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The convergence of implicit Mann and Ishikawa iterations for weak generalized φ-hemicontractive mappings in real Banach spaces

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Abstract
Let E be a real Banach space and let D be a nonempty closed convex subset of E, let $T: D \to D$ be a continuous weak generalized φ-hemicontractive mapping. The existence theorem of a fixed point of a weak generalized φ-pseudocontractive mapping is obtained. And we also prove that implicit Mann and Ishikawa iterations converge strongly to the unique fixed point of T. Our results extend the corresponding results of Xiang (Nonlinear Anal. 70(6): 2277-2279, 2009).
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1 Introduction

Throughout the paper we assume that E is an arbitrary real Banach space and E^* is its dual space. Let D be a nonempty closed convex subset of E and let $F(T) = \{x \in D : Tx = x\} \neq \emptyset$ be a fixed point set of T. We denote that the normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}, \quad \forall x \in E,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. We denote the single-valued normalized duality mapping by j.

Definition 1.1 [1] Let $T : D \to D$ be a mapping.

T is said to be strongly pseudocontractive if there exists a constant $k \in (0,1)$ such that for any $x, y \in D$, there exists $j(x-y) \in J(x-y)$ satisfying

$$\langle Tx - Ty, j(x-y) \rangle \leq k\|x-y\|^2. \tag{1.1}$$

T is called ϕ-strongly pseudocontractive if there exists a strictly increasing continuous function $\phi : [0, +\infty) \to [0, +\infty)$ with $\phi(0) = 0$ such that for any $x, y \in D$, there exists

$j(x-y) \in J(x-y)$ satisfying

$$\langle Tx - Ty, j(x-y) \rangle \le \|x-y\|^2 - \phi(\|x-y\|)\|x-y\|. \tag{1.2}$$

T is called generalized Φ-pseudocontractive if there exists a strictly increasing continuous function $\Phi : [0,+\infty) \to [0,+\infty)$ with $\Phi(0) = 0$ such that for any $x, y \in D$, there exists $j(x-y) \in J(x-y)$ satisfying

$$\langle Tx - Ty, j(x-y) \rangle \le \|x-y\|^2 - \Phi(\|x-y\|). \tag{1.3}$$

Furthermore, if the inequalities (1.1), (1.2) and (1.3) hold for any $x \in D$ and $y \in F(T)$, then the corresponding mapping T is called strongly hemicontractive, ϕ-strongly hemicontractive and generalized Φ-hemicontractive, respectively. Clearly, the generalized Φ-hemicontractive mappings not only include strongly hemicontractive and ϕ-strongly hemicontractive mappings, but also strongly pseudocontractive, ϕ-strongly pseudocontractive and Φ-pseudocontractive mappings. Thus, the class of generalized Φ-hemicontractive mappings is the most general in the class of above pseudocontractive mappings, *i.e.*, {strongly hemicontractive mappings set} $\subset$ {ϕ-strongly hemicontractive mappings set} $\subset$ {generalized Φ-hemicontractive mappings set}. The converse is not true in general. The counterexamples are as follows. (See [2].)

Example 1.2 Let $E = R$ be a real numbers space with the usual norm and $D = [0,+\infty)$. Define $T : D \to D$ by

$$Tx = \frac{x^2}{1+x}, \quad \forall x \in D.$$

Observe that T has a fixed point $q = 0 \in D$. Define $\phi : [0,+\infty) \to [0,+\infty)$ by $\phi(t) = \frac{t}{1+t}$. And ϕ is a strictly increasing function with $\phi(0) = 0$. Then T is a ϕ-strongly hemicontractive mapping. Indeed, for all $x \in D$, $q \in F(T)$, we have

$$\begin{aligned}
&\langle Tx - Tq, j(x-q) \rangle \\
&\quad = \left\langle \frac{x^2}{1+x} - 0, j(x-0) \right\rangle \\
&\quad = \left\langle \frac{x^2}{1+x}, x \right\rangle \\
&\quad = \frac{x^3}{1+x} \\
&\quad = |x-q|^2 - \frac{|x-q|}{1+|x-q|} \cdot |x-q| \\
&\quad = |x-q|^2 - \phi(|x-q|) \cdot |x-q|.
\end{aligned} \tag{1.4}$$

Hence, T is a ϕ-strongly hemicontractive mapping. But T is not a strongly hemicontractive mapping.

