Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

New generalized Hermite-Hadamard type inequalities and applications to special means

JinRong Wang[1,2*], Chun Zhu[2] and Yong Zhou[3]

*Correspondence:
wjr9668@126.com;
wangjinrong@gznc.edu.cn
[1]School of Mathematics and Computer Science, Guizhou Normal College, Guiyang, Guizhou 550018, P.R. China
[2]Department of Mathematics, Guizhou University, Guiyang, Guizhou 550025, P.R. China
Full list of author information is available at the end of the article

Abstract

In this paper, Hermite-Hadamard type inequalities involving Hadamard fractional integrals for the functions satisfying monotonicity, convexity and *s*-*e*-condition are studied. Three classes of left-type Hadamard fractional integral identities including the first-order derivative are firstly established. Some interesting Hermite-Hadamard type integral inequalities involving Hadamard fractional integrals are also presented by using the established integral identities. Finally, some applications to special means of real numbers are given.

MSC: 26A33; 26A51; 26D15

Keywords: generalized Hermite-Hadamard inequalities; Hadamard fractional integrals; integral inequalities; monotonicity; *s*-*e*-condition

1 Preliminaries

Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a convex function, then it is integrable in the sense of Riemann and

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{f(a)+f(b)}{2}.$$

The above inequality is the so-called classical Hermite-Hadamard type inequality which provides lower and upper estimations for the integral average of any convex function defined on a compact interval, involving the midpoint and the endpoints of the domain. This interesting inequality was firstly discovered by Hermite in 1881 in the journal *Mathesis* (see Mitrinović and Lacković [1]). However, this beautiful result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result (see Pečarić *et al.* [2]). For more recent results which generalize, improve and extend this classical Hermite-Hadamard inequality, one can see [3–13] and references therein.

Meanwhile, fractional integrals and derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. They are greatly applied in nonlinear oscillations of earthquakes, in many physical phenomena such as seepage flow in porous media and in a fluid-dynamic traffic model. For more recent development on fractional calculus, one can see the monographs [14–21].

Because of the wide application of Hermite-Hadamard type inequalities and fractional integrals, many researchers extend their studies to Hermite-Hadamard type inequalities involving fractional integrals not limited to integer integrals. Recently, more and more

Hermite-Hadamard inequalities involving fractional integrals have been obtained for different classes of functions; see [22, 23] for nondecreasing functions and convex functions, [24, 25] for m-convex functions and (s,m)-convex functions and the references therein.

It is remarkable that Wang *et al.* [23] adopted some idea in [22] to derive the following Hermite-Hadamard type inequalities involving Hadamard fractional integrals.

Theorem 1.1 (see Theorem 2.1, [23]) *Let $f:[a,b]\to\mathbb{R}$ be a positive function with $0<a<b$ and $f\in L[a,b]$. If f is a nondecreasing and convex function on $[a,b]$, then the following inequality for fractional integrals holds*:

$$f(\sqrt{ab})\leq\frac{\Gamma(\alpha+1)}{2(\ln b-\ln a)^{\alpha}}\left[{}_HJ^{\alpha}_{a^+}f(b)+{}_HJ^{\alpha}_{b^-}f(a)\right]\leq f(b).$$

We remark that the symbols denoting the left-sided and right-sided Hadamard fractional integrals of order $\alpha\in\mathbb{R}^+$ of function $f(x)$ are defined by

$$\left({}_HJ^{\alpha}_{a^+}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_a^x\left(\ln\frac{x}{t}\right)^{\alpha-1}f(t)\frac{dt}{t}\quad(0<a<x\leq b),$$

and

$$\left({}_HJ^{\alpha}_{b^-}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_x^b\left(\ln\frac{t}{x}\right)^{\alpha-1}f(t)\frac{dt}{t}\quad(0<a\leq x<b),$$

where $\Gamma(\cdot)$ is the gamma function.

Moreover, the following important right-type Hadamard fractional integral identities including the first-order derivative of f are also established.

Lemma 1.2 (see Lemma 3.1, [23]) *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $0<a<b$. If $f'\in L[a,b]$, then the following equality for fractional integrals holds*:

$$\begin{aligned}&\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(\ln b-\ln a)^{\alpha}}\left[{}_HJ^{\alpha}_{a^+}f(b)+{}_HJ^{\alpha}_{b^-}f(a)\right]\\&\quad=\frac{\ln b-\ln a}{2}\int_0^1\left[(1-t)^{\alpha}-t^{\alpha}\right]e^{t\ln a+(1-t)\ln b}f'\left(e^{t\ln a+(1-t)\ln b}\right)dt.\end{aligned}$$

In the sequence, some new Hermite-Hadamard type integral inequalities involving Hadamard fractional integrals are presented in [23].

However, other interesting left-type Hadamard fractional integral identities including the first-order derivative of f have not been reported. Thus, the first purpose of this paper is to find some possible presentation including the first-order derivative of f for the following equalities:

$$\frac{\Gamma(\alpha+1)}{2(\ln b-\ln a)^{\alpha}}\left[{}_HJ^{\alpha}_{a^+}f(b)+{}_HJ^{\alpha}_{b^-}f(a)\right]-f\left(\frac{a+b}{2}\right)=\text{What?}$$

$$\frac{\Gamma(\alpha+1)}{2(\ln b-\ln a)^{\alpha}}\left[{}_HJ^{\alpha}_{a^+}f(b)+{}_HJ^{\alpha}_{b^-}f(a)\right]-f(\sqrt{ab})=\text{What?}$$

$$\Gamma(\alpha+1)\left[{}_HJ^{\alpha}_{x^-}f(a)+{}_HJ^{\alpha}_{x^+}f(b)\right]-\left[f(a)(\ln x-\ln a)^{\alpha}+f(b)(\ln b-\ln x)^{\alpha}\right]=\text{What?}$$

The second purpose of this paper is to establish some interesting Hermite-Hadamard type inequalities involving Hadamard fractional integrals for the functions satisfying monotonicity, convexity and *s*-*e*-condition by utilizing the left-type Hadamard fractional integral identities. In [26], Wang *et al.* introduced the concept of *s*-*e*-condition to overcome some essential difficulties from the singular kernels in Hadamard fractional integrals.

Definition 1.3 A function $f : I \subset (0,\infty) \to \mathbb{R}$ is said to satisfy *s*-*e*-condition if

$$f\big(e^{\lambda x+(1-\lambda)y}\big) \le \lambda^s f\big(e^x\big) + (1-\lambda)^s f\big(e^y\big)$$

for all $x, y \in I$, $\lambda \in [0,1]$ and for some fixed $s \in (0,1]$.

Remark 1.4 If $f : I \subset (0,\infty) \to \mathbb{R}$ is a nondecreasing and convex function, then f satisfies

$$f\big(e^{\lambda x+(1-\lambda)y}\big) \le f\big(\lambda e^x + (1-\lambda)e^y\big) \le \lambda f\big(e^x\big) + (1-\lambda) f\big(e^y\big)$$

for all $x, y \in I$, $\lambda \in [0,1]$, which implies the *s*-*e*-condition above.

In the following, we recall the following two basic inequalities [27].

