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Generalized nonlinear vector mixed quasi-variational-like inequality governed by a multi-valued map

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Abstract
In this paper, we introduce and study a new class of generalized nonlinear vector mixed quasi-variational-like inequalities governed by a multi-valued map in Hausdorff topological vector spaces which includes generalized vector mixed general quasi-variational-like inequalities, generalized nonlinear mixed variational-like inequalities, and so on. By using the fixed point theorem, we prove some existence theorems for the proposed inequality.

Keywords: generalized nonlinear vector mixed quasi-variational-like inequality; multi-valued map; fixed point theorem; open lower section; 0-diagonally convex; locally convex topological vector space

1 Introduction

Variational inequality theory has appeared as an effective and powerful tool to study and investigate a wide class of problems arising in pure and applied sciences including elasticity, optimization, economics, transportation, and structural analysis; see, for instance, [1–4] and the references therein. A vector variational inequality in a finite-dimensional Euclidean space was first introduced by Giannessi [5]. This is a generalization of scalar variational inequality to the vector case by virtue of multi-criterion consideration. In 1966, Browder [6] first introduced and proved the basic existence theorems of solutions to a class of nonlinear variational inequalities. The Browder's results was extended to more generalized nonlinear variational inequalities by Liu *et al.* [7], Ahmad and Irfan [8], Husain and Gupta [9] and Xiao *et al.* [10], Zhao *et al.* [11].

In this paper, we consider a generalized nonlinear vector mixed quasi-variational-like inequality governed by a multi-valued map and establish some existence results in locally convex topological vector spaces by using the fixed point theorem.

Let Y be a locally convex Hausdorff topological vector space (l.c.s., in short) and let K be a nonempty convex subset of a Hausdorff topological vector space (t.v.s., in short) E. We denote by $L(E,Y)$ the space of all continuous linear operators from E into Y, where $L(E,Y)$ is equipped with a σ-topology, and by $\langle l,x\rangle$ the evaluation of $l \in L(E,Y)$ at $x \in E$. Let $X \subseteq L(E,Y)$. From the corollary of Schaefer [12], $L(E,Y)$ becomes a l.c.s. By Ding and Tarafdar [13], we have the bilinear map $\langle\cdot,\cdot\rangle : L(K,Y) \times K \to Y$ is continuous. Let $\operatorname{int} A$ and $\operatorname{co}(A)$ represent the interior and convex hull of a set A, respectively. Let $C : K \to 2^Y$

be a set-valued mapping such that $\operatorname{int} C(x) \neq \emptyset$ for each $x \in K$, let $\eta : K \times K \to E$ be a vector-valued mapping.

Let $N : L(E,Y) \times L(E,Y) \times L(E,Y) \to 2^{L(E,Y)}$ be a set-valued mapping, $H : K \times K \to 2^Y$, $D : K \to 2^K$ and $T, A, M : K \to 2^X$ be set-valued mappings. For each $\omega^* \in L(E,Y)$ and $g : K \to K$ a single-valued mapping, we consider the following class of generalized nonlinear vector mixed quasi-variational-like inequality governed by a multi-valued map :

$$(\mathcal{P}) \begin{cases} \text{find } u \in K \text{ such that } u \in D(u) \text{ and for each } v \in D(u), \\ \text{there exist } x \in T(u), y \in A(u) \text{ and } z \in M(u) \text{ satisfying} \\ \langle N(x,y,z) - \omega^*, \eta(v, g(u))\rangle + H(g(u), v) \nsubseteq -\operatorname{int} C(u). \end{cases} \tag{1.1}$$

The problem $(\mathcal{P})$ encompasses many models of variational inequality problems. The following problems are the special cases of $(\mathcal{P})$.

