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S-iteration process for quasi-contractive mappings

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Abstract

In this note, we show that the S-iteration process due to Sahu and Petrusel (Nonlinear Anal. TMA 74(17):6012-6023, 2011) is faster than the Picard, Mann, Ishikawa and Noor iteration processes for Zamfirescu operators. Also, using computer programs in C++, we present some examples to compare the convergence rate of iterative processes due to Picard, Mann, Ishikawa, Noor, Agarwal *et al*. and Sahu and Petrusel.

Keywords: iteration processes; quasi-contractive operator; Zamfirescu operator

1 Introduction

During the last many years, much attention has been given to the following iteration processes (see, for example, [1–7]).

For a nonempty convex subset C of a normed space E and $T : C \to C$,

(a) The Mann iteration process [8] is defined by the following sequence $\{x_n\}$:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1 - b_n)x_n + b_n Tx_n, \quad n \geq 0, \end{cases} \tag{M_n}$$

where $\{b_n\}$ is a sequence in $[0,1]$.

(b) The sequence $\{x_n\}$ defined by

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1 - b_n)x_n + b_n Ty_n, \\ y_n = (1 - b'_n)x_n + b'_n Tx_n, \quad n \geq 0, \end{cases} \tag{I_n}$$

where $\{b_n\}$, $\{b'_n\}$ are sequences in $[0,1]$, is known as the Ishikawa [9] iteration process.

(c) The sequence $\{x_n\}$ defined by

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1 - b_n)Tx_n + b_n Ty_n, \\ y_n = (1 - b'_n)x_n + b'_n Tx_n, \quad n \geq 0, \end{cases} \tag{ARS_n}$$

where $\{b_n\}$, $\{b'_n\}$ are sequences in $[0,1]$, is known as the Agarwal-O'Regan-Sahu [10] iteration process.

(d) The sequence $\{x_n\}$ defined by

$$\begin{cases} x_0 \in C, \\ y_n = (1 - b'_n)x_n + b'_n T x_n, \\ x_{n+1} = T y_n, \quad n \geq 0, \end{cases} \tag{S_n}$$

where $\{b'_n\}$ is sequence in $[0,1]$, is known as the S-iteration process [11].

(e) The sequence $\{x_n\}$ defined by

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1 - b_n)x_n + b_n T y_n, \\ y_n = (1 - b'_n)x_n + b'_n T z_n, \\ z_n = (1 - b''_n)x_n + b''_n T x_n, \quad n \geq 0, \end{cases} \tag{N_n}$$

where $\{b_n\}$, $\{b'_n\}$ and $\{b''_n\}$ are sequences of positive numbers in $[0,1]$ and denoted by N_n, is known as the Noor multi-step iteration process [12].

Definition 1 [13] Suppose $\{a_n\}$ and $\{b_n\}$ are two real convergent sequences with limits a and b, respectively. Then $\{a_n\}$ is said to converge faster than $\{b_n\}$ if

$$\lim_{n\to\infty} \left| \frac{a_n - a}{b_n - b} \right| = 0.$$

Theorem 2 [14] *Let (X,d) be a complete metric space, and let $T : X \to X$ be a mapping for which there exist real numbers a, b and c satisfying* $0 < a < 1$, $0 < b, c < \frac{1}{2}$ *such that for each pair $x, y \in X$ at least one of the following is true*:

(z1) $d(Tx, Ty) \leq a d(x, y)$,

(z2) $d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)]$,

(z3) $d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)]$.

Then T has a unique fixed point p and the Picard iteration process $\{x_n\}$ defined by

$$x_{n+1} = T x_n, \quad n = 0, 1, \ldots, \tag{P_n}$$

converges to p for any $x_0 \in X$.

Remark 3 An operator T which satisfies the contraction conditions (z1)-(z3) of Theorem 2 is called a Zamfirescu operator [13, 15, 16] and is denoted by Z.