Lemma 1.5 *For* $0 < \sigma \le 1$ *and* $0 \le a < b$, *we have*

$$\big|a^\sigma - b^\sigma\big| \le (b-a)^\sigma.$$

Lemma 1.6 *For all* $\lambda, \upsilon, \omega > 0$, *then for any* $t > 0$, *we have*

$$\int_0^t (t-s)^{\upsilon-1} s^{\lambda-1} e^{-\omega s}\, ds \le \max\big\{1, 2^{1-\upsilon}\big\}\Gamma(\lambda)\left(1+\frac{\lambda}{\upsilon}\right)\omega^{-\lambda} t^{\upsilon-1}.$$

2 Left-type Hadamard fractional integral identities

In this section, we establish some important left-type Hadamard fractional integral identities including the first-order derivative of a given function.

Lemma 2.1 *Let* $f : [a,b] \to \mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $0 < a < b$. *If* $f' \in L[a,b]$, *then the following equality for fractional integrals holds*:

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^\alpha}\big[{}_H J^\alpha_{a^+} f(b) + {}_H J^\alpha_{b^-} f(a)\big] - f\left(\frac{a+b}{2}\right) \\
&\quad = \frac{b-a}{2}\int_0^1 k f'\big(ta + (1-t)b\big)\, dt \\
&\qquad - \frac{\ln b - \ln a}{2}\int_0^1 \big[(1-t)^\alpha - t^\alpha\big] e^{t\ln a + (1-t)\ln b} f'\big(e^{t\ln a+(1-t)\ln b}\big)\, dt,
\end{aligned} \tag{1}$$

where

$$k = \begin{cases} 1, & 0 \le t < \frac{1}{2}, \\ -1, & \frac{1}{2} \le t < 1. \end{cases}$$

Proof Denote

$$\begin{aligned} I &= \frac{b-a}{2}\int_0^{\frac{1}{2}} f'\bigl(ta+(1-t)b\bigr)\,dt - \frac{b-a}{2}\int_{\frac{1}{2}}^1 f'\bigl(ta+(1-t)b\bigr)\,dt \\ &\quad - \frac{\ln b-\ln a}{2}\int_0^1 \bigl[(1-t)^\alpha - t^\alpha\bigr] e^{t\ln a+(1-t)\ln b} f'\bigl(e^{t\ln a+(1-t)\ln b}\bigr)\,dt \\ &:= I_1+I_2+I_3+I_4, \end{aligned} \tag{2}$$

where

$$\begin{aligned} I_1 &= \frac{b-a}{2}\int_0^{\frac{1}{2}} f'\bigl(ta+(1-t)b\bigr)\,dt, \\ I_2 &= -\frac{b-a}{2}\int_{\frac{1}{2}}^1 f'\bigl(ta+(1-t)b\bigr)\,dt, \\ I_3 &= -\frac{\ln b-\ln a}{2}\int_0^1 (1-t)^\alpha e^{\ln b-t(\ln b-\ln a)} f'\bigl(e^{\ln b-t(\ln b-\ln a)}\bigr)\,dt, \\ I_4 &= \frac{\ln b-\ln a}{2}\int_0^1 t^\alpha e^{\ln b-t(\ln b-\ln a)} f'\bigl(e^{\ln b-t(\ln b-\ln a)}\bigr)\,dt. \end{aligned}$$

Integrating by parts, we have

$$\begin{aligned} I_1 &= \frac{b-a}{2}\int_0^{\frac{1}{2}} f'\bigl(ta+(1-t)b\bigr)\,dt \\ &= -\frac{f(ta+(1-t)b)}{2}\bigg|_0^{\frac{1}{2}} = \frac{1}{2}\biggl[f(b)-f\biggl(\frac{a+b}{2}\biggr)\biggr], \end{aligned} \tag{3}$$

and

$$\begin{aligned} I_2 &= -\frac{b-a}{2}\int_{\frac{1}{2}}^1 f'\bigl(ta+(1-t)b\bigr)\,dt \\ &= \frac{f(ta+(1-t)b)}{2}\bigg|_{\frac{1}{2}}^1 = \frac{1}{2}\biggl[f(a)-f\biggl(\frac{a+b}{2}\biggr)\biggr], \end{aligned} \tag{4}$$

and

$$\begin{aligned} I_3 &= -\frac{\ln b-\ln a}{2}\int_0^1 (1-t)^\alpha e^{\ln b-t(\ln b-\ln a)} f'\bigl(e^{\ln b-t(\ln b-\ln a)}\bigr)\,dt \\ &= (1-t)^\alpha \frac{f(e^{\ln b-t(\ln b-\ln a)})}{2}\bigg|_0^1 + \frac{\alpha}{2}\int_0^1 (1-t)^{\alpha-1} f\bigl(e^{\ln b-t(\ln b-\ln a)}\bigr)\,dt \\ &= -\frac{f(b)}{2} + \frac{\alpha}{2(\ln a-\ln b)}\int_b^a \biggl(\frac{\ln u-\ln a}{\ln b-\ln a}\biggr)^{\alpha-1} f(u)\frac{du}{u} \\ &= -\frac{f(b)}{2} + \frac{\alpha}{2(\ln b-\ln a)^\alpha}\int_a^b (\ln u-\ln a)^{\alpha-1} f(u)\frac{du}{u} \\ &= -\frac{f(b)}{2} + \frac{\Gamma(\alpha+1)}{2(\ln b-\ln a)^\alpha}\, {}_H J_{b-}^\alpha f(a), \end{aligned} \tag{5}$$

and

$$\begin{aligned} I_4 &= \frac{\ln b - \ln a}{2} \int_0^1 t^\alpha e^{\ln b - t(\ln b - \ln a)} f'\bigl(e^{\ln b - t(\ln b - \ln a)}\bigr)\,dt \\ &= -t^\alpha \frac{f(e^{\ln b - t(\ln b - \ln a)})}{2}\bigg|_0^1 + \frac{\alpha}{2}\int_0^1 t^{\alpha-1} f\bigl(e^{\ln b - t(\ln b - \ln a)}\bigr)\,dt \\ &= -\frac{f(a)}{2} + \frac{\alpha}{2(\ln a - \ln b)}\int_b^a \left(\frac{\ln b - \ln u}{\ln b - \ln a}\right)^{\alpha-1} f(u)\frac{du}{u} \\ &= -\frac{f(a)}{2} + \frac{\alpha}{2(\ln b - \ln a)^\alpha}\int_a^b (\ln b - \ln u)^{\alpha-1} f(u)\frac{du}{u} \\ &= -\frac{f(a)}{2} + \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^\alpha}\,{}_HJ^\alpha_{a^+} f(b). \end{aligned} \tag{6}$$

Submitting (3), (4), (5) and (6) into (2), it follows that

$$I = \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^\alpha}\bigl[{}_HJ^\alpha_{a^+} f(b) + {}_HJ^\alpha_{b^-} f(a)\bigr] - f\left(\frac{a+b}{2}\right). \tag{7}$$

This completes the proof. □

Lemma 2.2 *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $0 < a < b$. If $f' \in L[a,b]$, then the following equality for fractional integrals holds*:

$$\begin{aligned} &\frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^\alpha}\bigl[{}_HJ^\alpha_{a^+} f(b) + {}_HJ^\alpha_{b^-} f(a)\bigr] - f(\sqrt{ab}) \\ &\quad = \frac{\ln b - \ln a}{2}\biggl[\int_0^1 k e^{t\ln a + (1-t)\ln b} f'\bigl(e^{t\ln a + (1-t)\ln b}\bigr)\,dt \\ &\qquad - \int_0^1 \bigl[(1-t)^\alpha - t^\alpha\bigr] e^{t\ln a + (1-t)\ln b} f'\bigl(e^{t\ln a + (1-t)\ln b}\bigr)\,dt\biggr], \end{aligned} \tag{8}$$

where

$$k = \begin{cases} 1, & 0 \le t < \frac{1}{2}, \\ -1, & \frac{1}{2} \le t < 1. \end{cases}$$