(a) If $N : L(E,Y) \times L(E,Y) \times L(E,Y) \to L(E,Y)$ and $H : K \times K \to Y$ are two single-valued mappings, $N(x,y,z) = A(x)$, where $A : L(E,Y) \to L(E,Y)$ and $\omega^* = 0$, then the problem $(\mathcal{P})$ reduces to the following generalized vector mixed general quasi-variational-like inequality problem for finding $u \in K$ such that $u \in D(u)$ and for each $v \in D(u)$, there exists $x \in T(u)$ satisfying

$$\langle A(x), \eta(v, g(u))\rangle + H(g(u), v) \notin -\operatorname{int} C(u). \tag{1.2}$$

The problem (1.2) was studied by Ding and Salahuddin [14]. Some existence results of solutions are established under suitable assumptions without monotonicity and compactness.

(b) If g is an identity mapping and $\omega^* = 0$, then the problem $(\mathcal{P})$ reduces to the following generalized nonlinear vector quasi-variational-like inequality problem for finding $(u,x,y,z) \in K \times U \times V \times W$ such that $u \in D(u)$ and for each $v \in D(u)$, there exist $x \in T(u)$, $y \in A(u)$ and $z \in M(u)$ satisfying

$$\langle N(x,y,z), \eta(v,u)\rangle + H(u,v) \nsubseteq -\operatorname{int} C(u). \tag{1.3}$$

The problem (1.3) was studied by Husain and Gupta [15].

(c) If $D(u) = K$, then the problem (1.3) reduces to the problem of finding $u \in K$ such that there exist $x \in T(u)$, $y \in A(u)$ and $z \in M(u)$ satisfying

$$\langle N(x,y,z), \eta(v,u)\rangle + H(u,v) \nsubseteq -\operatorname{int} C(u), \quad \forall v \in K, \tag{1.4}$$

which is introduced and studied by Xiao *et al.* [5]. When $N : L(E,Y) \times L(E,Y) \times L(E,Y) \to L(E,Y)$ and $H : K \times K \to Y$ are two single-valued mappings, the problem (1.4) includes some generalized variational inequality problems investigated in [8, 11, 16–19] as special cases.

(d) If $T(u) = A(u) = \emptyset$ for all $u \in K$, and N is an identity mapping, the problem (1.3) reduces to the problem of finding $u \in K$ such that $u \in D(u)$ and for all $v \in D(u)$,

$$\langle T(u), \eta(v,u)\rangle + H(u,v) \nsubseteq -\operatorname{int} C(u),$$

which is introduced and studied by Peng and Yang [20].

For suitable and appropriate conditions imposed on the mappings C, N, H, D, T, A, M, η and g and by means of the fixed point theorem, we establish some existence results of solutions for the problem $(\mathcal{P})$. It is clear that the problem $(\mathcal{P})$ is the most general and unifying one, which is also one of the main motivations of this paper.

Definition 1.1 [21] Let A and B be two topological vector spaces and let $T : A \to 2^B$ be a multi-valued mapping, then

(i) T is said to be upper semicontinuous if for any $x_0 \in A$ and for each open set U in B containing $T(x_0)$, there is a neighborhood V of x_0 in A such that $T(x) \subset U$ for all $x \in V$.

(ii) T is said to have open lower sections if the set $T^{-1}(y) = \{x \in A : y \in T(x)\}$ is open in X for each $y \in B$.

(iii) T is said to be closed if any net $\{x_\alpha\}$ in A such that $x_\alpha \to x$ and any $\{y_\alpha\}$ in B such that $y_\alpha \to y$ and $y_\alpha \in T(x_\alpha)$ for any α, we have $y \in T(x)$.

(iv) T is said to be lower semicontinuous if for any $x_0 \in A$ and for each open set U in B containing $T(x_0)$, there is a neighborhood V of x_0 in A such that $T(x) \cap U \neq \emptyset$ for all $x \in V$.

(v) T is said to be continuous if it is both lower and upper semicontinuous.