In [15, 16], Berinde introduced a new class of operators on a normed space E satisfying

$$\|Tx - Ty\| \leq \delta \|x - y\| + L\|Tx - x\| \tag{B}$$

for any $x, y \in E$ and some $\delta \in [0,1)$, $L \geq 0$.

He proved that this class is wider than the class of Zamfirescu operators. The following results are proved in [15, 16].

Theorem 4 [16] *Let C be a nonempty closed convex subset of a normed space E. Let $T : C \to C$ be an operator satisfying (B). Let $\{x_n\}$ be defined through the iterative process (M_n). If $F(T) \neq \emptyset$ and $\sum b_n = \infty$, then $\{x_n\}$ converges strongly to the unique fixed point of T.*

Theorem 5 [16] *Let C be a nonempty closed convex subset of an arbitrary Banach space E, and let $T : C \to C$ be an operator satisfying (B). Let $\{x_n\}$ be defined through the Ishikawa iterative process (I_n) and $x_0 \in C$, where $\{b_n\}$ and $\{b'_n\}$ are sequences of positive real numbers in $[0,1]$ with $\{b_n\}$ satisfying $\sum b_n = \infty$. Then $\{x_n\}$ converges strongly to the fixed point of T.*

The following theorem was presented in [17].

Theorem 6 *Let C be a closed convex subset of an arbitrary Banach space E. Let the Mann and Ishikawa iteration processes denoted by M_n and I_n, respectively, with $\{b_n\}$ and $\{b'_n\}$ be real sequences satisfying*

(i) $0 \leq b_n, b'_n \leq 1$,
(ii) $\sum b_n = \infty$.

Then M_n and I_n converge strongly to the unique fixed point of a Zamfirescu operator $T : C \to C$, and, moreover, the Mann iteration process converges faster than the Ishikawa iteration process to the fixed point of T.

Remark 7 In [18], Qing and Rhoades, by taking a counter example, showed that the Ishikawa iteration process is faster than the Mann iteration process for Zamfirescu operators.

In this note, we establish a general theorem to approximate the fixed points of quasi-contractive operators in a Banach space through the S-iteration process due to Sahu and Petrusel [11]. Our result generalizes and improves upon, among others, the corresponding results of Babu and Prasad [17] and Berinde [13, 15, 16]. We also prove that the S-iteration process is faster than the Mann, Ishikawa, Picard and Noor iteration processes, respectively, for Zamfirescu operators.

2 Main results

We now prove our main results.

Theorem 8 *Let C be a nonempty closed convex subset of a normed space E. Let $T : C \to C$ be an operator satisfying (B). Let $\{x_n\}$ be defined through the iterative process (S_n) and $x_0 \in C$, where $\{b'_n\}$ is a sequence in $[0,1]$ satisfying $\sum b'_n = \infty$. Then $\{x_n\}$ converges strongly to the fixed point of T.*

Proof Assume that $F(T) \neq \emptyset$ and $w \in F(T)$. Then, using (S_n), we have

$$\|x_{n+1} - w\| = \|Ty_n - w\|. \tag{2.1}$$

Now, using (B) with $x = w$, $y = y_n$, we obtain the following inequality:

$$\|Ty_n - w\| \leq \delta \|y_n - w\|. \tag{2.2}$$

By substituting (2.2) in (2.1), we obtain

$$\|x_{n+1} - w\| \le \delta \|y_n - w\|, \tag{2.3}$$

where

$$\begin{aligned} \|y_n - w\| &= \|(1 - b'_n)x_n + b'_n Tx_n - w\| = \|(1 - b'_n)(x_n - w) + b'_n(Tx_n - w)\| \\ &\le (1 - b'_n)\|x_n - w\| + b'_n\|Tx_n - w\|. \end{aligned} \tag{2.4}$$