Example 1.3 Let $E = R$ be a real numbers space with the usual norm and $D = [0, +\infty)$. Define $T : D \to D$ by

$$Tx = \frac{x^3}{1 + x^2}, \quad \forall x \in D.$$

Then T has a fixed point $q = 0 \in D$. Define $\Phi : [0, +\infty) \to [0, +\infty)$ by $\Phi(t) = \frac{t^2}{1+t^2}$. Then Φ is a strictly increasing function with $\Phi(0) = 0$. For all $x \in D$, $q \in F(T)$, we obtain that

$$\begin{aligned} &\langle Tx - Tq, j(x - q)\rangle \\ &\quad = \left\langle \frac{x^3}{1 + x^2} - 0, j(x - 0)\right\rangle \\ &\quad = \left\langle \frac{x^3}{1 + x^2}, x\right\rangle \\ &\quad = \frac{x^4}{1 + x^2} \\ &\quad = |x - q|^2 - \frac{|x - q|^2}{1 + |x - q|^2} \\ &\quad = |x - q|^2 - \Phi(|x - q|). \end{aligned} \tag{1.5}$$

Therefore, T is a generalized Φ-hemicontractive mapping. However, T is not a ϕ-strongly hemicontractive mapping. If it is not the case, then there exists a strictly increasing function $\phi : [0, +\infty) \to [0, +\infty)$ with $\phi(0) = 0$ such that

$$\langle Tx - Tq, j(x - q)\rangle \le \|x - q\|^2 - \phi(\|x - q\|)\|x - q\|,$$

i.e., $\phi(x) \le \frac{x}{1+x^2}$ for all $x \in [0, +\infty)$. So, $\lim_{x\to+\infty} \phi(x) = 0$. This is a contradiction with a strictly increasing function ϕ. Hence it holds.

Recently, Xiang [1] discussed the relationship between generalized Φ-pseudocontractive mappings and ϕ-strongly pseudocontractive mappings. The results are as follows.

Theorem 1.4 [1, Proposition 1.1] *Let C be a bounded subset of E and let $T : C \to E$ be a mapping. Then T is generalized Φ-pseudocontractive if and only if T is Ψ-strongly pseudocontractive*.

Theorem 1.5 [1, Proposition 1.2] *Suppose that C is an unbounded subset of E and $T : C \to E$ is a generalized Φ-pseudocontractive mapping. Then T is ϕ-strongly pseudocontractive if and only if there exists a strictly increasing function $\Phi : [0, +\infty) \to [0, +\infty)$ with $\Phi(0) = 0$ such that* (1.1) *holds and* $\lim_{s\to\infty} \inf \frac{\Phi(s)}{s} = \sigma > 0$.

At the same time, Xiang [1] also proved the following existence theorem.

Theorem 1.6 [1, Theorem 2.1] *Let E be real Banach space, let C be a nonempty closed convex subset of E, and let $T : C \to C$ be a continuous generalized Φ-pseudocontractive mapping. Then T has a unique fixed point in C.*

In this paper, we extend the results of Xiang [1] and give the convergence of other iterative methods. For this, we need to introduce the following lemmas.

Lemma 1.7 [3, Corollary 1] *Let D be a nonempty closed convex subset of E, and let $T : D \to D$ be a continuous strongly pseudocontractive mapping. Then T has the unique fixed point in D.*

Lemma 1.8 [4] *Let E be a real Banach space, and let $J : E \to 2^{E^*}$ be a normalized duality mapping. Then*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle$$

for all $x, y \in E$ and each $j(x + y) \in J(x + y)$.

2 Main results

In the sequel, we give the main results.

Definition 2.1 The map $T : D \to D$ is called weak generalized φ-pseudocontractive if there exists a strictly increasing continuous function $\varphi : [0, +\infty) \to [0, +\infty)$ with $\varphi(0) = 0$ such that for any $x, y \in D$, there exists $j(x - y) \in J(x - y)$ satisfying

$$\langle Tx - Ty, j(x - y)\rangle \leq \|x - y\|^2 - \frac{\varphi(\|x - y\|)}{1 + \varphi(\|x - y\|) + \|x - y\|^2}. \tag{2.1}$$

In Definition 2.1, if for any $x \in D$, $y \in F(T)$ such that (2.1) holds, then T is called a weak generalized φ-hemicontractive mapping. (See [5, 6].)