Lemma 1.2 [22] *Let A and B be two topological spaces. Suppose $T : A \to 2^B$ and $H : A \to 2^B$ are multi-valued mappings having open lower sections, then*

(i) $G : A \to 2^B$ *defined by, for each* $x \in A$, $G(x) = \mathrm{co}(T(x))$ *has open lower sections*;

(ii) $\rho : A \to 2^B$ *defined by, for each* $x \in A$, $\rho(x) = T(x) \cap H(x)$ *has open lower sections.*

Lemma 1.3 [23] *Let A and B be two topological spaces. If $T : A \to 2^B$ is an upper semicontinuous mapping with closed values, then T is closed.*

Lemma 1.4 [24] *Let A and B be two topological spaces and let $T : A \to 2^B$ be an upper semicontinuous mapping with compact values. Suppose $\{x_\alpha\}$ is a net in A such that $x_\alpha \to x_0$. If $y_\alpha \in T(x_\alpha)$ for each α, then there is a $y_0 \in T(x_0)$ and a subset $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta \to y_0$.*

Let I be an index set, E_i be a Hausdorff topological vector space for each $i \in I$. Let K_i be a family of nonempty compact convex subsets in E_i. Let $K = \prod_{i\in I} K_i$ and $E = \prod_{i\in I} E_i$.

Lemma 1.5 [8] *For each $i \in I$, let $T_i : K \to 2^{K_i}$ be a set-valued mapping. Assume that the following conditions hold.*

(i) *For each $i \in I$, T_i is a convex set-valued mapping*;

(ii) $K = \cup\{\mathrm{int}\, T_i^{-1}(x_i) : x_i \in K_i\}$.

Then there exists $\bar{x} \in K$ such that $\bar{x} \in T(\bar{x}) = \prod_{i\in I} T_i(\bar{x}_i)$, that is, $\bar{x}_i \in T_i(\bar{x}_i)$ for each $i \in I$, where $\bar{x}_i$ is the projection of $\bar{x}$ onto K_i.

2 Main results

In this section, we shall derive the solvability for the problem $(\mathcal{P})$ under certain conditions.

First, we give the concept of 0-diagonally convex which is useful for establishing the existence theorem for the problem $(\mathcal{P})$.

Definition 2.1 Let K be a convex subset of a t.v.s. E and Y be a t.v.s. Let $C : K \to 2^Y$ be a set-valued mapping and $g : K \to K$ be a single-valued mapping. Then the multi-valued mapping $H : K \times K \to 2^Y$ is said to be 0-diagonally convex with respect to g in the second variable if for any finite subset $\{x_1, \ldots, x_n\}$ of K and any $x = \sum_{i=1}^n \alpha_i x_i$ with $\alpha_i \geq 0$ for $i = 1, \ldots, n$, and $\sum_{i=1}^n \alpha_i = 1$,

$$\sum_{i=1}^n \alpha_i H\bigl(g(x), x_i\bigr) \nsubseteq -\operatorname{int} C(x).$$

Remark 2.2

(i) If g is an identity mapping, then the concept in Definition 2.1 reduces to the corresponding concept of 0-diagonal convexity in [25].

(ii) If $H : K \times K \to Y$ is a single-valued mapping, then the concept in Definition 2.1 reduces to the corresponding concept of 0-diagonally convex with respect to g in the second variable in [14].

Theorem 2.3 *Let Y be a l.c.s., K be a nonempty convex subset of a Hausdorff t.v.s. E, X be a nonempty compact convex subset of $L(E, Y)$, which is equipped with a σ-topology. Let $g : K \to K$, $\omega^* \in L(E, Y)$ and $T, A, M : K \to 2^X$ be upper semicontinuous set-valued mappings with nonempty compact values. Assume that the following conditions are satisfied*:

(i) *$D : K \to 2^K$ is a nonempty convex set-valued mapping and has open lower sections*;

(ii) *for each $v \in K$, the mapping*

$$\bigl\langle N(\cdot,\cdot,\cdot) - \omega^*, \eta(v,\cdot)\bigr\rangle + H(\cdot, v) : L(E, Y) \times L(E, Y) \times L(E, Y) \times K \times K \to 2^Y$$

is an upper semicontinuous set-valued mapping with compact values;

(iii) *$C : K \to 2^Y$ is a convex set-valued mapping with $\operatorname{int} C(u) \neq \emptyset$ for all $u \in K$*;

(iv) *$\eta : K \times K \to E$ is affine in the first argument and for all $u \in K$, $\eta(u, g(u)) = 0$*;

(v) *$H : K \times K \to 2^Y$ is generalized vector 0-diagonally convex with respect to g*;