Again, by using (B), $x = w$, $y = x_n$, we get

$$\|Tx_n - w\| \le \delta \|x_n - w\|, \tag{2.5}$$

and substitution of (2.5) in (2.4) yields

$$\|y_n - w\| \le (1 - (1 - \delta)b'_n)\|x_n - w\|. \tag{2.6}$$

From (2.3) and (2.6), we have

$$\|x_{n+1} - w\| \le (1 - \delta(1 - \delta)b'_n)\|x_n - w\|. \tag{2.7}$$

By (2.7) we inductively obtain

$$\|x_{n+1} - w\| \le \prod_{k=0}^{n}(1 - \delta(1 - \delta)b'_k)\|x_0 - w\|, \quad n = 0, 1, 2, \ldots. \tag{2.8}$$

Using the fact that $0 \le \delta < 1$, $0 \le b_n \le 1$ and $\sum b'_n = \infty$, we get that

$$\lim_{n\to\infty} \prod_{k=0}^{n}(1 - \delta(1 - \delta)b'_k) = 0,$$

which by (2.8) implies $\lim_{n\to\infty} \|x_{n+1} - w\| = 0$.

Consequently, $x_n \to w \in F$ and this completes the proof. □

Now we present an example to show that the S-iteration process is faster than the Mann, Ishikawa, Picard and Noor iteration processes, respectively, for Zamfirescu operators.

Example 9 Let $T : [0, 1] \to [0, 1] := \frac{x}{2}$. Let $b_n = \frac{4}{\sqrt{n}} = b'_n = b''_n$.

It is clear that T is a Zamfirescu operator with a unique fixed point 0. Also, it is easy to see that Example 9 satisfies all the conditions of Theorem 8.

Note that

$$\begin{aligned} M_n &= (1 - b_n)x_n + b_n Tx_n \\ &= \left(1 - \frac{4}{\sqrt{n}}\right)x_n + \frac{4}{\sqrt{n}}\frac{x_n}{2} \\ &= \left(1 - \frac{2}{\sqrt{n}}\right)x_n \end{aligned}$$

$$\begin{aligned}
&= \cdots \\
&= \prod_{i=16}^{n}\left(1-\frac{2}{\sqrt{i}}\right)x_0, \\
I_n &= (1-b_n)x_n + b_n T\big((1-b'_n)x_n + b'_n Tx_n\big) \\
&= \left(1-\frac{4}{\sqrt{n}}\right)x_n + \frac{4}{\sqrt{n}}\frac{1}{2}\left(1-\frac{2}{\sqrt{n}}\right)x_n \\
&= \left(1-\frac{2}{\sqrt{n}}-\frac{4}{\sqrt{n}}\right)x_n \\
&= \cdots \\
&= \prod_{i=16}^{n}\left(1-\frac{2}{\sqrt{i}}-\frac{4}{\sqrt{i}}\right)x_0, \\
S_n &= T\big((1-b'_n)x_n + b'_n Tx_n\big) \\
&= \frac{1}{2}\left(\left(1-\frac{4}{\sqrt{n}}\right)x_n + \frac{4}{\sqrt{n}}\frac{x_n}{2}\right) \\
&= \left(\frac{1}{2}-\frac{1}{\sqrt{n}}\right)x_n \\
&= \cdots \\
&= \prod_{i=16}^{n}\left(\frac{1}{2}-\frac{1}{\sqrt{i}}\right)x_0
\end{aligned}$$

and

$$\begin{aligned}
N_n &= (1-b_n)x_n + b_n T\big((1-b'_n)x_n + b'_n T\big((1-b''_n)x_n + b''_n Tx_n\big)\big) \\
&= \left(1-\frac{4}{\sqrt{n}}\right)x_n + \frac{4}{\sqrt{n}}\frac{1}{2}\left(1-\frac{2}{\sqrt{n}}-\frac{4}{n}\right)x_n \\
&= \left(1-\frac{2}{\sqrt{n}}-\frac{4}{n}-\frac{8}{n^{\frac{3}{2}}}\right)x_n \\
&= \cdots \\
&= \prod_{i=16}^{n}\left(1-\frac{2}{\sqrt{i}}-\frac{4}{i}-\frac{8}{i^{\frac{3}{2}}}\right)x_0.
\end{aligned}$$

