G_1\| - 1)(\frac{D(G_1)+1}{3} + \frac{3}{D(G_1)+1}))}{2(W(G_1) + \frac{|G_1|(|G_1|-1)}{2} - 2\|G_1\|)}\right] \\ &\quad \times \left(\frac{|G_1|(|G_1|-1)}{2} - \|G_1\|\right). \end{aligned}$$

Proof Using Theorems 1 and 2 with Corollary 1 in Theorem 6, we get the lower and upper bounds on the Cartesian product $G_1 \times G_2$ of the graphs G_1 and G_2. Moreover, both inequalities are strict as $G_1 \ncong K_{|G_1|}$, $G_2 \ncong K_{|G_2|}$ and by Theorem 6. □

Theorem 8 *Let* G_1 *and* G_2 *be two connected graphs with diameter* $D(G_1)$ *of the graph* G_1. *Then*

$$\begin{aligned} &|G_1|H(G_2) + |G_2|H(G_1) + |G_1|\big(|G_1|-1\big)H_D(G_2) \\ &\quad \le H(G_1 \times G_2) \\ &\quad \le |G_1|H(G_2) + |G_2|H(G_1) + |G_1|\big(|G_1|-1\big)H_1(G_2), \end{aligned} \tag{10}$$

where

$$H_D(G_2) = \sum_{v_j, v_\ell \in V(G_2), j \neq \ell} \frac{1}{d_{G_2}(v_j, v_\ell) + D(G_1)} \quad \text{and}$$

$$H_1(G_2) = \sum_{v_j, v_\ell \in V(G_1), j \neq \ell} \frac{1}{d_{G_2}(v_j, v_\ell) + 1}.$$

Moreover, both sides of the equality hold in (10) *if and only if* G_1 *is isomorphic to a complete graph of order* $|G_1|$.

Proof Since $1 \leq d_{G_1}(u_i, u_k) \leq D(G_1)$, $i \neq k$, from (9) we get the required result in (10). Moreover, both sides of the equality hold in (10) if and only if G_1 is isomorphic to a complete graph of order $|G_1|$. □

Theorem 9 *Let G_1 ($\ncong K_{|G_1|}$) and G_2 ($\ncong K_{|G_2|}$) be two connected graphs with diameter $D(G_1)$ of the graph G_1 and $D(G_2)$ of the graph G_2. Then*

$$\begin{aligned} H(G_1 \times G_2) > {} & |G_1|\|G_2\| + |G_2|\|G_1\| + \frac{|G_1|(|G_1|-1)\|G_2\|}{D(G_1)+1} \\ & + \frac{(\frac{|G_1|(|G_1|-1)}{2} - \|G_1\|)^2|G_2|}{W(G_1) - \|G_1\|} \\ & + \left[\frac{|G_1|}{W(G_2) - \|G_2\|} + \frac{|G_1|(|G_1|-1)}{W(G_2) + \frac{|G_2|(|G_2|-1)}{2}D(G_1) - (D(G_1)+1)\|G_2\|}\right] \\ & \times \left(\frac{|G_2|(|G_2|-1)}{2} - \|G_2\|\right)^2. \end{aligned}$$

Moreover,

$$\begin{aligned} H(G_1 \times G_2) < {} & |G_1|\|G_2\| + |G_2|\|G_1\| + \frac{|G_1|(|G_1|-1)\|G_2\|}{2} \\ & + \frac{|G_2|(\frac{|G_1|(|G_1|-1)}{2} - \|G_1\|)[2 + (\frac{|G_1|(|G_1|-1)}{2} - \|G_1\|) \times (\frac{D(G_1)}{2} + \frac{2}{D(G_1)})]}{2(W(G_1) - \|G_1\|)} \\ & + \left[\frac{|G_1|(2 + (\frac{|G_2|(|G_2|-1)}{2} - \|G_2\| - 1)(\frac{D(G_2)}{2} + \frac{2}{D(G_2)}))}{2(W(G_2) - \|G_2\|)}\right. \\ & \left. + \frac{|G_1|(|G_1|-1)(2 + (\frac{|G_2|(|G_2|-1)}{2} - \|G_2\| - 1)(\frac{D(G_2)+1}{3} + \frac{3}{D(G_2)+1}))}{2(W(G_2) + \frac{|G_2|(|G_2|-1)}{2} - 2\|G_2\|)}\right] \\ & \times \left(\frac{|G_2|(|G_2|-1)}{2} - \|G_2\|\right). \end{aligned}$$