Proof It is not difficult to verify that

$$\begin{aligned} I &= \int_0^{\frac{1}{2}} e^{\ln b - t(\ln b - \ln a)} f'\bigl(e^{\ln b - t(\ln b - \ln a)}\bigr)\,dt - \int_{\frac{1}{2}}^1 e^{\ln b - t(\ln b - \ln a)} f'\bigl(e^{\ln b - t(\ln b - \ln a)}\bigr)\,dt \\ &\quad - \int_0^1 \bigl[(1-t)^\alpha - t^\alpha\bigr] e^{t\ln a + (1-t)\ln b} f'\bigl(e^{t\ln a + (1-t)\ln b}\bigr)\,dt \\ &= \frac{f(b) - f(\sqrt{ab})}{\ln b - \ln a} + \frac{f(a) - f(\sqrt{ab})}{\ln b - \ln a} - \frac{f(b)}{\ln b - \ln a} + \frac{\Gamma(\alpha+1)}{(\ln b - \ln a)^{\alpha+1}}\,{}_HJ^\alpha_{b^-} f(a) \\ &\quad - \frac{f(a)}{\ln b - \ln a} + \frac{\Gamma(\alpha+1)}{(\ln b - \ln a)^{\alpha+1}}\,{}_HJ^\alpha_{a^+} f(b) \\ &= \frac{\Gamma(\alpha+1)}{(\ln b - \ln a)^{\alpha+1}}\bigl[{}_HJ^\alpha_{a^+} f(b) + {}_HJ^\alpha_{b^-} f(a)\bigr] - \frac{2f(\sqrt{ab})}{\ln b - \ln a}. \end{aligned} \tag{9}$$

Thus, by multiplying both sides by $\frac{\ln b - \ln a}{2}$ in (9), we have conclusion (8) immediately. □

Lemma 2.3 *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $0 < a < b$. If $f' \in L[a,b]$, then the following equality for fractional integrals holds*:

$$\begin{aligned}
&\Gamma(\alpha+1)\big[{}_HJ^{\alpha}_{x^-}f(a) + {}_HJ^{\alpha}_{x^+}f(b)\big] - \big[f(a)(\ln x - \ln a)^{\alpha} + f(b)(\ln b - \ln x)^{\alpha}\big] \\
&\quad = (\ln b - \ln x)^{\alpha+1}\int_0^1 \big(t^{\alpha}-1\big)e^{t\ln x+(1-t)\ln b}f'\big(e^{t\ln x+(1-t)\ln b}\big)\,dt \\
&\qquad - (\ln x - \ln a)^{\alpha+1}\int_0^1 \big(t^{\alpha}-1\big)e^{t\ln x+(1-t)\ln a}f'\big(e^{t\ln x+(1-t)\ln a}\big)\,dt.
\end{aligned} \tag{10}$$

Proof Integrating by parts, we have

$$\begin{aligned}
&\int_0^1 \big(t^{\alpha}-1\big)e^{t\ln x+(1-t)\ln b}f'\big(e^{t\ln x+(1-t)\ln b}\big)\,dt \\
&\quad = \big(t^{\alpha}-1\big)\frac{f(e^{t\ln x+(1-t)\ln b})}{\ln x - \ln b}\bigg|_0^1 - \frac{\alpha}{\ln x - \ln b}\int_0^1 t^{\alpha-1}f\big(e^{t\ln x+(1-t)\ln b}\big)\,dt \\
&\quad = \frac{f(b)}{\ln x - \ln b} - \frac{\alpha}{(\ln b - \ln x)^{\alpha+1}}\int_b^x (\ln b - \ln u)^{\alpha-1}f(u)\frac{du}{u} \\
&\quad = -\frac{f(b)}{\ln b - \ln x} + \frac{\Gamma(\alpha+1)}{(\ln b - \ln x)^{\alpha+1}}{}_HJ^{\alpha}_{x^+}f(b),
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
&\int_0^1 \big(t^{\alpha}-1\big)e^{t\ln x+(1-t)\ln a}f'\big(e^{t\ln x+(1-t)\ln a}\big)\,dt \\
&\quad = \big(t^{\alpha}-1\big)\frac{f(e^{t\ln x+(1-t)\ln a})}{\ln x - \ln a}\bigg|_0^1 - \frac{\alpha}{\ln x - \ln a}\int_0^1 t^{\alpha-1}f\big(e^{t\ln x+(1-t)\ln a}\big)\,dt \\
&\quad = \frac{f(a)}{\ln x - \ln a} - \frac{\alpha}{(\ln x - \ln a)^{\alpha+1}}\int_a^x (\ln u - \ln a)^{\alpha-1}f(u)\frac{du}{u} \\
&\quad = \frac{f(a)}{\ln x - \ln a} - \frac{\Gamma(\alpha+1)}{(\ln x - \ln a)^{\alpha+1}}{}_HJ^{\alpha}_{x^-}f(a).
\end{aligned} \tag{12}$$

Multiplying both sides of (11) and (12) by $(\ln b - \ln x)^{\alpha+1}$ and $-(\ln x - \ln a)^{\alpha+1}$, respectively, we have

$$\begin{aligned}
&(\ln b - \ln x)^{\alpha+1}\int_0^1 \big(t^{\alpha}-1\big)e^{t\ln x+(1-t)\ln b}f'\big(e^{t\ln x+(1-t)\ln b}\big)\,dt \\
&\quad = -f(b)(\ln b - \ln x)^{\alpha} + \Gamma(\alpha+1)_HJ^{\alpha}_{x^+}f(b),
\end{aligned} \tag{13}$$

and

$$\begin{aligned}
&-(\ln x - \ln a)^{\alpha+1}\int_0^1 \big(t^{\alpha}-1\big)e^{t\ln x+(1-t)\ln a}f'\big(e^{t\ln x+(1-t)\ln a}\big)\,dt \\
&\quad = -f(a)(\ln x - \ln a)^{\alpha} + \Gamma(\alpha+1)_HJ^{\alpha}_{x^-}f(a).
\end{aligned} \tag{14}$$

From equalities (13) and (14), we obtain inequality (10). □

3 Hermite-Hadamard type inequalities involving Hadamard fractional integrals

In this section, we use the important Hadamard fractional integral identities including the first-order derivative of f in Section 2 to establish many interesting Hermite-Hadamard type inequalities involving Hadamard fractional integrals of the order $\alpha \in \mathbb{R}^+$ for the functions satisfying monotonicity, convexity and s-e-condition.