Now, for $n \geq 16$, consider

$$\begin{aligned}
\left|\frac{S_n-0}{M_n-0}\right| &= \left|\frac{\prod_{i=16}^{n}(\frac{1}{2}-\frac{1}{\sqrt{i}})x_0}{\prod_{i=16}^{n}(1-\frac{2}{\sqrt{i}})x_0}\right| \\
&= \left|\frac{\prod_{i=16}^{n}(\frac{1}{2}-\frac{1}{\sqrt{i}})}{\prod_{i=64}^{n}(1-\frac{2}{\sqrt{i}})}\right| \\
&= \left|\prod_{i=16}^{n}\left[1-\frac{\frac{1}{2}-\frac{1}{\sqrt{i}}}{1-\frac{2}{\sqrt{i}}}\right]\right| \\
&= \prod_{i=16}^{n}\left(\frac{1}{2}\right).
\end{aligned}$$

Hence, $\lim_{n\to\infty}|\frac{S_n-0}{M_n-0}| = 0$.

Similarly,

$$\begin{aligned}\left|\frac{S_n-0}{I_n-0}\right| &= \left|\frac{\prod_{i=16}^{n}(\frac{1}{2}-\frac{1}{\sqrt{i}})x_0}{\prod_{i=16}^{n}(1-\frac{2}{\sqrt{i}}-\frac{4}{i})x_0}\right| \\ &= \left|\prod_{i=16}^{n}\left[\frac{(\frac{1}{2}-\frac{1}{\sqrt{i}})}{(1-\frac{2}{\sqrt{i}}-\frac{4}{i})}\right]\right| \\ &= \left|\prod_{i=16}^{n}\left[1-\frac{\frac{1}{2}-\frac{1}{\sqrt{i}}-\frac{4}{i}}{1-\frac{2}{\sqrt{i}}-\frac{4}{i}}\right]\right| \\ &= \left|\prod_{i=16}^{n}\left[1-\frac{(i-2\sqrt{i}-8)}{2(i-2\sqrt{i}-4)}\right]\right|\end{aligned}$$

with

$$\begin{aligned}0 &\le \lim_{n\to\infty}\prod_{i=16}^{n}\left[1-\frac{i-2\sqrt{i}-8}{2(i-2\sqrt{i}-4)}\right] \\ &\le \lim_{n\to\infty}\prod_{i=16}^{n}\left(1-\frac{1}{i}\right) \\ &= \lim_{n\to\infty}\frac{15}{n} \\ &= 0\end{aligned}$$

implies

$$\lim_{n\to\infty}\left|\frac{S_n-0}{I_n-0}\right| = 0.$$

Again, let $n \ge 16$. Then

$$\begin{aligned}\left|\frac{S_n-0}{P_n-0}\right| &= \left|\frac{\prod_{i=16}^{n}(\frac{1}{2}-\frac{1}{\sqrt{i}})x_0}{(\frac{1}{2})^n x_0}\right| \\ &\le \left|\frac{\prod_{i=16}^{n}(\frac{1}{2}-\frac{1}{i})}{(\frac{1}{2})^n}\right| \\ &\le \left|\frac{\prod_{i=16}^{n}(\frac{1}{2}-\frac{1}{2i})}{(\frac{1}{2})^n}\right| \\ &= \left|\frac{(\frac{1}{2})^{n-15}\prod_{i=16}^{n}(1-\frac{1}{i})}{(\frac{1}{2})^n}\right| \\ &= \left|\left(\frac{1}{2}\right)^{-15}\prod_{i=16}^{n}\left(1-\frac{1}{i}\right)\right| \\ &= \left|\frac{\frac{15}{n}}{(\frac{1}{2})^{15}}\right|\end{aligned}$$