(vi) *$g : K \to K$ is continuous*;

(vii) *for each $u \in K$, the set $\{u \in K : \operatorname{co} \Lambda(u) \cap D(u) \neq \emptyset\}$ is closed in K, where $\Lambda(u)$ is defined as*

$$\begin{aligned}\Lambda(u) = \bigl\{&v \in K : \bigl\langle N(x,y,z) - \omega^*, \eta\bigl(v, g(u)\bigr)\bigr\rangle + H\bigl(g(u), v\bigr) \subseteq -\operatorname{int} C(u),\\ &\forall x \in T(u), y \in A(u), z \in M(u)\bigr\}.\end{aligned}$$

Then the problem $(\mathcal{P})$ admits at least one solution.

Proof Let $\omega^* \in L(E, Y)$. Define a set-valued mapping $Q : K \to 2^K$ by

$$\begin{aligned}Q(u) = \bigl\{&v \in K : \bigl\langle N(x,y,z) - \omega^*, \eta\bigl(v, g(u)\bigr)\bigr\rangle + H\bigl(g(u), v\bigr) \subseteq -\operatorname{int} C(u),\\ &\forall x \in T(u), y \in A(u), z \in M(u)\bigr\}\end{aligned}$$

for all $u \in K$. We first prove that $u \notin \operatorname{co} Q(u)$ for all $u \in K$. To see this, suppose, by the method of contradiction, that there exists some point $\bar{u} \in K$ such that $\bar{u} \in \operatorname{co} Q(\bar{u})$. Then there exists a finite subset $\{v_1, v_2, \dots, v_n\} \subset Q(\bar{u})$, for $\bar{u} \in \operatorname{co}\{v_1, v_2, \dots, v_n\}$, such that

$$\langle N(\bar{x}, \bar{y}, \bar{z}) - \omega^*, \eta(v_i, g(\bar{u}))\rangle + H(g(\bar{u}), v_i) \subseteq -\operatorname{int} C(\bar{u}), \quad i = 1, 2, \dots, n.$$

Since $\operatorname{int} C(\bar{u})$ is a convex set and η is affine in the first argument, for $i = 1, 2, \dots, n$, $\alpha_i \geq 0$ with $\sum_{i=1}^n \alpha_i = 1$, $\bar{u} = \sum_{i=1}^n \alpha_i v_i$, we have

$$\left\langle N(\bar{x}, \bar{y}, \bar{z}) - \omega^*, \eta\left(\sum_{i=1}^n \alpha_i v_i, g(\bar{u})\right)\right\rangle + \sum_{i=1}^n \alpha_i H(g(\bar{u}), v_i) \subseteq -\operatorname{int} C(\bar{u}).$$

Since $\eta(u, g(u)) = 0$, for all $u \in K$, we have

$$\sum_{i=1}^n \alpha_i H(g(\bar{u}), v_i) \subseteq -\operatorname{int} C(\bar{u}),$$

which contradicts the condition (v), so that $u \notin \operatorname{co} Q(u)$ for all $u \in K$.

We now prove that

$$Q^-(v) = \{u \in K : \langle N(x, y, z) - \omega^*, \eta(v, g(u))\rangle + H(g(u), v) \subseteq -\operatorname{int} C(u),$$
$$\forall x \in T(u), y \in A(u), z \in M(u)\}$$

is open for all $v \in K$, that is, Q has open lower sections.

Consider a set-valued mapping $J : K \to 2^K$ is defined by

$$J(v) = \{u \in K : \exists x \in T(u), y \in A(u), z \in M(u) \text{ such that}$$
$$\langle N(x, y, z,) - \omega^*, \eta(v, g(u))\rangle + H(g(u), v) \nsubseteq -\operatorname{int} C(u)\}.$$

We only need to prove that $J(v)$ is closed for all $v \in K$. Let $\{u_\alpha\}$ be a net in $J(v)$ such that

$$u_\alpha \to u^*.$$

Since g is continuous, we have

$$g(u_\alpha) \to g(u^*).$$

Then there exist $x_\alpha \in T(u_\alpha)$, $y_\alpha \in A(u_\alpha)$ and $z_\alpha \in M(u_\alpha)$ such that

$$\langle N(x_\alpha, y_\alpha, z_\alpha,) - \omega^*, \eta(v_\alpha, g(u_\alpha))\rangle + H(g(u_\alpha), v_\alpha) \nsubseteq -\operatorname{int} C(u_\alpha).$$

