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Convergence theorems for finding zero points of maximal monotone operators and equilibrium problems in Banach spaces

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Abstract

In this paper, we construct a new hybrid projection method for approximating a common element of the set of zeroes of a finite family of maximal monotone operators and the set of common solutions to a system of generalized equilibrium problems in a uniformly smooth and strictly convex Banach space. We prove strong convergence theorems of the algorithm to a common element of these two sets. As application, we also apply our results to find common solutions of variational inequalities and zeroes of maximal monotone operators.

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1 Introduction

Let E be a Banach space with the norm $\| \cdot \|$ and let E^* denote the dual space of E. Let $S = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. A Banach space E is said to be *smooth* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1.1}$$

exists for any $x, y \in S$. E is said to be *uniformly smooth* if the limit (1.1) is attained uniformly for (x, y) in $S \times S$. A Banach space E is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$ (see [1] for more details).

One of the major problems in the theory of monotone operators is as follows.

Find a point $z \in E$ such that

$$0 \in Bz, \tag{1.2}$$

where B is an operator from E into E^*. Such $z \in E$ is called a *zero point* of B. We denote the set of zeroes of the operator B by $B^{-1}0$.

An operator $B \subset E \times E^*$ is said to be *monotone* if

$$\langle x - y, x^* - y^* \rangle \geq 0, \quad \forall (x, x^*), (y, y^*) \in B.$$

A monotone B is said to be *maximal* if its graph $G(B) = \{(x, y^*) : y^* \in Bx\}$ is not properly contained in the graph of any other monotone operator. If B is maximal monotone, then the solution set $B^{-1}0$ is closed and convex. The *resolvent* of a monotone operator B is defined by

$$J_\lambda = (J + \lambda B)^{-1}J, \quad \forall \lambda > 0.$$

Let C be a closed convex subset of a Banach space E, a mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C.$$

Recall that a point $x \in C$ is a *fixed point* of T provided $Tx = x$. Let E be a Banach space with dual E^* and let $\langle \cdot, \cdot \rangle$ be the pairing between E and E^*. The *normalized duality mapping* $J : E \to 2^{E^*}$ is defined by

$$J(x) = \left\{ x^* \in E^* : \langle x, x^* \rangle = \|x\|^2, \|x^*\| = \|x\| \right\}, \quad \forall x \in E.$$

The Lyapunov functional is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2 \quad \text{for } x, y \in E. \tag{1.3}$$

It is obvious that

$$\left(\|y\| - \|x\| \right)^2 \le \phi(y, x) \le \left(\|y\| + \|x\| \right)^2, \quad \forall x, y \in E. \tag{1.4}$$

If E is a Hilbert space, then $\phi(x, y) = \|x - y\|^2$ for all $x, y \in E$.

A point p in C is said to be a *strongly asymptotic fixed point* of T [2] if C contains a sequence $\{x_n\}$ which converges strongly to p such that $\lim_{n \to \infty} \|x_n - Tx_n\| = 0$. The set of strong asymptotic fixed points of T will be denoted by $\widetilde{F}(T)$.

A mapping T from C into itself is said to be a *weak relatively nonexpansive mapping* if
1. $F(T)$ is nonempty;
2. $\phi(p, Tx) \le \phi(p, x)$ for all $x \in C$ and $p \in F(T)$;
3. $\widetilde{F}(T) = F(T)$.

Kohasaka and Takahashi [3] proved that if E is a smooth strictly convex and reflexive Banach space and B is a continuous monotone operator with $B^{-1}0 \ne \emptyset$, then J_λ is a weak relatively nonexpansive mapping. By Takahashi [4], we know that $F(J_\lambda)$ is closed and convex, where $F(J_\lambda)$ is the set of fixed points of J_λ.

Let B be a maximal monotone operator in a Hilbert space H. The proximal point algorithm generates, for starting $x_1 = x \in H$, a sequence $\{x_n\}$ in H by

$$x_{n+1} = J_{\lambda_n} x_n, \quad \forall n \ge 1, \tag{1.5}$$

where $\{\lambda_n\} \subset (0, \infty)$ and $J_{\lambda_n} = (I + \lambda_n B)^{-1}$.

Also, Rockafellar [5] proved that the sequence $\{x_n\}$ defined by (1.5) converges weakly to an element of $B^{-1}0$.

Let C be a nonempty closed convex subset of E and let $\mathbb{R}$ be the set of real numbers. Let $f_i : C \times C \to \mathbb{R}$ be a bifunction and $A_i : C \to E^*$ be a nonlinear mapping for $i = 1, 2, 3, \ldots, N$. The *system of generalized equilibrium problems* is as follows.

Find $u \in C$ such that for all $y \in C$,

$$\begin{cases} f_1(u, y) + \langle y - u, A_1 u \rangle \geq 0, \\ f_2(u, y) + \langle y - u, A_2 u \rangle \geq 0, \\ \cdots \\ f_N(u, y) + \langle y - u, A_N u \rangle \geq 0. \end{cases} \tag{1.6}$$

If $f_i = f$ and $A_i = A$ in (1.6), then from the problem (1.6) we have the following *generalized equilibrium problem* denoted by GEP(f, A).

Find $u \in C$ such that

$$f(u, y) + \langle y - u, Au \rangle \geq 0, \quad \forall y \in C. \tag{1.7}$$

The generalized equilibrium problems include fixed point problems, optimization problem, monotone inclusion problems, saddle point problems, variational inequality problems, minimization problems, vector equilibrium problems, Nash equilibria in noncooperative games and equilibrium problems as special cases (see, for example, [6]). Also, some solution methods have been proposed to solve the equilibrium problem (see, for example, [7–14]) and numerous problems in physics, optimization and economics reduce to finding a solution of problem (1.7).

Recently, Li and Su [15] introduced the hybrid iterative scheme for approximating a common solution of the equilibrium problems and the variational inequality problems in a 2-uniformly convex real Banach space which is also uniformly smooth. In 2010, Zegeye and Shahzad [16] introduced the iterative process which converges strongly to a common solution of the variational inequality problems for two monotone mappings in Banach spaces.

Quite recently, Shehu [13] introduced an iterative scheme by the hybrid method for approximating a common element of the set of zeroes of a finite family of α-inverse-strongly monotone operators and the set of common solutions of a system of generalized mixed equilibrium problems in a 2-uniformly convex real Banach space which is also uniformly smooth.