with

$$0 \le \lim_{n\to\infty} \frac{\frac{15}{n}}{(\frac{1}{2})^{15}} = 0$$

implies

$$\lim_{n\to\infty} \left| \frac{S_n - 0}{P_n - 0} \right| = 0.$$

Also, for $n \ge 25$, we have

$$\begin{aligned} \left| \frac{S_n - 0}{N_n - 0} \right| &= \left| \frac{\prod_{i=25}^{n} (\frac{1}{2} - \frac{1}{\sqrt{i}}) x_0}{\prod_{i=25}^{n} (1 - \frac{2}{\sqrt{i}} - \frac{4}{i} - \frac{8}{i^{\frac{3}{2}}}) x_0} \right| \\ &= \left| \frac{\prod_{i=25}^{n} (\frac{1}{2} - \frac{1}{\sqrt{i}})}{\prod_{i=25}^{n} (1 - \frac{2}{\sqrt{i}} - \frac{4}{i} - \frac{8}{i^{\frac{3}{2}}})} \right| \\ &= \left| \prod_{i=25}^{n} \left(1 - \frac{(\frac{1}{2} - \frac{1}{\sqrt{i}} - \frac{4}{i} - \frac{8}{i^{\frac{3}{2}}})}{(1 - \frac{2}{\sqrt{i}} - \frac{4}{i} - \frac{8}{i^{\frac{3}{2}}})} \right) \right| \\ &= \left| \prod_{i=25}^{n} \left(1 - \frac{(i^{\frac{3}{2}} - 8\sqrt{i} - 16 - 2i)}{(2i^{\frac{3}{2}} - 4i - 8\sqrt{i} - 16)} \right) \right|, \end{aligned}$$

with

$$\begin{aligned} 0 &\le \lim_{n\to\infty} \prod_{i=25}^{n} \left(1 - \frac{(i^{\frac{3}{2}} - 8\sqrt{i} - 16 - 2i)}{(2i^{\frac{3}{2}} - 4i - 8\sqrt{i} - 16)} \right) \\ &\le \lim_{n\to\infty} \prod_{i=25}^{n} \left(1 - \frac{1}{i} \right) \\ &= \lim_{n\to\infty} \frac{24}{n} \\ &= 0 \end{aligned}$$

implies

$$\lim_{n\to\infty} \left| \frac{S_n - 0}{N_n - 0} \right| = 0.$$

3 Applications

In this section, using computer programs in C++, we compare the convergence rate of Picard, Mann, Ishikawa, Noor, Agarwal *et al.* and S-iterative processes through examples. The outcome is listed in the form of Tables 1-4 by taking initial approximation $x_0 = 0.8$ for all iterative processes.

Decreasing *cum* sublinear functions

Let $f : [0,1] \to [0,1]$ be defined by $f(x) = (1-x)^m$, $m = 7, 8 \ldots$. Then f is a decreasing function. By taking $m = 8$ and $\alpha_n = \beta_n = \gamma_n = \frac{1}{(1+n)^{\frac{1}{4}}}$, the comparison of convergence of the above-mentioned iterative processes to the exact fixed point $p = 0.188348$ is listed in Table 1.

Increasing functions

Let $f : [0,8] \to [0,8]$ be defined as $f(x) = \frac{x^2+9}{10}$. Then f is an increasing function. By taking $\alpha_n = \beta_n = \gamma_n = \frac{1}{(1+n)^{\frac{1}{2}}}$, the comparison of convergence of the above-mentioned iterative processes to the exact fixed point $p = 1$ is listed in Table 2.

Superlinear functions with multiple roots

The function defined by $f(x) = 2x^3 - 7x^2 + 8x - 2$ is a superlinear function with multiple real roots. By taking $\alpha_n = \beta_n = \gamma_n = \frac{1}{(1+n)^{\frac{1}{2}}}$, the comparison of convergence of the above-mentioned iterative processes to the exact fixed point $p = 1$ is listed in Table 3.

Oscillatory functions

The function defined by $f(x) = \frac{1}{x}$ is an oscillatory function. By taking $\alpha_n = \beta_n = \gamma_n = \frac{1}{(1+n)^{\frac{1}{4}}}$, the comparison of convergence of the above-mentioned iterative processes to the exact fixed point $p = 1$ is listed in Table 4.