Remark 2.2 If T is generalized φ-hemicontractive, then T must be weak generalized φ-hemicontractive. That is,

$$\begin{aligned}\langle Tx - Tq, j(x - q)\rangle &\leq \|x - q\|^2 - \varphi\bigl(\|x - q\|\bigr) \\ &\leq \|x - q\|^2 - \frac{\varphi(\|x - q\|)}{1 + \varphi(\|x - q\|) + \|x - q\|^2}.\end{aligned}$$

However, the converse is not true in general. See the following example.

Counterexample 2.3 Let $E = R$ be a real numbers space with the usual norm and $D = R^+ = [0, +\infty)$. Define $T : R^+ \to R^+$ by

$$Tx = \begin{cases} \frac{2}{3}x, & x \in [0, 1]; \\ \frac{x + x^3 + x^2\sqrt{x} - \sqrt{x}}{1 + x\sqrt{x} + x^2}, & x \in (1, +\infty). \end{cases}$$

Then T has a fixed point $q = 0 \in R^+$. Set $\Phi : [0, +\infty) \to [0, +\infty)$ by

$$\Phi(t) = \begin{cases} t^4, & t \in [0, 1]; \\ t^{3/2}, & t \in (1, +\infty). \end{cases}$$

Then Φ is a strictly increasing continuous function with $\Phi(0) = 0$. And for any $x \in [0,1]$, $q \in F(T)$, we obtain that

$$\begin{aligned}
&\langle Tx - Tq, j(x-q)\rangle \\
&\quad = \left\langle \frac{2}{3}x - 0, j(x-0)\right\rangle \\
&\quad = \frac{2}{3}x^2 \\
&\quad \le x^2 - \frac{x^4}{1 + x^4 + x^2} \\
&\quad = |x-q|^2 - \frac{|x-q|^4}{1 + |x-q|^4 + |x-q|^2} \\
&\quad = |x-q|^2 - \frac{\Phi(|x-q|)}{1 + \Phi(|x-q|) + |x-q|^2}.
\end{aligned} \tag{2.2}$$

For any $x \in (1, +\infty)$, $q \in F(T)$, we have

$$\begin{aligned}
&\langle Tx - Tq, j(x-q)\rangle \\
&\quad = \left\langle \frac{x + x^3 + x^2\sqrt{x} - \sqrt{x}}{1 + x\sqrt{x} + x^2} - 0, j(x-0)\right\rangle \\
&\quad = \left\langle \frac{x + x^3 + x^2\sqrt{x} - \sqrt{x}}{1 + x\sqrt{x} + x^2}, x\right\rangle \\
&\quad = \frac{x^2 + x^4 + x^3\sqrt{x} - x\sqrt{x}}{1 + x\sqrt{x} + x^2} \\
&\quad = x^2 - \frac{x^{3/2}}{1 + x^{3/2} + x^2} \\
&\quad = |x-q|^2 - \frac{|x-q|^{3/2}}{1 + |x-q|^{3/2} + |x-q|^2} \\
&\quad = |x-q|^2 - \frac{\Phi(|x-q|)}{1 + \Phi(|x-q|) + |x-q|^2}.
\end{aligned} \tag{2.3}$$

Then T is a weak generalized Φ-hemicontractive mapping. But T is not a generalized φ-hemicontractive mapping. Therefore, it has more practical significance to research of the class of mappings in fixed point theory and applications. For this, we firstly give the existence theorem.

Theorem 2.4 *Let E be a real Banach space, let D be a nonempty closed convex subset of E, and let $T : D \to D$ be a continuous weak generalized φ-pseudocontractive mapping. Then T has a unique fixed point in D.*

Proof Similar, using the proof method of Xiang [1].

Step I. Construct the sequence $\{x_n\}$.