Motivated by the results of Shehu [13], we prove some strong convergence theorems for finding a common zero of a finite family of continuous monotone mappings and a solution of the system of generalized equilibrium problems in a uniformly smooth and strictly convex real Banach space with the Kadec-Klee property.

2 Preliminaries

Throughout this paper, let E be a Banach space with its dual space E^*. For a sequence $\{x_n\}$ of E and a point $x \in E$, the weak convergence of $\{x_n\}$ to x is denoted by $x_n \rightharpoonup x$ and the strong convergence of $\{x_n\}$ to x is denoted by $x_n \to x$.

The *normalized duality mapping* $J : E \to 2^{E^*}$ is defined by

$$J(x) = \left\{ x^* \in E^* : \langle x, x^* \rangle = \|x\|^2, \|x^*\| = \|x\| \right\}, \quad \forall x \in E,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing.

Cioranescu [17] proved the following properties:

(1) If E is an arbitrary Banach space, then J is monotone and bounded;

(2) If E is a strictly convex, then J is strictly monotone;

(3) If E is a smooth, then J is single-valued and semi-continuous;

(4) If E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E;

(5) If E is reflexive, smooth and strictly convex, then the normalized duality mapping J is single-valued, one-to-one and onto;

(6) If E is a reflexive, strictly convex and smooth Banach space and J is the duality mapping from E into E^*, then J^{-1} is also single-valued, bijective and is also the duality mapping from E^* into E and thus $JJ^{-1} = I_{E^*}$ and $J^{-1}J = I_E$;

(7) If E is uniformly smooth, then E is smooth and reflexive;

(8) E is uniformly smooth if and only if E^* is uniformly convex.

A Banach space E has the Kadec-Klee property [1, 17] if, for any sequence $\{x_n\} \subset E$ and $x \in E$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$ as $n \to \infty$.

Consider the functional $\phi : E \times E \to \mathbb{R}$ defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E, \tag{2.1}$$

where J is the normalized duality mapping from E to 2^{E^*}.

It is obvious from the definition of the function ϕ that

$$\left(\|y\| - \|x\|\right)^2 \le \phi(y, x) \le \left(\|y\| + \|x\|\right)^2, \quad \forall x, y \in E. \tag{2.2}$$

If E is a Hilbert space, then $\phi(y, x) = \|y - x\|^2$.

Remark 2.1 If E is a reflexive, strictly convex and smooth Banach space, then, for any $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x, y) = 0$, then $x = y$. From (1.4) we have $\|x\| = \|y\|$. This implies that $\langle x, Jy \rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, one has $Jx = Jy$. Therefore, we have $x = y$ (see [1, 17] for more details).

Let C be a nonempty closed convex subset of a reflexive, strictly convex and smooth Banach space E. The *generalized projection* $\Pi_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \inf_{y \in C} \phi(y, x). \tag{2.3}$$

The existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(y, x)$ and the strict monotonicity of the mapping J (see, for example, [1, 17–20]). If E is a Hilbert space, then Π_C becomes the metric projection of E onto C.

Example 2.2 (Qin *et al.* [21]) Let Π_C be the generalized projection from a smooth strictly convex and reflexive Banach space E onto a nonempty closed convex subset C of E. Then Π_C is a closed relatively quasi-nonexpansive mapping from E onto C with $F(\Pi_C) = C$.

We also need the following lemmas for the proof of our main results.

Lemma 2.3 (Alber [19]) *Let C be a nonempty closed convex subset of a smooth Banach space E and let $x \in E$. Then $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.4 (Alber [19]) *Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C.$$

Let E be a smooth strictly convex and reflexive Banach space, C be a nonempty closed convex subset of E and $B \subset E \times E^*$ be a monotone operator satisfying the following:

$$D(B) \subset C \subset J^{-1}\left(\bigcap_{\lambda > 0} R(J + \lambda B)\right).$$

Then the *resolvent* $J_\lambda : C \to D(B)$ of B is defined by

$$J_\lambda x = \{z \in D(B) : Jx \in Jz + \lambda Bz, \forall x \in C\}.$$

J_λ is a single-valued mapping from E to $D(B)$. For any $\lambda > 0$, the *Yosida approximation* $B_\lambda : C \to E^*$ of B is defined by $B_\lambda x = \frac{Jx - JJ_\lambda x}{\lambda}$ for all $x \in C$. We know that $B_\lambda x \in B(J_\lambda x)$ for all $\lambda > 0$ and $x \in E$.

Lemma 2.5 (Kohsaka and Takahashi [22]) *Let E be a smooth strictly convex and reflexive Banach space, let C be a nonempty closed convex subset of E and let $B \subset E \times E^*$ be a monotone operator satisfying $D(B) \subset C \subset J^{-1}(\bigcap_{\lambda > 0} R(J + \lambda B))$. For any $\lambda > 0$, let J_λ and B_λ be the resolvent and the Yosida approximation of B, respectively. Then the following hold:*

(1) $\phi(p, J_\lambda x) + \phi(J_\lambda x, x) \leq \phi(p, x)$ *for all $x \in C$ and $p \in B^{-1}0$;*

(2) $(J_\lambda x, B_\lambda x) \in B$ *for all $x \in C$;*

(3) $F(J_\lambda) = B^{-1}0$.

Lemma 2.6 (Rockafellar [23]) *Let E be a reflexive strictly convex and smooth Banach space. Then an operator $B \subset E \times E^*$ is maximal monotone if and only if $R(J + \lambda B) = E^*$ for all $\lambda > 0$.*

For solving the equilibrium problem for a bifunction $f : C \times C \to \mathbb{R}$, assume that f satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;

(A2) f is monotone, *i.e.*, $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\lim_{t \downarrow 0} f(tz + (1 - t)x, y) \leq f(x, y);$$

(A4) for each $x \in C$, $y \mapsto f(x, y)$ is convex and lower semi-continuous.

The following result is given in Blum and Oettli [6].