3.1 Using monotonicity

Theorem 3.1 *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with $0 < a < b$. If $\alpha \in (0, 1]$, $f' \in L[a, b]$ and is nondecreasing, then the following inequality for fractional integrals holds*:

$$\begin{aligned}
&\left| \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^\alpha} \left[{}_HJ^\alpha_{a^+} f(b) + {}_HJ^\alpha_{b^-} f(a) \right] - f\left(\frac{a+b}{2} \right) \right| \\
&\quad \le \frac{1}{2} \left[b - a + \frac{b(\alpha+2)}{\alpha+1} + \frac{\sqrt{ab}(\ln b - \ln a)}{2(\alpha+1)} \right] \left| f'(b) \right|. \qquad (15)
\end{aligned}$$

Proof Using Lemma 2.1 and the nondecreasing property of f', we find

$$\begin{aligned}
&\left| \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^\alpha} \left[{}_HJ^\alpha_{a^+} f(b) + {}_HJ^\alpha_{b^-} f(a) \right] - f\left(\frac{a+b}{2} \right) \right| \\
&\quad \le \frac{b-a}{2} \int_0^1 \left| f'\big(b - t(b-a)\big) \right| dt \\
&\qquad + \frac{\ln b - \ln a}{2} \int_0^1 \left| (1-t)^\alpha - t^\alpha \right| e^{\ln b - t(\ln b - \ln a)} \left| f'\big(e^{\ln b - t(\ln b - \ln a)}\big) \right| dt \\
&\quad \le \frac{b-a}{2} \left| f'(b) \right| + \frac{\ln b - \ln a}{2} \int_0^{\frac{1}{2}} \left[(1-t)^\alpha - t^\alpha \right] e^{\ln b - t(\ln b - \ln a)} \left| f'(b) \right| dt \\
&\qquad + \frac{\ln b - \ln a}{2} \int_{\frac{1}{2}}^1 \left[t^\alpha - (1-t)^\alpha \right] e^{\ln b - t(\ln b - \ln a)} \left| f'(b) \right| dt \\
&\quad = \frac{b-a}{2} \left| f'(b) \right| + \frac{b(\ln b - \ln a)}{2} \left| f'(b) \right| (K_1 + K_2), \qquad (16)
\end{aligned}$$

where

$$\begin{aligned}
K_1 &= \int_0^{\frac{1}{2}} \left[(1-t)^\alpha - t^\alpha \right] e^{-t(\ln b - \ln a)}\, dt, \\
K_2 &= \int_{\frac{1}{2}}^1 \left[t^\alpha - (1-t)^\alpha \right] e^{-t(\ln b - \ln a)}\, dt.
\end{aligned}$$

Calculating K_1 and K_2, we have

$$\begin{aligned}
K_1 &= \int_0^{\frac{1}{2}} \left[(1-t)^\alpha - t^\alpha \right] e^{-t(\ln b - \ln a)}\, dt \\
&\le \int_0^{\frac{1}{2}} (1-2t)^\alpha e^{-t(\ln b - \ln a)}\, dt \\
&= \frac{1}{2} \int_0^1 (1-s)^{(\alpha+1)-1} e^{-\frac{1}{2}(\ln b - \ln a)s}\, ds
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2}\max\{1, 2^{-\alpha}\}\left(1 + \frac{1}{\alpha+1}\right)\left(\frac{\ln b - \ln a}{2}\right)^{-1} \\
&\leq \frac{\alpha+2}{(\alpha+1)(\ln b - \ln a)},
\end{aligned} \tag{17}$$

and

$$\begin{aligned}
K_2 &= \int_{\frac{1}{2}}^{1} \left[t^\alpha - (1-t)^\alpha\right] e^{-t(\ln b - \ln a)}\, dt \\
&\leq \int_{\frac{1}{2}}^{1} (2t-1)^\alpha e^{-t(\ln b - \ln a)}\, dt \\
&= \frac{1}{2}\int_1^2 (s-1)^\alpha e^{-\frac{\ln b - \ln a}{2}s}\, dt \\
&= \frac{1}{2} e^{-(\ln b - \ln a)} \int_0^1 (1-\tau)^\alpha e^{\frac{\ln b - \ln a}{2}\tau}\, d\tau \\
&\leq \frac{1}{2} e^{-\frac{\ln b - \ln a}{2}} \int_0^1 (1-\tau)^\alpha\, d\tau \\
&= \frac{\sqrt{\frac{a}{b}}}{2(\alpha+1)},
\end{aligned} \tag{18}$$

where Lemmas 1.5 and 1.6 are used.

Thus, if we use (17) and (18) in (16), we obtain the inequality of (15). This completes the proof. □

Theorem 3.2 *Let* $f : [a,b] \to \mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $0 < a < b$. *If* $\alpha \in (0,1]$, $f' \in L[a,b]$ *and is nondecreasing, then the following inequality for fractional integrals holds*:

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^\alpha}\left[{}_H J_{a^+}^\alpha f(b) + {}_H J_{b^-}^\alpha f(a)\right] - f\left(\frac{a+b}{2}\right)\right| \\
&\quad \leq \left[\frac{b-a}{2} + \frac{b(\ln b - \ln a)}{\alpha+1}\left(1 - \frac{1}{2^\alpha}\right)\right]\left|f'(b)\right|.
\end{aligned}$$

Proof Using Lemma 2.1 and the nondecreasing property of f', one can obtain

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^\alpha}\left[{}_H J_{a^+}^\alpha f(b) + {}_H J_{b^-}^\alpha f(a)\right] - f\left(\frac{a+b}{2}\right)\right| \\
&\quad \leq \frac{b-a}{2}\left|f'(b)\right| + \frac{b(\ln b - \ln a)}{2}\left|f'(b)\right|\left[\int_0^{\frac{1}{2}}\left[(1-t)^\alpha - t^\alpha\right] dt + \int_{\frac{1}{2}}^1 \left[t^\alpha - (1-t)^\alpha\right] dt\right] \\
&\quad = \left[\frac{b-a}{2} + \frac{b(\ln b - \ln a)}{\alpha+1}\left(1 - \frac{1}{2^\alpha}\right)\right]\left|f'(b)\right|.
\end{aligned}$$

The proof is completed. □

Theorem 3.3 *Let* $f : [a,b] \to \mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $0 < a < b$. *If* $\alpha \in (0,1]$, $f' \in L[a,b]$ *and is nondecreasing, then the following inequality for fractional integrals* *holds*:

$$\left|\frac{\Gamma(\alpha+1)}{2(\ln b-\ln a)^{\alpha}}\left[{}_HJ^{\alpha}_{a^+}f(b)+{}_HJ^{\alpha}_{b^-}f(a)\right]-f(\sqrt{ab})\right|$$
$$\leq \frac{b(\ln b-\ln a)}{2}\left[1+\frac{\alpha+2}{(\alpha+1)(\ln b-\ln a)}+\frac{\sqrt{\frac{a}{b}}}{2(\alpha+1)}\right]\left|f'(b)\right|.$$

Proof Using Lemma 2.2 and the nondecreasing property of f', we find

$$\left|\frac{\Gamma(\alpha+1)}{2(\ln b-\ln a)^{\alpha}}\left[{}_HJ^{\alpha}_{a^+}f(b)+{}_HJ^{\alpha}_{b^-}f(a)\right]-f(\sqrt{ab})\right|$$
$$\leq \frac{\ln b-\ln a}{2}\left[\int_0^1 b\left|f'(b)\right|dt+\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|e^{t\ln a+(1-t)\ln b}f'\left(e^{t\ln a+(1-t)\ln b}\right)dt\right]$$
$$\leq \frac{b(\ln b-\ln a)}{2}\left[1+\frac{\alpha+2}{(\alpha+1)(\ln b-\ln a)}+\frac{\sqrt{\frac{a}{b}}}{2(\alpha+1)}\right]\left|f'(b)\right|.$$