For any given $x_0 \in D$, the mapping $S_1 : D \to D$ is defined by $S_1 x = \frac{1}{2}x_0 + \frac{1}{2}Tx$ for all $x \in D$, then S_1 is a continuous strongly pseudocontractive mapping. So, there exists $x_1 \in D$ such that $S_1 x_1 = x_1$, *i.e.*, $x_1 = \frac{1}{2}x_0 + \frac{1}{2}Tx_1$. The mapping $S_2 : D \to D$ is defined by $S_2 x = \frac{1}{2}x_1 + \frac{1}{2}Tx$

for all $x \in D$, then S_2 is a continuous strongly pseudocontractive mapping. So, there exists $x_2 \in D$ such that $S_2 x_2 = x_2$, *i.e.*, $x_2 = \frac{1}{2}x_1 + \frac{1}{2}Tx_2, \ldots$, we obtain the sequence $\{x_n\}$ by $x_{n+1} = \frac{1}{2}x_n + \frac{1}{2}Tx_{n+1}$.

Step II. Show that $\lim_{n\to\infty} \|x_n - x_{n-1}\| = 0$.

From the above sequence $\{x_n\}$, we notice that

$$x_{n+1} = x_n - x_{n+1} + Tx_{n+1}, \qquad x_n = x_{n-1} - x_n + Tx_n.$$

Using the equalities above and Lemma 1.8, we have

$$\begin{aligned} &\|x_{n+1} - x_n\|^2 \\ &\quad = \big\|(x_n - x_{n-1}) - (x_{n+1} - x_n) + (Tx_{n+1} - Tx_n)\big\|^2 \\ &\quad \le \|x_n - x_{n-1}\|^2 - 2\|x_{n+1} - x_n\|^2 \\ &\qquad + 2\left[\|x_{n+1} - x_n\|^2 - \frac{\varphi(\|x_{n+1} - x_n\|)}{1 + \varphi(\|x_{n+1} - x_n\|) + \|x_{n+1} - x_n\|^2}\right] \\ &\quad \le \|x_n - x_{n-1}\|^2 - \frac{\varphi(\|x_{n+1} - x_n\|)}{1 + \varphi(\|x_{n+1} - x_n\|) + \|x_{n+1} - x_n\|^2} \\ &\quad \le \|x_n - x_{n-1}\|^2. \end{aligned} \tag{2.4}$$

Based on the monotone bounded principle, then $\lim_{n\to\infty} \|x_n - x_{n-1}\|$ exists. And

$$\lim_{n\to\infty} \|x_n - x_{n-1}\| = \lim_{n\to\infty} \|Tx_n - x_n\| = A.$$

Denote $M = \sup_n\{\|x_n - x_{n-1}\|\}$. From (2.4), we have

$$\begin{aligned} &\|x_{n+1} - x_n\|^2 \\ &\quad \le \|x_n - x_{n-1}\|^2 - \frac{\varphi(\|x_{n+1} - x_n\|)}{1 + \varphi(\|x_{n+1} - x_n\|) + \|x_{n+1} - x_n\|^2}. \end{aligned} \tag{2.5}$$

Let $\inf_{n\ge 0} \frac{\varphi(\|x_{n+1}-x_n\|)}{1+\varphi(\|x_{n+1}-x_n\|)+\|x_{n+1}-x_n\|^2} = \delta$, then $\delta = 0$. If this is not the case, then $\delta > 0$. We have

$$\frac{\varphi(\|x_{n+1} - x_n\|)}{1 + \varphi(\|x_{n+1} - x_n\|) + \|x_{n+1} - x_n\|^2} \ge \delta$$

for all $n \ge 0$. It follows from (2.5) that

$$\delta \le \|x_n - x_{n-1}\|^2 - \|x_{n+1} - x_n\|^2, \tag{2.6}$$

which implies that $\sum_{n=1}^{\infty} \delta \le \|x_1 - x_0\|^2 < \infty$, which is a contradiction. Then $\delta = 0$. Thus there exists an infinite subsequence $\{\frac{\varphi(\|x_{n_i+1}-x_{n_i}\|)}{1+\varphi(\|x_{n_i+1}-x_{n_i}\|)+\|x_{n_i+1}-x_{n_i}\|^2}\}$ such that

$$\lim_{i\to\infty} \frac{\varphi(\|x_{n_i+1} - x_{n_i}\|)}{1 + \varphi(\|x_{n_i+1} - x_{n_i}\|) + \|x_{n_i+1} - x_{n_i}\|^2} = 0.$$

Since $0 \le \frac{\varphi(\|x_{n_i+1}-x_{n_i}\|)}{1+\varphi(M)+M^2} \le \frac{\varphi(\|x_{n_i+1}-x_{n_i}\|)}{1+\varphi(\|x_{n_i+1}-x_{n_i}\|)+\|x_{n_i+1}-x_{n_i}\|^2}$, then $\lim_{i\to\infty} \varphi(\|x_{n_i+1} - x_{n_i}\|) = 0$. It leads to $\lim_{i\to\infty} \|x_{n_i+1} - x_{n_i}\| = 0$ by the strict increase and continuity of φ. Hence $A = 0$.