Proof Using Theorems 1 and 2 with Corollary 1 in Theorem 8, we get the lower and upper bound on the Cartesian product $G_1 \times G_2$ of graphs G_1 and G_2. Moreover, both inequalities are strict as $G_1 \ncong K_{|G_1|}$, $G_2 \ncong K_{|G_2|}$ and by Theorem 8. □

Corollary 3 *Let G be a connected graph of order $|G|$. Then*

$$H(G \times K_2) = 2H(G) + |G| + 2H_1(G),$$

where $H_1(G) = \sum_{v_i,v_j \in V(G), i \neq j} \frac{1}{d_G(v_i,v_j)+1}$.

Proof Choosing $G_1 = G$ and $G_2 = K_2$ in Theorem 6, this theorem follows immediately. □

The lattice graph $L_{2,n}$ (see [25]) is just $P_n \times K_2$. It is well known that [2] $H(P_n) = n\sum_{i=1}^{n-1} \frac{1}{i} - n + 1$. So, we have the following example.

Example 3

$$H(L_{2,n}) = 2H(P_n) + n + 2H_1(P_n) = 2n\sum_{i=1}^{n-1}\frac{1}{i} - 2n + 2 + n + 2H_1(P_n)$$
$$= 4n\sum_{i=3}^{n-1}\frac{1}{i} + n + 6.$$

The composition (also called *lexicographic product* [26]) $G = G_1[G_2]$ of graphs G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$ is the graph with vertex set $V(G_1) \times V(G_2)$ and (u_i, v_j) is adjacent with (u_k, v_ℓ) whenever u_i is adjacent with u_k, or $u_i = u_k$ and v_j is adjacent with v_ℓ.

Theorem 10 *Let G_1 and G_2 be two connected graphs. Then*

$$H\big(G_1[G_2]\big) = \frac{1}{4}|G_1||G_2|\big(|G_2| - 1\big) + \frac{1}{2}|G_1|\|G_2\| + |G_2|^2 H(G_1).$$

Proof By the definition of the Harary index, we have

$$\begin{aligned}
H\big(G_1[G_2]\big) &= \sum_{(u_i,v_j),(u_k,v_\ell)\in V(G_1[G_2]),(u_i,v_j)\neq(u_k,v_\ell)} \frac{1}{d_{G_1[G_2]}((u_i,v_j),(u_k,v_\ell))} \\
&= \sum_{(u_i,v_j),(u_i,v_\ell)\in V(G_1[G_2]),j\neq\ell} \frac{1}{d_{G_1[G_2]}((u_i,v_j),(u_i,v_\ell))} \\
&\quad + \sum_{(u_i,v_j),(u_k,v_\ell)\in V(G_1[G_2]),i\neq k} \frac{1}{d_{G_1[G_2]}((u_i,v_j),(u_k,v_\ell))} \\
&= \sum_{u_i\in V(G_1)} \frac{1}{2}\sum_{j=1}^{|G_2|}\left[d_{G_2}(v_j) + \frac{1}{2}\big(|G_2| - d_{G_2}(v_j) - 1\big)\right] \\
&\quad + \sum_{v_j\in V(G_2)}\sum_{u_i,u_k\in V(G_1),i\neq k} \frac{|G_2|}{d_{G_1}(u_i,u_k)} \\
&\quad \text{as } d_{G_1[G_2]}\big((u_i,v_j),(u_i,v_\ell)\big) = \begin{cases} 0 & \text{if } v_j = v_\ell, \\ 1 & \text{if } v_jv_\ell \in E(G_2), \\ 2 & \text{otherwise} \end{cases} \\
&= \frac{|G_1|}{4}\sum_{j=1}^{|G_2|}\big(|G_2| + d_{G_2}(v_j) - 1\big) + |G_2|^2 \sum_{u_i,u_k\in V(G_1),i\neq k}\frac{1}{d_{G_1}(u_i,u_k)} \\
&= \frac{1}{4}|G_1||G_2|\big(|G_2| - 1\big) + \frac{1}{2}|G_1|\|G_2\| + |G_2|^2 H(G_1).
\end{aligned}$$