Lemma 2.7 *Let C be a closed convex subset of a smooth strictly convex and reflexive Banach space E and let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying the conditions* (A1)-

(A4). *Then, for any $r > 0$ and $x \in E$, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.8 (Takahashi and Zembayashi [24]) *Let C be a closed convex subset of a uniformly smooth strictly convex and reflexive Banach space E and let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying the conditions* (A1)-(A4). *For any $r > 0$ and $x \in E$, define a mapping $T_r : E \to C$ as follows:*

$$T_r x = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0, \forall y \in C \right\}, \quad \forall x \in C.$$

Then the following hold:
   (1) *T_r is single-valued*;
   (2) *T_r is a firmly nonexpansive-type mapping for all $x, y \in E$, that is,*

$$\langle T_r x - T_r y, JT_r x - JT_r y \rangle \leq \langle T_r x - T_r y, Jx - Jy \rangle;$$

   (3) *$F(T_r) = \mathrm{EP}(f)$*;
   (4) *$\mathrm{EP}(f)$ is closed and convex.*

Lemma 2.9 (Takahashi and Zembayashi [24]) *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E, f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying the conditions* (A1)-(A4) *and let $r > 0$. Then, for any $x \in E$ and $p \in F(T_r)$,*

$$\phi(p, T_r x) + \phi(T_r x, x) \leq \phi(p, x).$$

Lemma 2.10 *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. Let $A : C \to E^*$ be a continuous monotone mapping and f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying the conditions* (A1)-(A4). *Then, for any $r > 0$ and $x \in E$, there exists $z \in C$ such that*

$$f(z, y) + \langle y - z, Az \rangle + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C.$$

Proof Define a bifunction $\Theta : C \times C \to \mathbb{R}$ by $\Theta(x, y) = f(x, y) + \langle y - x, Ax \rangle$ for all $x, y \in C$. We show that Θ satisfies the conditions (A1)-(A4).

First, we show that Θ satisfies the condition (A1). Since

$$\Theta(x, x) = f(x, x) + \langle x - x, Ax \rangle = 0, \quad \forall x \in C,$$

the condition (A2) is satisfied.

Next, we show that Θ satisfies the condition (A2). Since A is a continuous monotone mapping and f satisfies the condition (A2), for any $x, y \in C$, we have

$$\Theta(x, y) + \Theta(y, x) = f(x, y) + f(y, x) + \langle y - x, Ax \rangle + \langle x - y, Ay \rangle$$

$$\leq 0 + \langle y - x, Ax - Ay \rangle \leq 0.$$

So, the condition (A2) is satisfied.

Thirdly, we show that Θ satisfies the condition (A3). Since f satisfies the condition (A3) and A is a continuous monotone mapping, for any $x, y, z \in C$, we have

$$
\limsup_{t \downarrow 0} \Theta\big(x + t(z - x), y\big)
$$

$$
= \limsup_{t \downarrow 0} f\big(x + t(z - x), y\big) + \big\langle y - \big(x + t(z - x)\big), A\big(x + t(z - x)\big)\big\rangle
$$

$$
\leq \limsup_{t \downarrow 0} f\big(x + t(z - x), y\big) + \limsup_{t \downarrow 0}\big\langle y - \big(x + t(z - x)\big), A\big(x + t(z - x)\big)\big\rangle
$$

$$
\leq f(x, y) + \limsup_{t \downarrow 0}\big\langle y - \big(x + t(z - x)\big), A\big(x + t(z - x)\big)\big\rangle
$$

$$
= f(x, y) + \langle y - x, Ax \rangle
$$

$$
= \Theta(x, y).
$$

The condition (A3) is satisfied.

Finally, we show that Θ satisfies the condition (A4) since $y \mapsto \langle y - x, Ax \rangle$ is convex and continuous; that is, $y \mapsto \langle y - x, Ax \rangle$ is convex and lower semi-continuous. Since $y \mapsto f(x, y)$ is convex and lower semi-continuous, $y \mapsto \Theta(x, y)$ is convex and lower semi-continuous.

Therefore, $\Theta(x, y)$ satisfies the conditions (A1)-(A4). From Lemma 2.7, we can conclude that there exists $z \in C$ such that

$$
f(z, y) + \langle y - z, Az \rangle + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C.
$$

This completes the proof. $\qquad \square$

Lemma 2.11 *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. Let $A : C \to E^*$ be a continuous and monotone mapping and f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying the conditions* (A1)-(A4). *Then, for any $r > 0$ and $x \in E$, there exists $z \in C$ such that*

$$
f(z, y) + \langle y - z, Az \rangle + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C.
$$

Define a mapping $K_r : C \to C$ as follows:

$$
K_r(x) = \left\{ z \in C : f(z, y) + \langle y - z, Az \rangle + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0, \forall y \in C \right\}, \quad \forall x \in C. \tag{2.4}
$$

Then we have the following:

(1) K_r *is single-valued*;

(2) K_r *is firmly nonexpansive, i.e., for all $x, y \in E$,*

$$
\langle K_r x - K_r y, JK_r x - JK_r y \rangle \leq \langle K_r x - K_r y, Jx - Jy \rangle;
$$

(3) $F(K_r) = \mathrm{GEP}(f, A)$;

(4) $\mathrm{GEP}(f, A)$ *is closed and convex*;

(5) $\phi(p, K_r z) + \phi(K_r z, z) \leq \phi(p, z)$ *for all $p \in F(K_r)$ and $z \in E$.*

Proof Define a bifunction $\Theta : C \times C \to \mathbb{R}$ by $\Theta(u, y) = f(u, y) + \langle y - u, Au \rangle$ for all $u, y \in C$. From Lemma 2.8, it follows that Θ satisfies the conditions (A1)-(A4). Now, we can rewrite the mapping $K_r : C \to C$ given in (2.4) as follows:

$$K_r(x) = \left\{ z \in C : \Theta(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \forall y \in C \right\}, \quad \forall x \in C. \tag{2.5}$$

Thus, from Lemmas 2.8 and 2.9, we obtain the conclusion. This completes the proof. □

Throughout this paper, we define a mapping $K_{r_i}^{\Theta_i}(x) : C \to C$ by

$$K_{r_i}^{\Theta_i}(x) = \left\{ z \in C : \Theta_i(z, y) + \frac{1}{r_i} \langle y - z, Jz - Jx \rangle \geq 0, \forall y \in C \right\}, \quad \forall x \in C, \tag{2.6}$$

where $\Theta_i(z, y) = f_i(z, y) + \langle y - z, A_i z \rangle$ for all $z, y \in C$ and $i = 1, 2, 3, \ldots, m$.