Step III. $\{x_n\}$ is a Cauchy sequence.

Since $\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}\|Tx_n - x_n\| = 0$. For $\forall\epsilon \in (0,1)$, $\exists N$ such that

$$\|x_{n+1}-x_n\| < \epsilon, \qquad \|Tx_n - x_n\|, \|Tx_m - x_m\| < \frac{\varphi(\epsilon)}{2[1+\varphi(2\epsilon)+4\epsilon^2](1+2\epsilon)}$$

for all $m,n \ge N$. By the induction method, we prove that $\|x_m - x_n\| < \epsilon$ for all $m,n \ge N$. If $m = n+1$, then $\|x_{n+1}-x_n\| < \epsilon$. Suppose that $\|x_m - x_n\| < \epsilon$ holds for some $m \ge N$, then $\|x_{m+1}-x_n\| \le \|x_{m+1}-x_m\| + \|x_m - x_n\| < 2\epsilon$ (*). Next we want to show that $\|x_{m+1}-x_n\| < \epsilon$. Since T is a weak generalized φ pseudocontractive mapping, then

$$\langle Tx_{m+1} - Tx_n, j(x_{m+1}-x_n)\rangle \le \|x_{m+1}-x_n\|^2 - \frac{\varphi(\|x_{m+1}-x_n\|)}{1+\varphi(\|x_{m+1}-x_n\|)+\|x_{m+1}-x_n\|^2},$$

i.e., $\frac{\varphi(\|x_{m+1}-x_n\|)}{1+\varphi(\|x_{m+1}-x_n\|)+\|x_{m+1}-x_n\|^2} \le \|x_{m+1}-x_n\|^2 - \langle Tx_{m+1}-Tx_n, j(x_{m+1}-x_n)\rangle \le [\|x_{m+1}-Tx_{m+1}\| + \|x_n - Tx_n\|]\cdot\|x_{m+1}-x_n\|$. By the above inequalities, we have

$$\frac{\varphi(\|x_{m+1}-x_n\|)}{1+\varphi(2\epsilon)+4\epsilon^2} < \frac{2\epsilon\varphi(\epsilon)}{[1+\varphi(2\epsilon)+4\epsilon^2](1+2\epsilon)} < \frac{\varphi(\epsilon)}{1+\varphi(2\epsilon)+4\epsilon^2},$$

which implies that $\|x_{m+1}-x_n\| < \epsilon$ by the strict increase of φ. Therefore $\{x_n\}$ is a Cauchy sequence. Since D is closed in Banach space E, then D is complete. Hence, there exists a point $q \in D$ such that $x_n \to q$ as $n \to \infty$. Since T is continuous, then $q = Tq$. The uniqueness is obvious. □

3 Applications of the weak generalized φ-hemicontractive mappings

Now that the weak generalized φ-hemicontractive mappings are much more general mappings. Hence it is of interest to study the convergence of an iteration process of fixed points of the class mappings.

Definition 3.1 Let $T: D \to D$ be a mapping. For any given $u_1 \in D$, define the sequence $\{u_n\}_{n=1}^\infty \subset D$ by the iterative scheme

$$u_{n+1} = (1-a_n)u_n + a_n Tu_n, \quad n \ge 1, \tag{3.1}$$

which is called the Mann iterative process, where $\{a_n\}_{n=1}^\infty$ is a real sequence in $[0,1]$ satisfying certain conditions. Further, we assume that there exists $(I - tT)^{-1}$ for all $t \in (0,1)$. For any given $x_1 \in D$, define the sequence $\{x_n\}_{n=1}^\infty \subset D$ by the iterative scheme [3]

$$x_{n+1} = (1-a_n)x_n + a_n Tx_{n+1}, \quad n \ge 1, \tag{3.2}$$

which is called the implicit Mann iterative process.