□

The double graph of a given graph G, denoted by $G^{\circledast}$, is constructed by making two copies of G (including the initial edge set of each), denoted by G_1 and G_2, and adding edges u_1v_2 and u_2v_1 for every edge uv of G. From the definition of composition, we conclude that $G^{\circledast} \cong G[K_2]$ for any connected graph G. Therefore the following corollary can be easily obtained.

Corollary 4 *Let G be a connected graph. Then*

$$H\big(G[K_2]\big) = \frac{3|G|}{2} + 4H(G).$$

For a complete graph K_n, we find that $K_n^{\circledast}$ is a graph obtained by deleting a perfect matching from the complete graph K_{2n}, which is just the well-known cocktail party graph (see [27]).

Example 4

$$H\big(K_n^{\circledast}\big) = \frac{3n}{2} + 4H(K_n) = 2n^2 - \frac{n}{2}.$$

The disjunction $G_1 \otimes G_2$ of graphs G_1 and G_2 is the graph with vertex set $V(G_1) \times V(G_2)$ and (u_i, v_j) is adjacent with (u_k, v_ℓ) whenever $u_i u_k \in E(G_1)$ or $v_j v_\ell \in E(G_2)$.

Theorem 11 *Let G_1 and G_2 be two connected graphs. Then*

$$H(G_1 \otimes G_2) = \frac{1}{4}|G_1||G_2|\big(|G_1||G_2| - 1\big) + \frac{1}{2}\big(\|G_1\||G_2|^2 + \|G_2\||G_1|^2\big) - \|G_1\|\|G_2\|.$$

Proof In [22], it has been proved that

$$d_{G_1 \otimes G_2}\big((u_i, v_j), (u_k, v_\ell)\big) = \begin{cases} 0 & u_i = u_k \text{ and } v_j = v_\ell, \\ 1 & u_i u_k \in E(G) \text{ or } v_j v_\ell \in E(H), \\ 2 & \text{otherwise.} \end{cases} \tag{11}$$

Moreover, it has been showed that

$$\begin{aligned} &\big|\big\{v \in V(G_1 \otimes G_2) | d_{G_1 \otimes G_2}\big((u_i, v_j), v\big) = 1\big\}\big| \\ &\quad = d_{G_1}(u_i)|G_2| + d_{G_2}(v_j)|G_1| - d_{G_1}(u_i) d_{G_2}(v_j). \end{aligned} \tag{12}$$