For detailed study, these programs are again executed after changing the parameters, and the readings are recorded as follows.

4 Observations

Decreasing *cum* sublinear functions

1. For $m = 8$ and $x_0 = 0.8$, the Picard process never converges (oscillates between 0 and 1), the Mann process converges in 23 iterations, the Ishikawa process converges in 56 iterations, the Noor process converges in 21 iterations, the Agarwal *et al.* process converges in 12 iterations and the S-iterative process converges in 34 iterations.

2. For $m = 30$ and $x_0 = 0.8$, the Picard process never converges (oscillates between 0 and 1), the Mann process converges in 37 iterations, the Ishikawa process converges in 118 iterations, the Noor process converges in 47 iterations, the Agarwal *et al.* process converges in 15 iterations while the S-iterative process never converges.

3. Taking initial guess $x_0 = 0.2$ (nearer to the fixed point), the Picard process never converges (oscillates between 0 and 1), the Mann process converges in 24 iterations, the Ishikawa process converges in 56 iterations, the Noor process converges in 22 iterations, the Agarwal *et al.* process converges in 12 iterations and the S-iterative process converges in 34 iterations.

4. Taking $\alpha_n = \beta_n = \gamma_n = \frac{1}{(1+n)^{\frac{1}{6}}}$ and $x_0 = 0.8$, we find that the Mann process converges in 72 iterations, the Ishikawa process converges in 289 iterations, the Noor process converges in 105 iterations, the Agarwal *et al.* process converges in 36 iterations, while the S-iterative process converges in 240 iterations.

Increasing functions

1. For $x_0 = 0.8$, the Picard process converges to a fixed point in 8 iterations, the Mann process converges in 69 iterations, the Ishikawa process converges in 34 iterations, the Noor process converges in 24 iterations, the Agarwal *et al.* process converges in 7 iterations and the S-iterative process converges in 6 iterations.