3 Main results

Theorem 3.1 *Let C be a nonempty closed and convex subset of a uniformly smooth and strictly convex Banach space E with the Kadec-Klee property. For any $i = 1, 2, 3, \ldots, m$, let f_i be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying the conditions* (A1)-(A4) *and let $\{A_i\}$ be a finite family of continuous and monotone mappings from C to E^*. Let $B_j \subset E \times E^*$ be maximal monotone operators satisfying $D(B_j) \subset C$ and $J_{\lambda_{j,n}}^{B_j} = (J + \lambda_{j,n} B_j)^{-1} J$ for all $\lambda_{j,n} > 0$ and $j = 1, 2, \ldots, l$. Assume that $F := (\bigcap_{i=1}^m \mathrm{GEP}(f_i, A_i)) \cap (\bigcap_{j=1}^l B_j^{-1} 0) \neq \emptyset$. For arbitrary $x_1 \in C$ and $C_1 = C$, generate a sequence $\{x_n\}$ by*

$$\begin{cases} z_n = J_{\lambda_{l,n}}^{B_l} \circ J_{\lambda_{l-1,n}}^{B_{l-1}} \circ \cdots \circ J_{\lambda_{1,n}}^{B_1} x_n, \\ u_n = K_{r_{m,n}}^{\Theta_m} \circ K_{r_{m-1,n}}^{\Theta_{m-1}} \circ \cdots \circ K_{r_{1,n}}^{\Theta_1} z_n, \\ C_{n+1} = \{ z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) \}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases} \tag{3.1}$$

where $\{r_{i,n}\} \subset [a, \infty)$ for some $a > 0$ for all $i = 1, 2, \ldots, m$ and $\liminf_{n \to \infty} \lambda_{j,n} > 0$ for all $j = 1, 2, \ldots, l$. Then the sequence $\{x_n\}$ converges strongly to a point $p \in F$, where $p = \Pi_F x_1$.

Proof We split the proof into five steps as follows.

Step 1. We first show that C_{n+1} is closed and convex for all $n \geq 1$. Clearly, $C_1 = C$ is closed and convex. Suppose that C_n is closed and convex for all $n \geq 1$. Since, for any $z \in C_n$, we know that $\phi(z, u_n) \leq \phi(z, x_n)$ is equivalent to the following:

$$2 \langle z, Jx_n - Ju_n \rangle \leq \|x_n\|^2 - \|u_n\|^2.$$

Thus C_{n+1} is closed and convex for all $n \geq 1$.

Step 2. We show that $F \subset C_n$ for all $n \geq 1$ and $\{x_n\}$ is well defined. Since $F \subset C_1 = C$, suppose that $F \subset C_n$ for some $n \geq 1$. Let $q \in F$, from Lemma 2.11 and Lemma 2.5, we have that

$$\begin{aligned} \phi(q, u_n) &= \phi\big(q, K_{r_{m,n}}^{\Theta_m} \circ K_{r_{m-1,n}}^{\Theta_{m-1}} \circ \cdots \circ K_{r_{1,n}}^{\Theta_1} z_n\big) \\ &\leq \phi\big(q, K_{r_{m-1,n}}^{\Theta_{m-1}} \circ \cdots \circ K_{r_{1,n}}^{\Theta_1} z_n\big) \\ &\qquad \cdots \end{aligned}$$

$$\leq \phi\big(q, K_{r_{1,n}}^{\Theta_1} z_n\big)$$

$$\leq \phi(q, z_n)$$

$$= \phi\big(q, J_{\lambda_{l,n}}^l \circ J_{\lambda_{l-1,n}}^{l-1} \circ \cdots \circ J_{\lambda_{1,n}}^1 x_n\big)$$

$$\leq \phi\big(q, J_{\lambda_{l-1,n}}^{l-1} \circ \cdots \circ J_{\lambda_{1,n}}^1 x_n\big)$$

$$\cdots$$

$$\leq \phi\big(q, J_{\lambda_{1,n}}^1 x_n\big)$$

$$\leq \phi(q, x_n). \tag{3.2}$$

This shows that $q \in C_{n+1}$, which implies that $F \subset C_{n+1}$. Hence $F \subset C_n$ for all $n \geq 1$. This implies that the sequence $\{x_n\}$ is well defined.

Step 3. We show that $\lim_{n\to\infty} \|u_n - x_n\| = 0$ and $\lim_{n\to\infty} \|Ju_n - Jx_n\| = 0$. By the definition of C_{n+1} with $x_n = \Pi_{C_n} x_1$ and $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$, it follows that

$$\phi(x_n, x_1) \leq \phi(x_{n+1}, x_1), \quad \forall n \geq 1, \tag{3.3}$$

that is, $\{\phi(x_n, x_1)\}$ is nondecreasing. By Lemma 2.4, we get

$$\phi(x_n, x_1) = \phi(\Pi_{C_n} x_1, x_1)$$

$$\leq \phi(q, x_1) - \phi(q, x_n)$$

$$\leq \phi(q, x_1), \quad \forall q \in F. \tag{3.4}$$

This implies that $\{\phi(x_n, x_1)\}$ is bounded and so $\lim_{n\to\infty} \phi(x_n, x_1)$ exists. In particular, by (1.4), the sequence $\{(\|x_n\| - \|x_1\|)^2\}$ is bounded. This implies $\{x_n\}$ is also bounded. So, $\{z_n\}$ and $\{u_n\}$ are also bounded. Since E is reflexive and C_n is closed and convex, without loss of generality, we may assume that there exists $p \in C_n$ such that $x_n \rightharpoonup p$.

Since $x_n = \Pi_{C_n} x_1$, we have

$$\phi(x_n, x_1) \leq \phi(p, x_1), \quad \forall p \in C_n.$$

On the other hand, since

$$\liminf_{n\to\infty} \phi(x_n, x_1) = \liminf_{n\to\infty}\big\{\|x_n\|^2 - 2\langle x_n, Jx_1\rangle + \|x_1\|^2\big\}$$

$$\geq \|p\|^2 - 2\langle p, Jx_1\rangle + \|x_1\|^2$$

$$= \phi(p, x_1),$$

it follows that

$$\phi(p, x_1) \leq \liminf_{n\to\infty} \phi(x_n, x_1) \leq \limsup_{n\to\infty} \phi(x_n, x_1) \leq \phi(p, x_1).$$