This completes the proof. □

Theorem 3.4 *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $0<a<b$. If $\alpha\in(0,1]$, $f'\in L[a,b]$ and is nondecreasing, then the following inequality for fractional integrals holds*:

$$\left|\frac{\Gamma(\alpha+1)}{2(\ln b-\ln a)^{\alpha}}\left[{}_HJ^{\alpha}_{a^+}f(b)+{}_HJ^{\alpha}_{b^-}f(a)\right]-f(\sqrt{ab})\right|$$
$$\leq \frac{b(\ln b-\ln a)}{2}\left[1+\frac{2}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)\right]\left|f'(b)\right|.$$

Proof Using Lemma 2.2 and the nondecreasing property of f', we find

$$\left|\frac{\Gamma(\alpha+1)}{2(\ln b-\ln a)^{\alpha}}\left[{}_HJ^{\alpha}_{a^+}f(b)+{}_HJ^{\alpha}_{b^-}f(a)\right]-f(\sqrt{ab})\right|$$
$$\leq \frac{\ln b-\ln a}{2}\left[\int_0^1 b\left|f'(b)\right|dt+\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|b\left|f'(b)\right|dt\right]$$
$$\leq \frac{b(\ln b-\ln a)}{2}\left[1+\frac{2}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)\right]\left|f'(b)\right|.$$

This completes the proof. □

3.2 Using *s*-*e*-condition

Theorem 3.5 *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a differentiable mapping on (a,b) with $0<a<b$ such that $f'\in L(a,b)$. If $|f'|$ satisfies s-e-condition on $[a,b]$ for some fixed $s\in(0,1]$ and $|f'(x)|\leq M$, $x\in[a,b]$, then the following inequality for fractional integrals with $\alpha>0$ holds*:

$$\left|\Gamma(\alpha+1)\left[{}_HJ^{\alpha}_{x^-}f(a)+{}_HJ^{\alpha}_{x^+}f(b)\right]-\left[f(a)(\ln x-\ln a)^{\alpha}+f(b)(\ln b-\ln x)^{\alpha}\right]\right|$$
$$\leq Mb\left[1+\frac{2\alpha}{s+1}-\frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+1)}\right]\frac{(\ln x-\ln a)^{\alpha+1}+(\ln b-\ln x)^{\alpha+1}}{\alpha+s+1}.$$

Proof From Lemma 2.3 and since $|f'|$ satisfies s-e-condition on $[a,b]$, we have

$$
\begin{aligned}
&\bigl|\Gamma(\alpha+1)\bigl[{}_HJ^{\alpha}_{x^-}f(a)+{}_HJ^{\alpha}_{x^+}f(b)\bigr]-\bigl[f(a)(\ln x-\ln a)^{\alpha}+f(b)(\ln b-\ln x)^{\alpha}\bigr]\bigr|\\
&\quad\le(\ln b-\ln x)^{\alpha+1}\int_0^1\bigl(1-t^{\alpha}\bigr)e^{t\ln x+(1-t)\ln b}\bigl|f'\bigl(e^{t\ln x+(1-t)\ln b}\bigr)\bigr|\,dt\\
&\qquad+(\ln x-\ln a)^{\alpha+1}\int_0^1\bigl(1-t^{\alpha}\bigr)e^{t\ln x+(1-t)\ln a}\bigl|f'\bigl(e^{t\ln x+(1-t)\ln a}\bigr)\bigr|\,dt\\
&\quad\le(\ln b-\ln x)^{\alpha+1}\int_0^1\bigl(1-t^{\alpha}\bigr)x^tb^{1-t}\bigl(t^s\bigl|f'(x)\bigr|+(1-t)^s\bigl|f'(b)\bigr|\bigr)\,dt\\
&\qquad+(\ln x-\ln a)^{\alpha+1}\int_0^1\bigl(1-t^{\alpha}\bigr)x^ta^{1-t}\bigl(t^s\bigl|f'(x)\bigr|+(1-t)^s\bigl|f'(a)\bigr|\bigr)\,dt\\
&\quad\le(\ln b-\ln x)^{\alpha+1}Mb\int_0^1\bigl[t^s\bigl(1-t^{\alpha}\bigr)+\bigl(1-t^{\alpha}\bigr)(1-t)^s\bigr]\,dt\\
&\qquad+(\ln x-\ln a)^{\alpha+1}Mb\int_0^1\bigl[t^s\bigl(1-t^{\alpha}\bigr)+\bigl(1-t^{\alpha}\bigr)(1-t)^s\bigr]\,dt\\
&\quad=Mb\bigl[(\ln x-\ln a)^{\alpha+1}+(\ln b-\ln x)^{\alpha+1}\bigr]\int_0^1\bigl[t^s\bigl(1-t^{\alpha}\bigr)+(1-t)^s-t^{\alpha}(1-t)^s\bigr]\,dt\\
&\quad=Mb\biggl[\frac{2}{s+1}-\frac{1}{s+\alpha+1}-\frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+2)}\biggr]\bigl[(\ln x-\ln a)^{\alpha+1}+(\ln b-\ln x)^{\alpha+1}\bigr]\\
&\quad=Mb\biggl[1+\frac{2\alpha}{s+1}-\frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+1)}\biggr]\frac{(\ln x-\ln a)^{\alpha+1}+(\ln b-\ln x)^{\alpha+1}}{\alpha+s+1},
\end{aligned}
$$

where we use the fact that $x^ta^{1-t}\le x\le b$ and $x^tb^{1-t}\le b$ via

$$\int_0^1 t^s\bigl(1-t^{\alpha}\bigr)\,dt=\frac{1}{s+1}-\frac{1}{s+\alpha+1},$$

and

$$\int_0^1\bigl[(1-t)^s-t^{\alpha}(1-t)^s\bigr]\,dt=\frac{1}{s+1}-\mathbb{B}(\alpha+1,s+1)=\frac{1}{s+1}-\frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+2)}.$$

So, using the reduction formula $\Gamma(n+1)=n\Gamma(n)$ $(n>0)$ for the Euler gamma function, the proof is completed. □