Since T, A, M are upper semicontinuous set-valued mappings with compact values, by Lemma 1.4, $\{x_\alpha\}$, $\{y_\alpha\}$, $\{z_\alpha\}$ have convergent subnets with limits, say x^*, y^*, z^* and $x^* \in T(u^*)$, $y^* \in A(u^*)$ and $z^* \in M(u^*)$. Without loss of generality, we may assume that $x_\alpha \to x^*$, $y_\alpha \to y^*$ and $z_\alpha \to z^*$. Suppose that

$$m_\alpha \in \{\langle N(x_\alpha, y_\alpha, z_\alpha,) - \omega^*, \eta(v_\alpha, g(u_\alpha))\rangle + H(g(u_\alpha), v_\alpha) \nsubseteq -\operatorname{int} C(u_\alpha)\}.$$

Since $\langle N(\cdot,\cdot,\cdot) - \omega^*, \eta(v,\cdot)\rangle + H(\cdot,v)$ is upper semicontinuous with compact values, by Lemma 1.4, there exist $m^* \in \langle N(x^*,y^*,z^*) - \omega^*, \eta(v^*,g(u^*))\rangle + H(g(u^*),v^*)$ and a subnet $\{m_\beta\}$ of $\{m_\alpha\}$ such that $m_\beta \to m^*$. Hence $J(v)$ is closed in K. So that $Q^-(v)$ is open for each $v \in K$. Therefore Q has open lower sections.

Consider a set-valued mapping $G : K \times U \times V \times W \to 2^K$ defined by

$$G(u) = \operatorname{co} Q(u) \cap D(u), \quad \forall u \in K.$$

Since D has open lower sections by hypothesis (i), we may apply Lemma 1.2 to assert that the set-valued mapping G has also open lower sections. Let

$$Z = \{u \in K : G(u) \neq \emptyset\}.$$

There are two cases to consider. In the case $Z = \emptyset$, we have

$$\operatorname{co} Q(u) \cap D(u) = \emptyset \quad \text{for each } u \in K.$$

This implies that for each $u \in K$,

$$Q(u) \cap D(u) = \emptyset.$$

On the other hand, by the condition (i), and the fact that K is a compact convex subset of Y, we can apply Lemma 1.5, in this case that $I = \{1\}$, to assert the existence of a fixed point $u^* \in D(u^*)$, we have

$$Q(u^*) \cap D(u^*) = \emptyset.$$

This implies $\forall v \in D(u^*)$, $v \notin Q(u^*)$. Hence, in this particular case, the assertion of the theorem holds.

We now consider the case $Z \neq \emptyset$. Define a set-valued mapping $S : K \to 2^K$ by

$$S(u) = \begin{cases} G(u), & u \in Z; \\ D(u), & u \in K \setminus Z. \end{cases}$$

Then, for each $u \in K$, $S(u)$ is a convex set and for each $t \in K$,

$$S^-(t) = G^-(t) \cup \bigl((K \setminus Z) \cap \bigl(D^-(t)\bigr)\bigr).$$

Since $D^-(t)$, $\operatorname{co} Q^-(t)$ are open in K and $K \setminus Z$ is open in K by the condition (vii), we have $S^-(t)$ is open in K. This implies that S has open lower sections. Therefore, there exists $u^* \in K$ such that $u^* \in S(u^*)$. Suppose that $u^* \in Z$, then

$$u^* \in \operatorname{co} Q(u^*) \cap D(u^*),$$

so that $u^* \in \operatorname{co} Q(u^*)$. This is a contradiction. Hence, $u^* \notin Z$. Therefore,

$$u^* \in D(u^*) \quad \text{and} \quad G(u^*) = \emptyset.$$

Thus

$$u^* \in D(u^*) \quad \text{and} \quad \operatorname{co} Q(u^*) \cap D(u^*) = \emptyset.$$