Definition 3.2 Let $T: D \to D$ be a mapping. For any given $u_1 \in D$, define the sequence $\{w_n\}_{n=1}^\infty \subset D$ by the iterative scheme

$$\begin{cases} w_1 \in D, \\ v_n = (1-b_n)w_n + b_n Tw_n, \quad n \ge 1, \\ w_{n+1} = (1-a_n)w_n + a_n Tv_n, \quad n \ge 1, \end{cases} \tag{3.3}$$

which is called the Ishikawa iterative process, where $\{a_n\}$ and $\{b_n\}$ are two real sequences in $[0,1]$ satisfying certain conditions. And for any given $z_1 \in D$, define the sequence $\{z_n\}_{n=1}^{\infty} \subset D$ by the iterative scheme

$$\begin{cases} z_1 \in D, \\ y_n = (1-b_n)z_n + b_n T y_n, \quad n \geq 1, \\ z_{n+1} = (1-a_n)y_n + a_n T z_{n+1}, \quad n \geq 1, \end{cases} \tag{3.4}$$

which is called the implicit Ishikawa iterative process. Especially, if $b_n = 0$, then the corresponding iterations (3.3) and (3.4) reduce to (3.1) and (3.2), respectively.

Lemma 3.3 [1] *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three nonnegative real sequences and satisfy*

$$a_{n+1} \leq (1+b_n)a_n + c_n, \quad n \geq 0.$$

If $\sum_{n=0}^{\infty} b_n < \infty$, $\sum_{n=0}^{\infty} c_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

In the following, we study the convergence of implicit Mann and Ishikawa iterative processes for weak generalized φ-hemicontractive mappings in general real Banach spaces.

Theorem 3.4 *Let E be a real Banach space and let D be a nonempty closed convex subset of E, let $T : D \to D$ be a weak generalized φ-hemicontractive mapping. Suppose that $\{x_n\}_{n=1}^{\infty}$ is defined by* (3.2) *with the iteration parameter $\{a_n\}_{n=1}^{\infty} \subset [0, \frac{1}{2})$ satisfying: $a_n \to 0$ as $n \to \infty$; $\sum_{n=1}^{\infty} \frac{a_n}{1-2a_n} = \infty$ and $\sum_{n=1}^{\infty} \frac{a_n^2}{1-2a_n} < \infty$. Then the implicit Mann iteration $\{x_n\}_{n=1}^{\infty}$ converges strongly to the unique fixed point of T.*

Proof Let $q \in F(T)$. Applying Lemma 1.8 and (3.4), we have

$$\begin{aligned} &\|x_{n+1} - q\|^2 \\ &\quad = \bigl\|(1-a_n)(x_n - q) + a_n(Tx_{n+1} - Tq)\bigr\|^2 \\ &\quad \leq (1-a_n)^2\|x_n - q\|^2 + 2a_n\bigl\langle Tx_{n+1} - Tq, j(x_{n+1} - q)\bigr\rangle \\ &\quad \leq (1-a_n)^2\|x_n - q\|^2 + 2a_n\left[\|x_{n+1} - q\|^2 - \frac{\varphi(\|x_{n+1} - q\|)}{1 + \varphi(\|x_{n+1} - q\|) + \|x_{n+1} - q\|^2}\right], \end{aligned} \tag{3.5}$$

which implies that

$$\begin{aligned} \|x_{n+1} - q\|^2 &\leq \frac{(1-a_n)^2}{1-2a_n}\|x_n - q\|^2 - \frac{2a_n}{1-2a_n} \cdot \frac{\varphi(\|x_{n+1} - q\|)}{1 + \varphi(\|x_{n+1} - q\|) + \|x_{n+1} - q\|^2} \\ &= \left(1 + \frac{a_n^2}{1-2a_n}\right)\|x_n - q\|^2 - \frac{2a_n}{1-2a_n} \cdot \frac{\varphi(\|x_{n+1} - q\|)}{1 + \varphi(\|x_{n+1} - q\|) + \|x_{n+1} - q\|^2} \\ &\leq \left(1 + \frac{a_n^2}{1-2a_n}\right)\|x_n - q\|^2. \end{aligned} \tag{3.6}$$

By Lemma 3.3, we obtain that $\lim_{n\to\infty} \|x_n - q\|^2$ exists. Let $M = \sup_{n\geq 1}\{\|x_n - q\|\}$.