This implies that $\lim_{n\to\infty} \phi(x_n, x_1) = \phi(p, x_1)$. Hence we get $\|x_n\| \to \|p\|$ as $n \to \infty$. In view of the Kadec-Klee property of E, we obtain

$$\lim_{n\to\infty} x_n = p. \tag{3.5}$$

Now, we claim that $\|Ju_n - Jx_n\| \to 0$ as $n \to \infty$. By the definition of $\Pi_{C_n} x_1$, it follows that

$$
\begin{aligned}
\phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{C_n} x_1) \\
&\leq \phi(x_{n+1}, x_1) - \phi(\Pi_{C_n} x_1, x_1) \\
&= \phi(x_{n+1}, x_1) - \phi(x_n, x_1).
\end{aligned}
$$

Since $\lim_{n \to \infty} \phi(x_n, x_1)$ exists, we obtain

$$
\lim_{n \to \infty} \phi(x_{n+1}, x_n) = 0. \tag{3.6}
$$

Since $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$ and the definition of C_{n+1}, we have $\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n)$. By (3.6) we also have

$$
\lim_{n \to \infty} \phi(x_{n+1}, u_n) = 0. \tag{3.7}
$$

From (2.2) it follows that

$$
\|u_n\| \to \|p\| \quad (n \to \infty). \tag{3.8}
$$

Since J is uniformly norm-to-norm continuous on bounded subsets of E, it follows that

$$
\|Ju_n\| \to \|Jp\| \quad (n \to \infty). \tag{3.9}
$$

This implies that $\{\|Ju_n\|\}$ is bounded in E^*. Note that E is reflexive and E^* is also reflexive, we can assume that $Ju_n \rightharpoonup x^* \in E^*$. Since E is reflexive, we see that $J(E) = E^*$. Hence there exists $x \in E$ such that $Jx = x^*$, and we have

$$
\begin{aligned}
\phi(x_{n+1}, u_n) &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Ju_n \rangle + \|u_n\|^2 \\
&= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Ju_n \rangle + \|Ju_n\|^2.
\end{aligned}
$$

Taking $\liminf_{n \to \infty}$ on both sides of the equation above, in view of the weak lower semi-continuity of the norm $\|\cdot\|$, it follows that

$$
\begin{aligned}
0 &\geq \|p\|^2 - 2\langle p, x^* \rangle + \|x^*\|^2 \\
&= \|p\|^2 - 2\langle p, Jx \rangle + \|Jx\|^2 \\
&= \|p\|^2 - 2\langle p, Jx \rangle + \|x\|^2 \\
&= \phi(p, x).
\end{aligned}
$$

From Remark 2.1 we have $p = x$, which implies that $x^* = Jp$, and so $Ju_n \rightharpoonup Jp \in E^*$. From the Kadec-Klee property, we have that

$$
Ju_n \rightharpoonup Jp \quad \text{as } n \to \infty.
$$

Note that $J^{-1} : E^* \to E$ is norm-weak*-continuous, that is,

$$u_n \rightharpoonup p \quad \text{as } n \to \infty. \tag{3.10}$$

From (3.8), (3.10) and the Kadec-Klee property of E, it follows that

$$\lim_{n \to \infty} u_n = p. \tag{3.11}$$

Since $\|x_n - u_n\| \le \|x_n - p\| + \|p - u_n\|$, it follows that

$$\lim_{n \to \infty} \|x_n - u_n\| = 0. \tag{3.12}$$

Since J is uniformly norm-to-norm continuous on bounded subsets of E, we obtain

$$\lim_{n \to \infty} \|Ju_n - Jx_n\| = 0. \tag{3.13}$$

Step 4. We show that $p \in F := (\bigcap_{i=1}^{m} \text{GEP}(f_i, A_i)) \cap (\bigcap_{j=1}^{l} B_j^{-1} 0)$. First, we show that $p \in \bigcap_{i=1}^{m} \text{GEP}(f_i, A_i)$. From (3.2), (3.5) and (3.11), it follows that for any $q \in F$,

$$\lim_{n \to \infty} \phi(q, z_n) = \phi(q, p). \tag{3.14}$$

Denote $\Delta_n^j := J_{\lambda_{j,n}}^{B_j} \circ J_{\lambda_{j-1,n}}^{B_{j-1}} \circ \cdots \circ J_{\lambda_{1,n}}^{B_1} x_n$ for each $j = 1, 2, \ldots, l$ and $\Delta_n^0 = I$. We have that $z_n = \Delta_n^l x_n$ for all $n \ge 1$. From Lemma 2.5(1), it follows that

$$\begin{aligned}
\phi(z_n, x_n) &= \phi\big(\Delta_n^l x_n, x_n\big) \\
&\le \phi(q, x_n) - \phi\big(q, \Delta_n^l x_n\big) \\
&= \phi(q, x_n) - \phi(q, z_n).
\end{aligned}$$

Taking limit as $n \to \infty$ on both sides of the inequality, we have

$$\lim_{n \to \infty} \phi(z_n, x_n) = 0.$$

From (2.2) it follows that $(\|x_n\| - \|z_n\|)^2 \to 0$. Since $\|x_n\| \to \|p\|$, we have

$$\|z_n\| \to \|p\| \quad (n \to \infty). \tag{3.15}$$

Since J is uniformly norm-to-norm continuous on bounded subsets of E, it follows that

$$\|Jz_n\| \to \|Jp\| \quad (n \to \infty). \tag{3.16}$$

This implies that $\{\|Jz_n\|\}$ is bounded in E^* and E^* is reflexive, we can assume that $Jz_n \rightharpoonup z^* \in E^*$. In view of $J(E) = E^*$, there exists $z \in E$ such that $Jz = z^*$, and so

$$\begin{aligned}
\phi(x_n, z_n) &= \|x_n\|^2 - 2\langle x_n, Jz_n \rangle + \|z_n\|^2 \\
&= \|x_n\|^2 - 2\langle x_n, Jz_n \rangle + \|Jz_n\|^2.
\end{aligned}$$

Taking $\liminf_{n \to \infty}$ on both sides of the equality above and in view of the weak lower semi-continuity of the norm $\| \cdot \|$, it follows that

$$
\begin{aligned}
0 &\geq \|p\|^2 - 2\langle p, z^* \rangle + \|z^*\|^2 \\
&= \|p\|^2 - 2\langle p, Jz \rangle + \|Jz\|^2 \\
&= \|p\|^2 - 2\langle p, Jz \rangle + \|z\|^2 \\
&= \phi(p, z).
\end{aligned}
$$