This implies

$$Q(u^*) \cap D(u^*) = \emptyset.$$

Consequently, the assertion of the theorem holds in this case. The problem $(\mathcal{P})$ admits at least one solution. □

Corollary 2.4 *Let Y be a l.c.s., K be a nonempty convex subset of a Hausdorff t.v.s. E, X be a nonempty compact convex subset of $L(E,Y)$, which is equipped with a σ-topology. Assume that N and H are single-valued mappings and $T, A, M : K \to 2^X$ are upper semicontinuous set-valued mappings with nonempty compact values. Let $\omega^* \in L(E,Y)$ and $g : K \to K$. Assume that the following conditions are satisfied*:

(i) *$D : K \to 2^K$ is a nonempty convex set-valued mapping and has open lower sections*;
(ii) *for each $v \in K$, the mapping*

$$\langle N(\cdot,\cdot,\cdot) - \omega^*, \eta(v,\cdot)\rangle + H(\cdot,v) : L(E,Y) \times L(E,Y) \times L(E,Y) \times K \times K \to 2^Y$$

is continuous;
(iii) *$C : K \to 2^Y$ is a convex set-valued mapping with $\operatorname{int} C(u) \neq \emptyset$ for all $u \in K$*;
(iv) *$\eta : K \times K \to E$ is affine in the first argument and for all $u \in K$, $\eta(u, g(u)) = 0$*;
(v) *$H : K \times K \to 2^Y$ is vector 0-diagonally convex with respect to g*;
(vi) *$g : K \to K$ is continuous*;
(vii) *for each $u \in K$, the set $\{u \in K : \operatorname{co} \Lambda(u) \cap D(u) \neq \emptyset\}$ is closed in K, where $\Lambda(u)$ is defined as*

$$\Lambda(u) = \bigl\{v \in K : \langle N(x,y,z) - \omega^*, \eta(v, g(u))\rangle + H(g(u), v) \subseteq -\operatorname{int} C(u),$$
$$\forall x \in T(u), y \in A(u), z \in M(u)\bigr\};$$

(viii) *$Y \setminus \{-\operatorname{int} C(u)\}$ is an upper semicontinuous set-valued mapping.*

Then there exists a point $\bar{u} \in K$ such that $\bar{u} \in D(\bar{u})$ and for each $v \in D(\bar{u})$, there exist $\bar{x} \in T(\bar{u})$, $\bar{y} \in A(\bar{u})$ and $\bar{z} \in M(\bar{u})$ such that

$$\langle N(\bar{x},\bar{y},\bar{z}) - \omega^*, \eta(v, g(\bar{u}))\rangle + H(g(\bar{u}), v) \notin -\operatorname{int} C(\bar{u}).$$

Proof

Define a set-valued mapping $Q : K \to 2^K$ by

$$Q(u) = \bigl\{v \in K : \langle N(x,y,z) - \omega^*, \eta(v, g(u))\rangle + H(g(u), v) \in -\operatorname{int} C(u),$$
$$\forall x \in T(u), y \in A(u), z \in M(u)\bigr\}$$

for all $u \in K$. We now prove that $Q^{-}(v)$ is open for each $v \in K$, that is,

$$\begin{aligned}(Q^{-1}(v))^c = \{u \in K : \exists x \in T(u), y \in A(u), z \in M(u) \text{ such that} \\ \langle N(x,y,z) - \omega^*, \eta(v, g(u))\rangle + H(g(u), v) \cap Y \setminus \{-\operatorname{int} C(u)\} \neq \emptyset\}\end{aligned}$$

is closed in K. Let $\{u_t\}$ be a net in $(Q^{-1}(v))^c$ such that

$$g(u_t) \to g(u^*) \in K.$$

Then there exist $x_t \in T(u_t)$, $y_t \in A(u_t)$ and $z_t \in M(u_t)$ such that

$$\langle N(x_t, y_t, z_t) - \omega^*, \eta(v, g(u_t))\rangle + H(g(u_t), v) \in Y \setminus \{-\operatorname{int} C(u_t)\}.$$