Set $\inf_{n\geq 1} \frac{\varphi(\|x_{n+1}-q\|)}{1+\varphi(\|x_{n+1}-q\|)+\|x_{n+1}-q\|^2} = \lambda$, then $\lambda = 0$. If this is not the case, we assume that $\lambda > 0$, then $\frac{\varphi(\|x_{n+1}-q\|)}{1+\varphi(\|x_{n+1}-q\|)+\|x_{n+1}-q\|^2} \geq \lambda$ for any n. From (3.6), we get

$$\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 + \frac{a_n^2}{1-2a_n} M^2 - \frac{2\lambda a_n}{1-2a_n}, \tag{3.7}$$

which implies that

$$\sum_{n=1}^{\infty} \frac{2\lambda a_n}{1-2a_n} \leq \|x_1 - q\|^2 + \sum_{n=1}^{\infty} \frac{a_n^2}{1-2a_n} M^2 < \infty, \tag{3.8}$$

which is a contradiction, and so $\lambda = 0$. Consequently, there exists an infinite subsequence such that $\frac{\varphi(\|x_{n_i+1}-q\|)}{1+\varphi(\|x_{n_i+1}-q\|)+\|x_{n_i+1}-q\|^2} \to 0$ as $i \to \infty$. Then we have

$$0 \leq \frac{\varphi(\|x_{n_i+1} - q\|)}{1 + \varphi(M) + M^2} \leq \frac{\varphi(\|x_{n_i+1} - q\|)}{1 + \varphi(\|x_{n_i+1} - q\|) + \|x_{n_i+1} - q\|^2},$$

which implies that $\varphi(\|x_{n_i+1} - q\|) \to 0$ as $i \to \infty$. It leads to $\|x_{n_i+1} - q\| \to 0$ as $i \to \infty$ by the strict increase and continuity of φ. Thus, we obtain that $\|x_n - q\| \to 0$ as $n \to \infty$. This completes the proof. □

Theorem 3.5 *Let E be a real Banach space and let D be a nonempty closed convex subset of E, let $T : D \to D$ be a weak generalized φ-hemicontractive mapping. Suppose that $\{z_n\}_{n=1}^{\infty}$ is defined by* (3.4) *with the iteration parameters $a_n, b_n \in [0, \frac{1}{2})$ satisfying the conditions*:

(i) $a_n, b_n \to 0$ *as* $n \to \infty$;
(ii) $\sum_{n=1}^{\infty} \frac{a_n}{1-2a_n} = \infty$;
(iii) $\sum_{n=1}^{\infty} \frac{a_n^2}{1-2a_n} < \infty$, $\sum_{n=1}^{\infty} \frac{b_n^2}{1-2b_n} < \infty$.

Then the implicit Ishikawa iteration $\{z_n\}_{n=1}^{\infty}$ converges strongly to the unique fixed point of T.

Proof By the definition of a weak generalized φ-hemicontractive mapping, we know that the fixed point of T is unique. Denote q. And for any $x \in D$, we have

$$\langle Tx - Tq, j(x - q)\rangle \leq \|x - q\|^2 - \frac{\varphi(\|x - q\|)}{1 + \varphi(\|x - q\|) + \|x - q\|^2}. \tag{3.9}$$

Applying Lemma 1.8 and (3.2), we have

$$\begin{aligned}
&\|z_{n+1} - q\|^2 \\
&\quad = \big\|(1 - a_n)(y_n - q) + a_n(Tz_{n+1} - Tq)\big\|^2 \\
&\quad \leq (1 - a_n)^2 \|y_n - q\|^2 + 2a_n\langle Tz_{n+1} - Tq, j(z_{n+1} - q)\rangle \\
&\quad \leq (1 - a_n)^2 \|y_n - q\|^2 + 2a_n\left[\|z_{n+1} - q\|^2 - \frac{\varphi(\|z_{n+1} - q\|)}{1 + \varphi(\|z_{n+1} - q\|) + \|z_{n+1} - q\|^2}\right], \qquad (3.10) \\
&\|y_n - q\|^2 \\
&\quad = \big\|(1 - b_n)(z_n - q) + b_n(Ty_n - Tq)\big\|^2
\end{aligned}$$