Theorem 3.6 *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a differentiable mapping on (a,b) with $0<a<b$ such that $f'\in L(a,b)$. If $|f'|^q$ satisfies s-e-condition on $[a,b]$ for some fixed $s\in(0,1]$ and $|f'(x)|\le M$, $x\in[a,b]$, then the following inequality for fractional integrals with $\alpha>0$ holds*:

$$
\begin{aligned}
&\bigl|\Gamma(\alpha+1)\bigl[{}_HJ^{\alpha}_{x^-}f(a)+{}_HJ^{\alpha}_{x^+}f(b)\bigr]-\bigl[f(a)(\ln x-\ln a)^{\alpha}+f(b)(\ln b-\ln x)^{\alpha}\bigr]\bigr|\\
&\quad\le Mb\biggl(1-\frac{1}{\alpha+1}\biggr)^{\frac{1}{p}}\biggl[1+\frac{2\alpha}{s+1}-\frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+1)}\biggr]^{\frac{1}{q}}\\
&\qquad\times\frac{(\ln x-\ln a)^{\alpha+1}+(\ln b-\ln x)^{\alpha+1}}{(\alpha+s+1)^{\frac{1}{q}}},
\end{aligned}
$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof Using Lemma 2.3 and the well-known Hölder inequality, and since $|f'|^q$ satisfies s-e-condition on $[a,b]$, we have

$$\begin{aligned}
&\bigl|\Gamma(\alpha+1)\bigl[{}_HJ^{\alpha}_{x^-}f(a)+{}_HJ^{\alpha}_{x^+}f(b)\bigr]-\bigl[f(a)(\ln x-\ln a)^{\alpha}+f(b)(\ln b-\ln x)^{\alpha}\bigr]\bigr| \\
&\quad\le(\ln b-\ln x)^{\alpha+1}\int_0^1\bigl(1-t^{\alpha}\bigr)e^{t\ln x+(1-t)\ln b}\bigl|f'\bigl(e^{t\ln x+(1-t)\ln b}\bigr)\bigr|\,dt \\
&\qquad+(\ln x-\ln a)^{\alpha+1}\int_0^1\bigl(1-t^{\alpha}\bigr)e^{t\ln x+(1-t)\ln a}\bigl|f'\bigl(e^{t\ln x+(1-t)\ln a}\bigr)\bigr|\,dt \\
&\quad\le(\ln b-\ln x)^{\alpha+1}b\int_0^1\bigl(1-t^{\alpha}\bigr)\bigl|f'\bigl(e^{t\ln x+(1-t)\ln b}\bigr)\bigr|\,dt \\
&\qquad+(\ln x-\ln a)^{\alpha+1}b\int_0^1\bigl(1-t^{\alpha}\bigr)\bigl|f'\bigl(e^{t\ln x+(1-t)\ln a}\bigr)\bigr|\,dt \\
&\quad\le(\ln b-\ln x)^{\alpha+1}b\biggl(\int_0^1\bigl(1-t^{\alpha}\bigr)\,dt\biggr)^{\frac{1}{p}}\biggl(\int_0^1\bigl(1-t^{\alpha}\bigr)\bigl|f'\bigl(e^{t\ln x+(1-t)\ln b}\bigr)\bigr|^q\,dt\biggr)^{\frac{1}{q}} \\
&\qquad+(\ln x-\ln a)^{\alpha+1}b\biggl(\int_0^1\bigl(1-t^{\alpha}\bigr)\,dt\biggr)^{\frac{1}{p}}\biggl(\int_0^1\bigl(1-t^{\alpha}\bigr)\bigl|f'\bigl(e^{t\ln x+(1-t)\ln a}\bigr)\bigr|^q\,dt\biggr)^{\frac{1}{q}} \\
&\quad\le(\ln b-\ln x)^{\alpha+1}b\biggl(1-\frac{1}{\alpha+1}\biggr)^{\frac{1}{p}}\biggl(\int_0^1\bigl(1-t^{\alpha}\bigr)\bigl[t^s\bigl|f'(x)\bigr|^q+(1-t)^s\bigl|f'(b)\bigr|^q\bigr]\,dt\biggr)^{\frac{1}{q}} \\
&\qquad+(\ln x-\ln a)^{\alpha+1}b\biggl(1-\frac{1}{\alpha+1}\biggr)^{\frac{1}{p}}\biggl(\int_0^1\bigl(1-t^{\alpha}\bigr)\bigl[t^s\bigl|f'(x)\bigr|^q+(1-t)^s\bigl|f'(a)\bigr|^q\bigr]\,dt\biggr)^{\frac{1}{q}} \\
&\quad\le(\ln b-\ln x)^{\alpha+1}Mb\biggl(1-\frac{1}{\alpha+1}\biggr)^{\frac{1}{p}}\biggl(\int_0^1\bigl(1-t^{\alpha}\bigr)\bigl[t^s+(1-t)^s\bigr]\,dt\biggr)^{\frac{1}{q}} \\
&\qquad+(\ln x-\ln a)^{\alpha+1}Mb\biggl(1-\frac{1}{\alpha+1}\biggr)^{\frac{1}{p}}\biggl(\int_0^1\bigl(1-t^{\alpha}\bigr)\bigl[t^s+(1-t)^s\bigr]\,dt\biggr)^{\frac{1}{q}} \\
&\quad=Mb\biggl(1-\frac{1}{\alpha+1}\biggr)^{\frac{1}{p}}\biggl[1+\frac{2\alpha}{s+1}-\frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+1)}\biggr]^{\frac{1}{q}} \\
&\qquad\times\frac{(\ln x-\ln a)^{\alpha+1}+(\ln b-\ln x)^{\alpha+1}}{(\alpha+s+1)^{\frac{1}{q}}}.
\end{aligned}$$

This completes the proof. □

3.3 Using monotonicity and convexity

Noting Remark 1.4 and repeating the same procedures in Theorem 3.5 and Theorem 3.6, we can derive the following results immediately.

Theorem 3.7 *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a differentiable mapping on (a,b) with $0<a<b$ such that $f'\in L(a,b)$. If $|f'|$ is convex and nondecreasing on $[a,b]$ and $|f'(x)|\le M$, $x\in[a,b]$, then the following inequality for fractional integrals with $\alpha>0$ holds*:

$$\begin{aligned}
&\bigl|\Gamma(\alpha+1)\bigl[{}_HJ^{\alpha}_{x^-}f(a)+{}_HJ^{\alpha}_{x^+}f(b)\bigr]-\bigl[f(a)(\ln x-\ln a)^{\alpha}+f(b)(\ln b-\ln x)^{\alpha}\bigr]\bigr| \\
&\quad\le\alpha Mb\frac{(\ln x-\ln a)^{\alpha+1}+(\ln b-\ln x)^{\alpha+1}}{\alpha+1}.
\end{aligned}$$

Theorem 3.8 *Let* $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $0 < a < b$ *such that* $f' \in L(a,b)$. *If* $|f'|^q$ *is convex and nondecreasing on* $[a,b]$ *and* $|f'(x)| \le M$, $x \in [a,b]$, *then the following inequality for fractional integrals with* $\alpha > 0$ *holds*:

$$\begin{aligned}&\big|\Gamma(\alpha+1)\big[{}_HJ^{\alpha}_{x^-}f(a) + {}_HJ^{\alpha}_{x^+}f(b)\big] - \big[f(a)(\ln x - \ln a)^{\alpha} + f(b)(\ln b - \ln x)^{\alpha}\big]\big| \\ &\quad \le \alpha^{\frac{1}{q}} Mb\left(1 - \frac{1}{\alpha+1}\right)^{\frac{1}{p}} \frac{(\ln x - \ln a)^{\alpha+1} + (\ln b - \ln x)^{\alpha+1}}{(\alpha+1)^{\frac{1}{q}}},\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

4 Applications to special means

Consider the following special means (see Pearce and Pečarić [28]) for arbitrary real numbers α, β, $\alpha \ne \beta$ as follows:

(i) $H(\alpha,\beta) = \frac{2}{\frac{1}{\alpha}+\frac{1}{\beta}}$, $\alpha,\beta \in \mathbb{R} \setminus \{0\}$,

(ii) $A(\alpha,\beta) = \frac{\alpha+\beta}{2}$, $\alpha,\beta \in \mathbb{R}$,

(iii) $L(\alpha,\beta) = \frac{\beta-\alpha}{\ln|\beta|-\ln|\alpha|}$, $|\alpha| \ne |\beta|$, $\alpha\beta \ne 0$,

(iv) $L_n(\alpha,\beta) = [\frac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)}]^{\frac{1}{n}}$, $n \in \mathbb{Z} \setminus \{-1,0\}$, $\alpha,\beta \in \mathbb{R}$, $\alpha \ne \beta$.