The upper semicontinuity, compact values of T, A, M and Lemma 1.4 imply that there exist convergent subnets $\{x_{t_j}\}$, $\{y_{t_j}\}$ and $\{z_{t_j}\}$ such that

$$x_{t_j} \to x^*, \qquad y_{t_j} \to y^* \quad \text{and} \quad z_{t_j} \to z^*$$

for some $x^* \in T(u), y^* \in A(u)$ and $z^* \in M(u)$. Since $\langle N(\cdot,\cdot,\cdot) - \omega^*, \eta(v,\cdot)\rangle + H(\cdot, v)$ is continuous, we have

$$\begin{aligned}&\langle N(x_{t_j}, y_{t_j}, z_{t_j}) - \omega^*, \eta(v, g(u_{t_j}))\rangle + H(g(u_{t_j}), v) \\ &\quad \to \langle N(x^*, y^*, z^*) - \omega^*, \eta(v, g(u^*))\rangle + H(g(u^*), v).\end{aligned}$$

From Lemma 1.3 and upper semicontinuity of $Y \setminus (-\operatorname{int} C(u))$, we have

$$\langle N(x^*, y^*, z^*) - \omega^*, \eta(v, g(u^*))\rangle + H(g(u^*), v) \in Y \setminus (-\operatorname{int} C(u^*)),$$

and hence $u^* \in (Q^{-1}(v))^c$, which gives that $(Q^{-1}(v))^c$ is closed. Therefore Q has open lower sections. For the remainder of the proof, we can just follow that of Theorem 2.3. This completes the proof. □

Theorem 2.5 *Let Y be a l.c.s., K be a nonempty convex subset of a Hausdorff t.v.s. E, X be a nonempty compact convex subset of $L(E,Y)$, which is equipped with a σ-topology. Let $\omega^* \in L(E,Y)$, $g: K \to K$ and $T, A, M: K \to 2^X$ be upper semicontinuous set-valued mappings. Assume that the following conditions are satisfied.*

(i) *$D: K \to 2^K$ is a nonempty convex set-valued mapping and has open lower sections*;

(ii) *for each $y \in K$, the mapping*

$$\langle N(\cdot,\cdot,\cdot) - \omega^*, \eta(v,\cdot)\rangle + H(\cdot, v) : L(E,Y) \times L(E,Y) \times L(E,Y) \times K \times K \to 2^Y$$

is upper semicontinuous;

(iii) *$C: K \to 2^Y$ is a convex set-valued mapping with $\operatorname{int} C(u) \neq \emptyset$ for all $u \in K$*;

(iv) *$\eta: K \times K \to E$ is affine in the first argument and for all $x \in K$, $\eta(u, g(u)) = 0$*;

(v) *$H: K \times K \to 2^Y$ is generalized vector 0-diagonally convex with respect to g*;

(vi) *$g: K \to K$ is continuous*;

(vii) *For each* $u \in K$, *the set* $\{u \in K : \operatorname{co}\Lambda(u) \cap D(u) \neq \emptyset\}$ *is closed in* K, *where* $\Lambda(u)$ *is defined as*

$$\Lambda(u) = \bigl\{v \in K : \bigl\langle N(x,y,z) - \omega^*, \eta\bigl(v, g(u)\bigr)\bigr\rangle + H\bigl(g(u), v\bigr) \subseteq -\operatorname{int} C(u), \\ \forall x \in T(u), y \in A(u), z \in M(u)\bigr\};$$

(viii) *for a given* $u \in K$, *and a neighborhood* O *of* u, *for all* $t \in O$, $\operatorname{int} C(u) = \operatorname{int} C(t)$.