$$\begin{aligned}
&\le (1-b_n)^2\|z_n-q\|^2 + 2b_n\langle Ty_n - Tq, j(y_n-q)\rangle \\
&\le (1-b_n)^2\|z_n-q\|^2 + 2b_n\left[\|y_n-q\|^2 - \frac{\varphi(\|y_n-q\|)}{1+\varphi(\|y_n-q\|)+\|y_n-q\|^2}\right], \qquad (3.11)
\end{aligned}$$

which implies that

$$\|y_n-q\|^2 \le \frac{(1-b_n)^2}{1-2b_n}\|z_n-q\|^2. \qquad (3.12)$$

Substituting (3.14) into (3.12), we obtain that

$$\begin{aligned}
&\|z_{n+1}-q\|^2 \\
&\le \frac{(1-a_n)^2(1-b_n)^2}{(1-2a_n)(1-2b_n)}\|z_n-q\|^2 - \frac{2a_n}{1-2a_n}\cdot\frac{\varphi(\|z_{n+1}-q\|)}{1+\varphi(\|z_{n+1}-q\|)+\|z_{n+1}-q\|^2} \\
&\le \left(1+\frac{a_n^2}{1-2a_n}+\frac{b_n^2}{1-2b_n}\right)\|z_n-q\|^2 - \frac{2a_n}{1-2a_n}\cdot\frac{\varphi(\|z_{n+1}-q\|)}{1+\varphi(\|z_{n+1}-q\|)+\|z_{n+1}-q\|^2} \\
&\le \left(1+\frac{a_n^2}{1-2a_n}+\frac{b_n^2}{1-2b_n}\right)\|z_n-q\|^2. \qquad (3.13)
\end{aligned}$$

By Lemma 3.3, we obtain that $\lim_{n\to\infty}\|z_n-q\|^2$ exists. Let $M_1 = \sup_{n\ge1}\{\|z_n-q\|\}$.

Set $\inf_{n\ge1}\frac{\varphi(\|z_{n+1}-q\|)}{1+\varphi(\|z_{n+1}-q\|)+\|z_{n+1}-q\|^2} = \delta$, then $\delta = 0$. If this is not the case, we assume that $\delta > 0$, then $\frac{\varphi(\|z_{n+1}-q\|)}{1+\varphi(\|z_{n+1}-q\|)+\|z_{n+1}-q\|^2} \ge \delta$ for any n. From (3.13), we get

$$\|z_{n+1}-q\|^2 \le \|z_n-q\|^2 + \left(\frac{a_n^2}{1-2a_n}+\frac{b_n^2}{1-2b_n}\right)M_1^2 - \frac{2\delta a_n}{1-2a_n}, \qquad (3.14)$$

which implies that

$$\sum_{n=1}^{\infty}\frac{2\delta a_n}{1-2a_n} \le \|z_1-q\|^2 + \sum_{n=1}^{\infty}\left(\frac{a_n^2}{1-2a_n}+\frac{b_n^2}{1-2b_n}\right)M_1^2 < \infty, \qquad (3.15)$$

which is a contradiction, and so $\delta = 0$. Consequently, there exists an infinite subsequence such that $\frac{\varphi(\|z_{n_i+1}-q\|)}{1+\varphi(\|z_{n_i+1}-q\|)+\|z_{n_i+1}-q\|^2} \to 0$ as $i\to\infty$. Then we have

$$0 \le \frac{\varphi(\|z_{n_i+1}-q\|)}{1+\varphi(M_1)+M_1^2} \le \frac{\varphi(\|z_{n_i+1}-q\|)}{1+\varphi(\|z_{n_i+1}-q\|)+\|z_{n_i+1}-q\|^2},$$

which implies that $\varphi(\|z_{n_i+1}-q\|)\to 0$ as $i\to\infty$. It leads to $\|z_{n_i+1}-q\|\to 0$ as $i\to\infty$ by the strict increase and continuity of φ. Thus, we obtain that $\|z_n-q\|\to 0$ as $n\to\infty$. This completes the proof. □

Remark 3.6 Theorem 2.4 shows that the implicit iteration $\{x_n\}$ by $x_{n+1} = \frac{1}{2}x_n + \frac{1}{2}Tx_{n+1}$ is convergent, and it converges strongly to the fixed point of T. And Theorem 3.4 and Theorem 3.5 also yield that the implicit Mann iteration and the implicit Ishikawa iteration converge strongly to the fixed point of T, respectively.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally in writing this paper, and read and approved the final manuscript.

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