From Remark 2.1, we have $p = z$, which implies that $z^* = Jp$ and so $Jz_n \rightharpoonup Jp \in E^*$. From (3.16) and the Kadec-Klee property of E^*, we have $Jz_n \to Jp$ as $n \to \infty$. Note that J^{-1} is norm-weak*-continuous, that is, $z_n \rightharpoonup p$. From (3.15) and the Kadec-Klee property of E, we have

$$
\lim_{n \to \infty} z_n = p. \tag{3.17}
$$

For any $q \in F$, we note that

$$
\begin{aligned}
\phi(q, x_n) - \phi(q, u_n) &= \|x_n\|^2 - \|u_n\|^2 - 2\langle q, Jx_n - Ju_n \rangle \\
&\leq \|x_n - u_n\|\big(\|x_n\| + \|u_n\|\big) + 2\|q\|\|Jx_n - Ju_n\|.
\end{aligned}
$$

Thus it follows from $\|x_n - u_n\| \to 0$ and $\|Jx_n - Ju_n\| \to 0$ that

$$
\phi(q, x_n) - \phi(q, u_n) \to 0 \quad (n \to \infty). \tag{3.18}
$$

Denote $\Phi_n^i := K_{r_{i,n}}^{\Theta_i} \circ K_{r_{i-1,n}}^{\Theta_{i-1}} \circ \cdots \circ K_{r_1,n}^{\Theta_1}$ for each $i = 1, 2, \ldots, m$ and $\Phi_n^0 = I$. We can rewrite u_n as $u_n = \Phi_n^m z_n$. It follows that for each $i = 1, 2, \ldots, m$, we have

$$
\begin{aligned}
\phi(q, u_n) &= \phi\big(q, \Phi_n^m z_n\big) \\
&\leq \phi\big(q, \Phi_n^{m-1} z_n\big) \\
&\leq \phi\big(q, \Phi_n^{m-2} z_n\big) \\
&\quad \cdots \\
&\leq \phi\big(q, \Phi_n^i z_n\big).
\end{aligned} \tag{3.19}
$$

From Lemma 2.11(5), for each $i = 1, 2, \ldots, m$, we have

$$
\begin{aligned}
\phi\big(\Phi_n^i z_n, z_n\big) &\leq \phi(q, z_n) - \phi\big(q, \Phi_n^i z_n\big) \\
&\leq \phi(q, x_n) - \phi\big(q, \Phi_n^i z_n\big) \\
&\leq \phi(q, x_n) - \phi(q, u_n).
\end{aligned} \tag{3.20}
$$

From (3.18) it follows that $\phi(\Phi_n^i z_n, z_n) \to 0$ as $n \to \infty$ for each $i = 1, 2, \ldots, m$, and so from (2.2), that

$$
\big(\|\Phi_n^i z_n\| - \|z_n\|\big)^2 \to 0.
$$

Since $\|z_n\| \to \|p\|$, we also have, for each $i = 1, 2, \dots, m$,

$$\left\| \Phi_n^i z_n \right\| \to \|p\| \quad (n \to \infty). \tag{3.21}$$

Since $\{\Phi_n^i z_n\}$ is bounded for each $i = 1, 2, \dots, m$ and E is reflexive, without loss of generality, we may assume that $\Phi_n^i z_n \rightharpoonup h$ for all $i = 1, 2, \dots, m$. From the first step, since C_n is closed and convex for each $n \geq 1$, it is obvious that $h \in C_n$. Again, since

$$\phi\left(\Phi_n^i z_n, z_n\right) = \left\| \Phi_n^i z_n \right\|^2 - 2\left\langle \Phi_n^i z_n, J z_n \right\rangle + \|z_n\|^2,$$

taking $\liminf_{n\to\infty}$ on both sides of the equality above, we have

$$0 \geq \|h\|^2 - 2\langle h, Jp \rangle + \|p\|^2 = \phi(h, p).$$

This implies that $h = p$ for each $i = 1, 2, \dots, m$, and so

$$\Phi_n^i z_n \rightharpoonup p. \tag{3.22}$$

From (3.21), (3.22) and the Kadec-Klee property, for each $i = 1, 2, \dots, m$, we have

$$\lim_{n\to\infty} \Phi_n^i z_n = p. \tag{3.23}$$

By using the triangle inequality, for each $i = 1, 2, \dots, m$, we obtain

$$\left\| \Phi_n^i z_n - \Phi_n^{i-1} z_n \right\| \leq \left\| \Phi_n^i z_n - p \right\| + \left\| p - \Phi_n^{i-1} z_n \right\|.$$

Hence, for each $i = 1, 2, \dots, m$, we have

$$\lim_{n\to\infty} \left\| \Phi_n^i z_n - \Phi_n^{i-1} z_n \right\| = 0. \tag{3.24}$$

Since $\{r_{i,n}\} \subset [a, \infty)$ and J is uniformly norm-to-norm continuous on bounded subsets of E, for each $i = 1, 2, 3, \dots, m$, we have

$$\lim_{n\to\infty} \frac{\|J\Phi_n^i z_n - J\Phi_n^{i-1} z_n\|}{r_{i,n}} = 0. \tag{3.25}$$

From Lemma 2.11 we have, for each $i = 1, 2, \dots, m$,

$$\Theta_i\left(\Phi_n^i z_n, y\right) + \frac{1}{r_{i,n}}\left\langle y - \Phi_n^i z_n, J\Phi_n^i z_n - J\Phi_n^{i-1} z_n \right\rangle \geq 0, \quad \forall y \in C,$$

where $\Theta_i(u_n, y) = f_i(u_n, y) + \langle y - u_n, A_i u_n \rangle$ for all $u_n, y \in C$. From the condition (A2), it follows that for each $i = 1, 2, \dots, m$,

$$\frac{1}{r_{i,n}}\left\langle y - \Phi_n^i z_n, J\Phi_n^i z_n - J\Phi_n^{i-1} z_n \right\rangle \geq \Theta_i\left(y, \Phi_n^i z_n\right), \quad \forall y \in C.$$

From (3.23) and (3.25), for each $i = 1, 2, \dots, m$, we have

$$0 \geq \Theta_i(y, p), \quad \forall y \in C. \tag{3.26}$$

For any $t \in [0,1]$ and $y \in C$, let $y_t = ty + (1-t)p$. Then we get $y_t \in C$. From (3.26) it follows that for each $i = 1, 2, \ldots, m$,

$$\Theta_i(y_t, p) \le 0, \quad \forall y_t \in C. \tag{3.27}$$

By the conditions (A1) and (A4), for each $i = 1, 2, \ldots, m$, we have

$$\begin{aligned}
0 &= \Theta_i(y_t, y_t) \\
&\le t\Theta_i(y_t, y) + (1-t)\Theta_i(y_t, p) \\
&\le t\Theta_i(y_t, y) \\
&\le \Theta_i(y_t, y).
\end{aligned} \tag{3.28}$$