Now, using the results obtained in Section 3, we give some applications to special means of real numbers.

Proposition 4.1 *Let* $a,b \in \mathbb{R}^+ \setminus \{0\}$, $a < b$. *Then*

$$\big|L(a,b) - A(a,b)\big| \le \frac{1}{2}\left[\frac{5b}{2} - a + \frac{\sqrt{ab}(\ln b - \ln a)}{4}\right], \tag{19}$$

$$\big|L(a,b) - A(a,b)\big| \le \frac{1}{2}\left[b - a + \frac{b(\ln b - \ln a)}{2}\right], \tag{20}$$

$$\big|L(a,b) - \big[A(a,b)H(a,b)\big]^{\frac{1}{2}}\big| \le \frac{b(\ln b - \ln a)}{8}\left(4 + \frac{6}{\ln b - \ln a} + \sqrt{\frac{a}{b}}\right), \tag{21}$$

and

$$\big|L(a,b) - \big[A(a,b)H(a,b)\big]^{\frac{1}{2}}\big| \le \frac{3b(\ln b - \ln a)}{4}. \tag{22}$$

Proof Applying Theorems 3.1, 3.2, 3.3 and 3.4 for $f(x) = x$ and $\alpha = 1$, one can obtain the results immediately. □

Proposition 4.2 *Let* $a,b \in \mathbb{R}^+ \setminus \{0\}$, $a < b$. *Then*

$$\big|L(a^n,b^n) - A^n(a,b)\big| \le \frac{nb^{n-1}}{2}\left[\frac{5b}{2} - a + \frac{\sqrt{ab}(\ln b - \ln a)}{4}\right], \tag{23}$$

$$\big|L(a^n,b^n) - A^n(a,b)\big| \le \frac{nb^{n-1}}{2}\left[b - a + \frac{b(\ln b - \ln a)}{2}\right], \tag{24}$$

$$\big|L(a^n,b^n) - \big[A(a,b)H(a,b)\big]^{\frac{n}{2}}\big| \le \frac{nb^n(\ln b - \ln a)}{8}\left(4 + \frac{6}{\ln b - \ln a} + \sqrt{\frac{a}{b}}\right), \tag{25}$$

and

$$\left|L(a^n,b^n)-\left[A(a,b)H(a,b)\right]^{\frac{n}{2}}\right|\le\frac{3nb^n(\ln b-\ln a)}{4}. \tag{26}$$

Proof Applying Theorems 3.1, 3.2, 3.3 and 3.4 for $f(x)=x^n$ and $\alpha=1$, one can obtain the results immediately. □

Proposition 4.3 *Let $a,b\in\mathbb{R}^+\setminus\{0\}$, $a<b$ and $n\in\mathbb{Z}$, $|n|\ge 2$. Then*

$$\left|L(a,b)L_n^n(a,b)-A^{n+1}(a,b)\right|\le\frac{(n+1)b^n}{2}\left[\frac{5b}{2}-a+\frac{\sqrt{ab}(\ln b-\ln a)}{4}\right], \tag{27}$$

$$\left|L(a,b)L_n^n(a,b)-A^{n+1}(a,b)\right|\le\frac{(n+1)b^n}{2}\left[b-a+\frac{b(\ln b-\ln a)}{2}\right], \tag{28}$$

$$\begin{aligned}&\left|L(a,b)L_n^n(a,b)-\left[A(a,b)H(a,b)\right]^{\frac{n+1}{2}}\right|\\&\quad\le\frac{(n+1)b^{n+1}(\ln b-\ln a)}{8}\left(4+\frac{6}{\ln b-\ln a}+\sqrt{\frac{a}{b}}\right),\end{aligned} \tag{29}$$

and

$$\left|L(a,b)L_n^n(a,b)-\left[A(a,b)H(a,b)\right]^{\frac{n+1}{2}}\right|\le\frac{3(n+1)b^{n+1}(\ln b-\ln a)}{4}. \tag{30}$$

Proof Applying Theorems 3.1, 3.2, 3.3 and 3.4 for $f(x)=x^{n+1}$ and $\alpha=1$, $x\in\mathbb{R}$, $z\in\mathbb{Z}$, $|n|\ge 2$, one can obtain the result immediately. □

Proposition 4.4 *Let $a,b\in\mathbb{R}^+\setminus\{0\}$ $(a<b)$, $a^{-1}>b^{-1}$. Then we have, for $n\in\mathbb{Z}$, $|n|\ge 2$*

(c1)

$$\left|L(b^{-1},a^{-1})-H^{-1}(b,a)\right|\le\frac{1}{2}\left(\frac{5}{2a}-\frac{1}{b}+\frac{\ln b-\ln a}{4\sqrt{ab}}\right),$$

(c2)

$$\left|L(b^{-1},a^{-1})-H^{-1}(b,a)\right|\le\frac{1}{2a}\left(\frac{b-a}{b}+\frac{\ln b-\ln a}{2}\right),$$

(c3)

$$\left|L(b^{-1},a^{-1})-\left[A(a,b)H(a,b)\right]^{-\frac{1}{2}}\right|\le\frac{\ln b-\ln a}{8a}\left(4+\frac{6}{\ln b-\ln a}+\sqrt{\frac{a}{b}}\right),$$

(c4)

$$\left|L(b^{-1},a^{-1})-\left[A(a,b)H(a,b)\right]^{-\frac{1}{2}}\right|\le\frac{3(\ln b-\ln a)}{4a},$$

(c5)

$$\left|L(b^{-n},a^{-n})-H^{-n}(b,a)\right|\le\frac{n}{2a^{n-1}}\left(\frac{5}{2a}-\frac{1}{b}+\frac{\ln b-\ln a}{4\sqrt{ab}}\right),$$

(c6)

$$\left|L\left(b^{-n},a^{-n}\right)-H^{-n}(b,a)\right|\leq\frac{n}{2a^{n}}\left(\frac{b-a}{b}+\frac{\ln b-\ln a}{2}\right),$$

(c7)

$$\left|L\left(b^{-n},a^{-n}\right)-\left[A(a,b)H(a,b)\right]^{-\frac{n}{2}}\right|\leq\frac{n(\ln b-\ln a)}{8a^{n}}\left(4+\frac{6}{\ln b-\ln a}+\sqrt{\frac{a}{b}}\right),$$

(c8)

$$\left|L\left(b^{-n},a^{-n}\right)-\left[A(a,b)H(a,b)\right]^{-\frac{n}{2}}\right|\leq\frac{3n(\ln b-\ln a)}{4a^{n}},$$

(c9)

$$\left|L\left(b^{-1},a^{-1}\right)L_n^n\left(b^{-1},a^{-1}\right)-H^{-n-1}(b,a)\right|\leq\frac{n+1}{2a^{n}}\left(\frac{5}{2a}-\frac{1}{b}+\frac{\ln b-\ln a}{4\sqrt{ab}}\right),$$