Then the problem $(\mathcal{P})$ *admits at least one solution.*

Proof Define a set-valued mapping $Q : K \to 2^K$ by

$$Q(u) = \bigl\{v \in K : \bigl\langle N(x,y,z) - \omega^*, \eta\bigl(v, g(u)\bigr)\bigr\rangle + H\bigl(g(u), v\bigr) \subseteq -\operatorname{int} C(u), \\ \forall x \in T(u), y \in A(u), z \in M(u)\bigr\}$$

for all $u \in K$. We now prove that for each $v \in K$,

$$Q^{-1}(v) = \bigl\{u \in K : \bigl\langle N(x,y,z) - \omega^*, \eta\bigl(v, g(u)\bigr)\bigr\rangle + H\bigl(g(u), v\bigr) \subseteq -\operatorname{int} C(u), \\ \forall x \in T(u), y \in A(u), z \in M(u)\bigr\}$$

is open. That is, Q has open lower sections in K. Indeed, let $\bar{u} \in Q^-(v)$, that is,

$$\bigl\langle N(x,y,z) - \omega^*, \eta\bigl(v, g(\bar{u})\bigr)\bigr\rangle + H\bigl(g(\bar{u}), v\bigr) \subseteq -\operatorname{int} C(\bar{u}).$$

Since $\langle N(\cdot,\cdot,\cdot) - \omega^*, \eta(y, g(\cdot))\rangle + H(g(\cdot), y)$ is upper semicontinuous, there exists a neighborhood O of $\bar{u}$ such that

$$\bigl\langle N(x,y,z) - \omega^*, \eta\bigl(v, g(u)\bigr)\bigr\rangle + H\bigl(g(u), v\bigr) \subseteq -\operatorname{int} C(u), \quad \forall u \in O.$$

By (vii),

$$\bigl\langle N(x,y,z) - \omega^*, \eta\bigl(v, g(u)\bigr)\bigr\rangle + H\bigl(g(u), v\bigr) \subseteq -\operatorname{int} C(\bar{u}), \quad \forall u \in O.$$

Hence, $O \subset Q^-(v)$. This implies $Q^-(v)$ is open for each $v \in K$, and so Q has open lower sections. For the remainder of the proof, we can just follow that of Theorem 2.3. This completes the proof. □

Corollary 2.6 *Let* Y *be a l.c.s.,* K *be a nonempty convex subset of a Hausdorff t.v.s.* E, X *be a nonempty compact convex subset of* $L(E,Y)$, *which is equipped with a* σ*-topology. Let* $\omega^* \in L(E,Y)$, $g : K \to K$ *and* $T, A, M : K \to 2^X$ *be upper semicontinuous set-valued mappings. Assume that the following conditions are satisfied.*

(i) $D : K \to 2^K$ *is a nonempty convex set-valued mapping and has open lower sections*;

(ii) *for each* $y \in K$, *the mapping*

$$\bigl\langle N(\cdot,\cdot,\cdot) - \omega^*, \eta\bigl(v, g(\cdot)\bigr)\bigr\rangle + H\bigl(g(\cdot), v\bigr) : L(E,Y) \times L(E,Y) \times L(E,Y) \times K \times K \to 2^Y$$

is upper semicontinuous;

(iii) $C: K \to 2^Y$ *is a convex set-valued mapping such that for each* $u \in K$, $C(u) = C$ *is a convex cone with* $\operatorname{int} C(u) \neq \emptyset$ *for all* $u \in K$;

(iv) $\eta: K \times K \to E$ *is affine in the first argument and for all* $u \in K$, $\eta(u, g(u)) = 0$;

(v) $H: K \times K \to 2^Y$ *is generalized vector* 0*-diagonally convex with respect to* g;

(vi) $g: K \to K$ *is continuous*;

(vii) *for each* $u \in K$, *the set* $\{u \in K : \operatorname{co} \Lambda(u) \cap D(u) \neq \emptyset\}$ *is closed in* K, *where* $\Lambda(u)$ *is defined as*

$$\Lambda(u) = \big\{v \in K : \big\langle N(x,y,z) - \omega^*, \eta\big(v, g(u)\big)\big\rangle + H\big(g(u), v\big) \subseteq -\operatorname{int} C(u), \\ \forall x \in T(u), y \in A(u), z \in M(u)\big\}.$$

Then the problem $(\mathcal{P})$ *admits at least one solution*.

Proof By hypothesis (iii), the condition (vii) in Theorem 2.5 is satisfied. Hence, all the conditions in Theorem 2.5 are satisfied. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors read and approved the final manuscript.

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