From the condition (A3), we get

$$0 \le \Theta_i(y_t, y) = \Theta_i\big(ty + (1-t)p, y\big).$$

Taking $t \to 0$ in the equality above, for each $i = 1, 2, \ldots, m$, we have

$$0 = \lim_{t \to 0} 0 \le \lim_{t \to 0} \Theta_i\big(ty + (1-t)p, y\big) = \Theta_i(p, y), \quad \forall y \in C,$$

that is, $f_i(p, y) + \langle y - p, A_i p \rangle \ge 0$ for all $y \in C$ and $i = 1, 2, \ldots, m$. This implies that $p \in \mathrm{GEP}(f_i, A_i)$ for each $i = 1, 2, \ldots, m$. Therefore, $p \in \bigcap_{i=1}^{m} \mathrm{GEP}(f_i, A_i)$.

Next, we show that $p \in \bigcap_{j=1}^{l} B_j^{-1} 0$. Let $z_n = \Delta_n^l x_n$ for each $n \ge 1$. For any $q \in F$, it follows that for each $j = 1, 2, \ldots, l$,

$$\begin{aligned}
\phi(q, z_n) &= \phi\big(q, \Delta_n^l x_n\big) \\
&\le \phi\big(q, \Delta_n^{l-1} x_n\big) \\
&\le \phi\big(q, \Delta_n^{l-2} x_n\big) \\
&\quad \cdots \\
&\le \phi\big(q, \Delta_n^j x_n\big).
\end{aligned} \tag{3.29}$$

By Lemma 2.5 we have, for $j = 1, 2, 3, \ldots, m$,

$$\begin{aligned}
\phi\big(\Delta_n^j x_n, x_n\big) &\le \phi(q, x_n) - \phi\big(q, \Delta_n^j x_n\big) \\
&\le \phi(q, x_n) - \phi(q, z_n).
\end{aligned} \tag{3.30}$$

Since $x_n \to p$ and $z_n \to p$ as $n \to \infty$, we get $\phi(\Delta_n^j x_n, x_n) \to 0$ as $n \to \infty$ for $j = 1, 2, 3, \ldots, m$. From (2.2) it follows that

$$\big(\big\| \Delta_n^j x_n \big\| - \|x_n\|\big)^2 \to 0.$$

Since $\|x_n\| \to \|p\|$, we also have

$$\big\| \Delta_n^j x_n \big\| \to \|p\| \quad \text{as } n \to \infty, \forall j = 1, 2, 3, \ldots, m. \tag{3.31}$$

This implies that for each $j = 1, 2, 3, \ldots, m$, $\{\Delta_n^j x_n\}$ is bounded and E is reflexive, without loss of generality, we assume that $\Delta_n^j x_n \rightharpoonup k$. We know that C_n is closed and convex for each $n \geq 1$, it is obvious that $k \in C_n$. Again, since

$$\phi\left(\Delta_n^j x_n, x_n\right) = \left\|\Delta_n^j x_n\right\|^2 - 2\left\langle\Delta_n^j x_n, Jx_n\right\rangle + \|x_n\|^2,$$

taking $\liminf_{n\to\infty}$ on both sides of equality above, we have

$$\begin{aligned}
0 &\geq \|k\|^2 - 2\langle k, Jp\rangle + \|p\|^2 \\
&= \phi(k, p).
\end{aligned}$$

That is, $k = p$, $\forall j = 1, 2, 3, \ldots, l$, it follows that

$$\Delta_n^j x_n \rightharpoonup p. \tag{3.32}$$

From (3.31), (3.32) and the Kadec-Klee property, it follows that

$$\lim_{n\to\infty} \Delta_n^j x_n = p, \quad \forall j = 1, 2, 3, \ldots, m. \tag{3.33}$$

We also have

$$\lim_{n\to\infty} \Delta_n^{j-1} x_n = p, \quad \forall j = 1, 2, 3, \ldots, m. \tag{3.34}$$

It follows that

$$\lim_{n\to\infty} \left\|\Delta_n^j x_n - \Delta_n^{j-1} x_n\right\| = 0, \quad \forall j = 1, 2, 3, \ldots, m. \tag{3.35}$$

Since J is uniformly norm-to-norm continuous on bounded subsets of E and

$$\liminf_{n\to\infty} \lambda_{j,n} > 0$$

for each $j = 1, 2, \ldots, l$, we have

$$\lim_{n\to\infty} \frac{1}{\lambda_{j,n}} \left\|J\Delta_n^j x_n - J\Delta_n^{j-1} x_n\right\| = 0.$$

Let $\Delta_n^j x_n = J_{\lambda_{j,n}}^j \Delta_n^{j-1} x_n$ for each $j = 1, 2, \ldots, l$. Then we have

$$\lim_{n\to\infty} \left\|B_{\lambda_{j,n}} \Delta_n^{j-1} x_n\right\| = \lim_{n\to\infty} \frac{1}{\lambda_{j,n}} \left\|J\Delta_n^j x_n - J\Delta_n^{j-1} x_n\right\| = 0.$$

For any $(w, w^*) \in G(B_j)$ and $(\Delta_n^j x_n, A_{\lambda_{j,n}} \Delta_n^{j-1} x_n) \in G(B_j)$ for each $j = 1, 2, \ldots, l$, it follows from the monotonicity of B_j that for all $n \geq 0$,

$$\left\langle w - \Delta_n^j x_n, w^* - B_{\lambda_{j,n}} \Delta_n^{j-1} x_n\right\rangle \geq 0.$$

Letting $n \to \infty$ in the inequality above, we get $\langle w - p, w^* \rangle \geq 0$. Since B_j is maximal monotone for each $j = 1, 2, \ldots, l$, we obtain $p \in \bigcap_{j=1}^{l} B_j^{-1} 0$.