(c10)

$$\left|L\left(b^{-1},a^{-1}\right)L_n^n\left(b^{-1},a^{-1}\right)-H^{-n-1}(b,a)\right|\leq\frac{n+1}{2a^{n+1}}\left(\frac{b-a}{b}+\frac{\ln b-\ln a}{2}\right),$$

(c11)

$$\begin{aligned}&\left|L\left(b^{-1},a^{-1}\right)L_n^n\left(b^{-1},a^{-1}\right)-\left[A(a,b)H(a,b)\right]^{-\frac{n+1}{2}}\right|\\&\quad\leq\frac{(n+1)(\ln b-\ln a)}{8a^{n+1}}\left(4+\frac{6}{\ln b-\ln a}+\sqrt{\frac{a}{b}}\right),\end{aligned}$$

(c12)

$$\left|L\left(b^{-1},a^{-1}\right)L_n^n\left(b^{-1},a^{-1}\right)-\left[A(a,b)H(a,b)\right]^{-\frac{n+1}{2}}\right|\leq\frac{3(n+1)(\ln b-\ln a)}{4a^{n+1}}.$$

Proof Making the substitutions $a\to b^{-1}$, $b\to a^{-1}$ in (19)-(30), one can obtain desired inequalities respectively, where $A^{-1}(a^{-1},b^{-1})=H(a,b)=\frac{2}{\frac{1}{a}+\frac{1}{b}}$, $b^{-1}<a^{-1}$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
This work was carried out in collaboration between all authors. JRW and YZ raised these interesting problems in the research. JRW, CZ and YZ proved the theorems, interpreted the results and wrote the article. All authors defined the research theme, read and approved the manuscript.

Author details
[1]School of Mathematics and Computer Science, Guizhou Normal College, Guiyang, Guizhou 550018, P.R. China. [2]Department of Mathematics, Guizhou University, Guiyang, Guizhou 550025, P.R. China. [3]Department of Mathematics, Xiangtan University, Xiangtan, Hunan 411105, P.R. China.

Acknowledgements

The first author acknowledges the support by the National Natural Science Foundation of China (11201091), Key Projects of Science and Technology Research in the Chinese Ministry of Education (211169) and Key Support Subject (Applied Mathematics) of Guizhou Normal College. The third author acknowledges the support by Key Projects of Science and Technology Research in the Chinese Ministry of Education (211169) and the third author acknowledges the support by the National Natural Science Foundation of China (10971173) and Specialized Research Fund for the Doctoral Program of Higher Education (20114301110001).

Received: 19 April 2013 Accepted: 1 July 2013 Published: 16 July 2013

References

1. Mitrinović, DS, Lacković, IB: Hermite and convexity. Aequ. Math. **28**, 229-232 (1985)
2. Pečarić, JE, Proschan, F, Tong, YL: Convex Functions, Partial Orderings, and Statistical Applications. Academic Press, San Diego (1992)
3. Abramovich, S, Barić, J, Pečarić, J: Fejer and Hermite-Hadamard type inequalities for superquadratic functions. J. Math. Anal. Appl. **344**, 1048-1056 (2008)
4. Cal, J, Carcamob, J, Escauriaza, L: A general multidimensional Hermite-Hadamard type inequality. J. Math. Anal. Appl. **356**, 659-663 (2009)
5. Ödemir, ME, Avci, M, Set, E: On some inequalities of Hermite-Hadamard type via *m*-convexity. Appl. Math. Lett. **23**, 1065-1070 (2010)
6. Ödemir, ME, Avci, M, Kavurmaci, H: Hermite-Hadamard-type inequalities via (α, m)-convexity. Comput. Math. Appl. **61**, 2614-2620 (2011)
7. Dragomir, SS: Hermite-Hadamard's type inequalities for operator convex functions. Appl. Math. Comput. **218**, 766-772 (2011)
8. Dragomir, SS: Hermite-Hadamard's type inequalities for convex functions of selfadjoint operators in Hilbert spaces. Linear Algebra Appl. **436**, 1503-1515 (2012)
9. Sarikaya, MZ, Aktan, N: On the generalization of some integral inequalities and their applications. Math. Comput. Model. **54**, 2175-2182 (2011)
10. Xiao, Z, Zhang, Z, Wu, Y: On weighted Hermite-Hadamard inequalities. Appl. Math. Comput. **218**, 1147-1152 (2011)
11. Bessenyei, M: The Hermite-Hadamard inequality in Beckenbach's setting. J. Math. Anal. Appl. **364**, 366-383 (2010)
12. Tseng, K, Hwang, S, Hsu, K: Hadamard-type and Bullen-type inequalities for Lipschitzian functions and their applications. Comput. Math. Appl. **64**, 651-660 (2012)
13. Niculescu, CP: The Hermite-Hadamard inequality for log-convex functions. Nonlinear Anal. TMA **75**, 662-669 (2012)
14. Baleanu, D, Machado, JAT, Luo, AC-J: Fractional Dynamics and Control. Springer, New York (2012)
15. Diethelm, K: The Analysis of Fractional Differential Equations. Lecture Notes in Mathematics. Springer, Berlin (2010)
16. Kilbas, AA, Srivastava, HM, Trujillo, JJ: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
17. Lakshmikantham, V, Leela, S, Devi, JV: Theory of Fractional Dynamic Systems. Cambridge Scientific Publishers, Cambridge (2009)
18. Miller, KS, Ross, B: An Introduction to the Fractional Calculus and Differential Equations. Wiley, New York (1993)
19. Michalski, MW: Derivatives of noninteger order and their applications. Dissertationes Mathematicae **CCCXXVIII**. Inst. Math., Polish Acad. Sci. (1993)
20. Podlubny, I: Fractional Differential Equations. Academic Press, San Diego (1999)
21. Tarasov, VE: Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles, Fields and Media. Springer, Berlin (2011)
22. Sarikaya, MZ, Set, E, Yaldiz, H, Başak, N: Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities. Math. Comput. Model. **57**, 2403-2407 (2013)
23. Wang, J, Li, X, Zhu, C: Refinements of Hermite-Hadamard type inequalities involving fractional integrals. Bull. Belg. Math. Soc. Simon Stevin **20** (2013, in press)
24. Set, E: New inequalities of Ostrowski type for mappings whose derivatives are *s*-convex in the second sense via fractional integrals. Comput. Math. Appl. **63**, 1147-1154 (2012)
25. Wang, J, Li, X, Fečkan, M, Zhou, Y: Hermite-Hadamard-type inequalities for Riemann-Liouville fractional integrals via two kinds of convexity. Appl. Anal. (2012). doi:10.1080/00036811.2012.727986
26. Wang, J, Deng, J, Fĕckan, M: Exploring *s*-*e*-condition and applications to some Ostrowski type inequalities via Hadamard fractional integrals. Math. Slovaca (2013, in press)
27. Prudnikov, AP, Brychkov, YA, Marichev, OI: Integral and series. In: Elementary Functions, vol. 1. Nauka, Moscow (1981)
28. Pearce, CEM, Pečarić, JE: Inequalities for differentiable mappings with application to special means and quadrature formula. Appl. Math. Lett. **13**, 51-55 (2000)

doi:10.1186/1029-242X-2013-325
Cite this article as: Wang et al.: **New generalized Hermite-Hadamard type inequalities and applications to special means.** *Journal of Inequalities and Applications* 2013 **2013**:325.