By the definition of the Harary index, we have

$$\begin{aligned} H(G_1 \otimes G_2) &= \sum_{(u_i,v_j),(u_k,v_\ell) \in V(G_1 \otimes G_2),(u_i,v_j) \neq (u_k,v_\ell)} \frac{1}{d_{G_1 \otimes G_2}((u_i, v_j), (u_k, v_\ell))} \\ &= \frac{1}{2} \sum_{(u_i,v_j) \in V(G_1 \otimes G_2)} \left[d_{G_1 \otimes G_2}\big((u_i, v_j)\big) + \frac{1}{2}\big(|G_1||G_2| - d_{G_1 \otimes G_2}\big((u_i, v_j)\big) - 1\big) \right] \\ &\quad \text{by (11)} \\ &= \frac{1}{4}|G_1||G_2|\big(|G_1||G_2| - 1\big) \\ &\quad + \frac{1}{4} \sum_{(u_i,v_j) \in V(G_1 \otimes G_2)} \big(d_{G_1}(u_i)|G_2| + d_{G_2}(v_j)|G_1| - d_{G_1}(u_i) d_{G_2}(v_j)\big) \quad \text{by (12)} \\ &= \frac{1}{4}|G_1||G_2|\big(|G_1||G_2| - 1\big) \end{aligned}$$

$$+\frac{1}{4}\sum_{u_i\in V(G_1)}\sum_{v_j\in V(G_2)}\bigl(d_{G_1}(u_i)|G_2|+d_{G_2}(v_j)|G_1|-d_{G_1}(u_i)d_{G_2}(v_j)\bigr)$$

$$=\frac{1}{4}|G_1||G_2|\bigl(|G_1||G_2|-1\bigr)+\frac{1}{2}\bigl(\|G_1\||G_2|^2+\|G_2\||G_1|^2\bigr)-\|G_1\|\|G_2\|. \qquad \square$$

The construction of the extended double cover was introduced by Alon [28] in 1986. For a simple graph G with vertex set $V=\{v_1,v_2,\dots,v_n\}$, the extended double cover of G, denoted by $G^\star$, is the bipartite graph with bipartition $(X;Y)$ where $X=\{x_1,x_2,\dots,x_n\}$ and $Y=\{y_1,y_2,\dots,y_n\}$, in which x_i and y_j are adjacent if and only if $i=j$ or v_i and v_j are adjacent in G. Note that for a graph G, $G^\star\cong G\otimes K_2$. So, the corollary below follows immediately.

Corollary 5 *Let G be a connected graph of order n. Then*

$$H(G\otimes K_2)=\frac{1}{2}n(2n-1)+\frac{1}{2}\bigl(4\|G\|+n^2\bigr)-\|G\|.$$

For a complete graph K_n, by the definition listed above, we find that $K_n\otimes K_2$ is just $K_{n,n}$.

Example 5

$$H(K_n\otimes K_2)=\frac{1}{2}n(2n-1)+\frac{1}{2}\bigl(4\|K_n\|+n^2\bigr)-\|K_n\|=2n^2-n.$$

Let $G=(V,E)$ be a connected graph of $|G|$ vertices with $\|G\|$ edges. If we put two similar graphs G side by side, and any vertex of the first graph G is connected by edges with those vertices which are adjacent to the corresponding vertex of the second graph G and the resultant graph is denoted by $K_2\uplus G$, then we have $|K_2\uplus G|=2|G|$ and $\|K_2\uplus G\|=\|G\|+\|G\|+2\|G\|=4\|G\|$. Moreover, $K_2\uplus G$ is the graph of K_2 and G with the vertex set $V(K_2\times G)=V(K_2)\times V(G)$ and $(u_i,v_j)(u_k,v_\ell)$ is an edge of $K_2\times G$ whenever ($u_i=u_k$ and v_j is adjacent with v_ℓ) or ($u_i\neq u_k$ and v_j is adjacent with v_ℓ).

Theorem 12 *Let G be a connected graph. Then*

$$H(K_2\uplus G)=4H(G)+\frac{|G|}{2}.$$

Proof By the definition of the Harary index, we have

$$H(K_2\uplus G)=\sum_{(u_i,v_j),(u_k,v_\ell)\in V(K_2\uplus G_2),(u_i,v_j)\neq(u_k,v_\ell)}\frac{1}{d_{K_2\uplus G}((u_i,v_j),(u_k,v_\ell))}$$