Step 5. We show that $p = \Pi_F x_1$. From $x_n = \Pi_{C_n} x_1$, we have $\langle Jx_1 - Jx_n, x_n - z \rangle \geq 0$ for all $z \in C_n$. Since $F \subset C_n$, we also have

$$\langle Jx_1 - Jx_n, x_n - y \rangle \geq 0, \quad \forall y \in F,$$

and so, taking limit $n \to \infty$, we get

$$\langle Jx_1 - Jp, p - y \rangle \geq 0, \quad \forall y \in F.$$

Therefore, by Lemma 2.3, we can conclude that $p = \Pi_F x_1$ and $x_n \to p$ as $n \to \infty$. The proof is completed. □

If $i = 1$ and $j = 1$, we have the following.

Corollary 3.2 *Let C be a nonempty closed and convex subset of a uniformly smooth and strictly convex Banach space E with the Kadec-Klee property. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying the conditions* (A1)-(A4) *and let $A : C \to E^*$ be a continuous and monotone mapping. Let $B \subset E \times E^*$ be a maximal monotone operator satisfying $D(B) \subset C$ and $J_{\lambda_n} = (J + \lambda_n B)^{-1} J$ for all $\lambda_n > 0$. Assume that $F := \text{GEP}(f, A) \cap B^{-1} 0 \neq \emptyset$. For arbitrary $x_1 \in C$ and $C_1 = C$, generate a sequence $\{x_n\}$ by*

$$\begin{cases} z_n = J_{\lambda_n} x_n, \\ u_n = K_{r_n} z_n, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases} \tag{3.36}$$

where $\{r_n\} \subset [a, \infty)$ for some $a > 0$ and $\liminf_{n \to \infty} \lambda_n > 0$. Then the sequence $\{x_n\}$ converges strongly to a point $p \in F$, where $p = \Pi_F x_1$.

4 Applications

In this section, we apply our result to find a common solution of the variational inequality problems and zeros of the maximal operators.

We need the following lemma for our result, which is a special case of Lemmas 2.8 and 2.9 of [24].

Lemma 4.1 (Zegeye and Shahzad [16]) *Let C be a closed convex subset of a uniformly smooth, strictly convex real Banach space E. Let $A : C \to E^*$ be a continuous monotone mapping. For any $r > 0$ and $x \in E$, define a mapping $T_r : E \to C$ as follows:*

$$T_r x = \left\{ z \in C : \langle y - z, Az \rangle + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \forall y \in C \right\}, \quad \forall x \in E.$$

Then we have the following:

  (1) *T_r is single-valued;*

(2) T_r *is a firmly nonexpansive-type mapping, i.e., for any* $x, y \in E$,

$$\langle T_r x - T_r y, J T_r x - J T_r y \rangle \leq \langle T_r x - T_r y, Jx - Jy \rangle;$$

(3) $F(T_r) = \mathrm{VI}(C, A)$;

(4) $\mathrm{VI}(C, A)$ *is closed and convex*;

(5) $\phi(p, T_r x) + \phi(T_r x, x) \leq \phi(p, x)$ *for any* $p \in F(T_r)$.

Theorem 4.2 *Let C be a nonempty closed and convex subset of a uniformly smooth and strictly convex Banach space E with the Kadec-Klee property. For each* $i = 1, 2, \dots, m$, *let* $\{A_i\}$ *be a finite family of continuous and monotone mappings* $C \to E^*$. *For* $r_n > 0$ *and* $x \in E$, *define a mapping* $T_{r_{i,n}} : E \to C$ *by*

$$T_{r_{i,n}} x := \left\{ z \in C : \langle y - z, A_i z \rangle + \frac{1}{r_n} \langle y - z, Jz - Jx \rangle \geq 0, \forall y \in C \right\}.$$

Let $B \subset E \times E^*$ *be a maximal monotone operator satisfying* $D(B) \subset C$ *and* $J_{\lambda_{j,n}}^{B_j} = (J + \lambda_{j,n} B_j)^{-1} J$ *for all* $\lambda > 0$ *and* $j = 1, 2, \dots, l$. *Assume that* $F := (\bigcap_{i=1}^{m} \mathrm{VI}(C, A_i)) \cap (\bigcap_{j=1}^{l} B_j^{-1} 0) \neq \emptyset$. *For an initial point* $x_1 \in E$ *with* $C_1 = C$, *define the sequence* $\{x_n\}$ *in C as follows*:

$$
\begin{cases}
z_n = J_{\lambda_{l,n}}^{B_l} \circ J_{\lambda_{l-1,n}}^{B_{l-1}} \circ \cdots \circ J_{\lambda_{1,n}}^{B_1} x_n, \\
u_n = T_{r_{m,n}} \circ T_{r_{m-1,n}} \circ \cdots \circ T_{r_{1,n}} z_n, \\
C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\
x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1,
\end{cases}
\tag{4.1}
$$

where $\{r_{i,n}\} \subset [a, \infty)$ *for some* $a > 0$ *for all* $i = 1, 2, \dots, m$ *and* $\liminf_{n \to \infty} \lambda_{j,n} > 0$ *for all* $j = 1, 2, \dots, l$. *Then the sequence* $\{x_n\}$ *converges strongly to a point* $p \in F$, *where* $p = \Pi_F x_1$.

Proof Taking $f_i(u_n, y) = 0$ for all $i = 1, 2, \dots, m$ in Theorem 3.1, we can get the desired conclusion. □

By Theorem 4.2, if we set $B_j \equiv 0$ for each $j = 1, 2, \dots, l$, we obtain the following.

Corollary 4.3 *Let C be a nonempty closed and convex subset of a uniformly smooth and strictly convex Banach space E with the Kadec-Klee property. For any* $i = 1, 2, \dots, m$, *let* $\{A_i\}$ *be a finite family of continuous and monotone mappings* $C \to E^*$. *Assume that* $F := \bigcap_{i=1}^{m} \mathrm{VI}(C, A_i) \neq \emptyset$. *For an initial point* $x_1 \in E$ *with* $C_1 = C$, *define the sequence* $\{x_n\}$ *in C as follows*:

$$
\begin{cases}
u_n = T_{r_{m,n}} \circ T_{r_{m-1,n}} \circ \cdots \circ T_{r_{1,n}} x_n, \\
C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\
x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1,
\end{cases}
\tag{4.2}
$$

where $\{r_{i,n}\} \subset [a, \infty)$ *for some* $a > 0$ *for each* $i = 1, 2, \dots, m$. *Then the sequence* $\{x_n\}$ *converges strongly to a point* $p \in F$, *where* $p = \Pi_F x_1$.

Remark 4.4 Corollary 4.3 extends and improves the result of Zegeye and Shahzad [16] to a common solution of the variational inequality problems.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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