$$=\sum_{(u_i,v_j),(u_i,v_\ell)\in V(K_2\uplus G),j\neq\ell}\frac{1}{d_{K_2\uplus G}((u_i,v_j),(u_i,v_\ell))}$$

$$+\sum_{(u_i,v_j),(u_k,v_\ell)\in V(K_2\uplus G),i\neq k}\frac{1}{d_{K_2\uplus G}((u_i,v_j),(u_k,v_\ell))}$$

$$=\sum_{u_i\in K_2}\sum_{v_j,v_\ell\in V(G),j\neq\ell}\frac{1}{d_G(v_j,v_\ell)}+\sum_{v_j\in V(G)}\left[\frac{1}{2}+\sum_{v_\ell\in V(G),\ell\neq j}\frac{1}{d_G(v_j,v_\ell)}\right]$$

$$=2\sum_{v_j,v_\ell\in V(G),j\neq\ell}\frac{1}{d_G(v_j,v_\ell)}+\frac{|G|}{2}+\sum_{v_j\in V(G)}\sum_{v_\ell\in V(G),\ell\neq j}\frac{1}{d_G(v_j,v_\ell)}$$

$$= 4H(G) + \frac{|G|}{2}$$

$$\text{as } H(G) = \sum_{v_j, v_\ell \in V(G), \ell \neq j} \frac{1}{d_G(v_j, v_\ell)} = \frac{1}{2} \sum_{v_j \in V(G)} \sum_{v_\ell \in V(G), \ell \neq j} \frac{1}{d_G(v_j, v_\ell)}. \qquad \square$$

Let $G = (V, E)$ be a connected graph of $|G|$ vertices with $\|G\|$ edges. If we put two similar graphs G side by side, and any vertex of the first graph G is connected by edges with those vertices which are nonadjacent to the corresponding vertex (including the corresponding vertex itself) of the second graph G and the resultant graph is denoted by $K_2 \sqcup G$, then we have $|K_2 \sqcup G| = 2|G|$ and $\|K_2 \sqcup G\| = \|G\| + \|G\| + |G|^2 - 2\|G\| = |G|^2$. Moreover, $K_2 \sqcup G$ is the graph of K_2 and G with the vertex set $V(K_2 \sqcup G) = V(K_2) \times V(G)$ and $(u_i, v_j)(u_k, v_\ell)$ is an edge of $K_2 \times G$ whenever ($u_i = u_k$ and v_j is adjacent with v_ℓ) or ($u_i \neq u_k$ and v_j is nonadjacent with v_ℓ).

Theorem 13 *Let G be a connected graph of order* $|G|$. *Then*

$$H(K_2 \sqcup G) = \frac{|G|(3|G| - 1)}{2}.$$

Proof In $K_2 \sqcup G$, for each vertex (u_i, v_j), there are $d_G(v_j) + |G| - 1 - d_G(v_j) + 1 = |G|$ neighbors, and $d_G(v_j) + n - 1 - d_G(v_j) = |G| - 1$ vertices with the distance 2 from itself. By the definition of the Harary index, we have

$$\begin{aligned} H(K_2 \sqcup G) &= \sum_{(u_i,v_j),(u_k,v_\ell) \in V(K_2 \sqcup G_2),(u_i,v_j) \neq (u_k,v_\ell)} \frac{1}{d_{K_2 \sqcup G}((u_i, v_j), (u_k, v_\ell))} \\ &= \sum_{u_i \in K_2} \frac{1}{2} \sum_{v_j \in G} \left(d_{K_2 \sqcup G}(u_i, v_j) + \frac{2|G| - d_{K_2 \sqcup G}(u_i, v_j) - 1}{2} \right) \\ &= \sum_{u_i \in K_2} \frac{1}{2} \sum_{v_j \in G} \left(|G| + \frac{|G| - 1}{2} \right) \\ &= \frac{|G|(3|G| - 1)}{2}. \qquad \square \end{aligned}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors completed the paper together. All authors read and approved the final manuscript.

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