Table 1 Deceasing *cum* sublinear functions

Number of iterations	Noor iteration		Picard iteration		Mann iteration		Ishikawa iteration		Agarwal *et al.* iteration		*S*-iteration	
n	fx_n	x_{n+1}	fx_n	x_{n+1}	fx_n	x_{n+1}	fx_n	x_{n+1}	fx_n	x_{n+1}	fx_n	x_{n+1}
0	2.56e–06	3.094628e–38	2.56e–06	2.56e–06	2.56e–06	2.56e–06	1.103589	1.137884	2.56e–06	0.99998	2.56e–06	0.99998
1	1	0.840894	0.99998	0.99998	0.99998	0.84088	1.150601	1.149315	3.094628e–38	0.210231	3.094628e–38	0.250008
2	06676e–07	0.233944	3.094628e–38	3.094628e–38	4.109669e–07	0.20195	–	–	0.151357	0.215059	0.100105	0.31023
3	0.1186	0.134251	1	1	0.164529	0.175489	1.157988	1.157426	0.14411	0.209484	0.051243	0.337071
4	0.315599	0.313519	0	0	0.213585	0.200965	1.158157	1.15772	0.152505	0.199147	0.037302	0.308803
5	0.049321	0.151679	1	1	0.166159	0.178726	1.158273	1.157929	0.169209	0.19168	0.052097	0.286162
–	–	–	–	–	–	–	–	–	–	–	–	–
8	0.240452	0.214091	0	0	0.198079	0.191799	1.158458	1.158278	0.188331	0.188347	0.096417	0.241306
9	0.145539	0.175829	1	1	0.182035	0.186308	1.158492	1.158344	0.188349	0.188348	0.109782	0.231058
10	0.212882	0.197132	0	0	0.192167	0.189525	1.158518	1.158395	0.188347	0.188348	0.122222	0.222431
11	0.172645	0.18423	1	1	0.186172	0.187724	1.158538	1.158436	**0.188348**	**0.188348**	0.133632	0.215183
12	0.196128	0.190408	0	0	0.189509	0.188664	1.158554	1.158468	**0.188348**	**0.188348**	0.143929	0.209132
13	0.184556	0.187511	1	1	0.187761	0.188197	1.158567	1.158494	**0.188348**	**0.188348**	0.153049	0.204138
–	–	–	–	–	–	–	–	–	**–**	**–**	–	–
18	0.188363	0.18835	0	0	0.188356	0.188349	0.165756	0.198778	**0.188348**	**0.188348**	0.180912	0.191033
19	0.188344	0.188347	1	1	0.188345	0.188347	0.169834	0.196745	**0.188348**	**0.188348**	0.183419	0.190053
20	**0.188348**	**0.188348**	0	0	0.188349	0.188348	0.173312	0.195058	**0.188348**	**0.188348**	0.185206	0.189388
21	**0.188348**	**0.188348**	1	1	0.188347	0.188348	0.176246	0.193668	**0.188348**	**0.188348**	0.186424	0.188957
22	**0.188348**	**0.188348**	0	0	**0.188348**	**0.188348**	0.178695	0.192533	**0.188348**	**0.188348**	0.187219	0.18869
–	**–**	**–**	–	–	**–**	**–**	–	–	**–**	**–**	**–**	**–**
34	**–**	**–**	–	–	**–**	**–**	–	–	**–**	**–**	**0.188348**	**0.188348**
35	**–**	**–**	–	–	**–**	**–**	–	–	**–**	**–**	**0.188348**	**0.188348**
–	**–**	**–**	–	–	**–**	**–**	–	–	**–**	**–**	**–**	**–**
53	**0.188348**	**0.188348**	1	1	**0.188348**	**0.188348**	0.188347	0.188348	**0.188348**	**0.188348**	**0.188348**	**0.188348**
54	**0.188348**	**0.188348**	0	0	**0.188348**	**0.188348**	0.188347	0.188348	**0.188348**	**0.188348**	**0.188348**	**0.188348**
55	**0.188348**	**0.188348**	1	1	**0.188348**	**0.188348**	**0.188348**	**0.188348**	**0.188348**	**0.188348**	**0.188348**	**0.188348**
56	**0.188348**	**0.188348**	0	0	**0.188348**	**0.188348**	**0.188348**	**0.188348**	**0.188348**	**0.188348**	**0.188348**	**0.188348**

Table 2 Increasing functions

Number of iterations	Noor iteration		Picard iteration		Mann iteration		Ishikawa iteration		Agarwal *et al.* iteration		*S*-iteration	
n	fx_n	x_{n+1}	fx_n	x_{n+1}	fx_n	x_{n+1}	fx_n	x_{n+1}	fx_n	x_{n+1}	fx_n	x_{n+1}
0	0.964	0.998591	0.9	0.9	0.964	0.964	0.99293	0.99293	0.964	0.99293	0.964	0.99293
–	–	–	–	–	–	–	–	–	–	–	–	–
4	0.999974	0.999922	0.99985	0.99985	0.994714	0.999996	0.999997	0.999937	0.999996	0.999997	0.999998	0.999999
5	0.999984	0.99995	0.99997	0.99997	0.996595	0.999999	0.999999	0.999973	0.999999	0.999999	**1**	**1**
6	0.99999	0.999966	0.999994	0.999994	0.997703	1	1	0.999937	**1**	**1**	**1**	**1**
7	0.999993	0.999976	0.999999	0.999999	0.998396	1	1	0.999988	**1**	**1**	**1**	**1**
8	0.999995	0.999983	**1**	**1**	0.998849	1	1	0.999994	**1**	**1**	**1**	**1**
–	–	–	–	–	–	**–**	**–**	–	**–**	**–**	**–**	**–**
23	**1**	**1**	**1**	**1**	0.999964	1	1	0.999997	**1**	**1**	**1**	**1**
24	**1**	**1**	**1**	**1**	0.99997	1	1	0.999998	**1**	**1**	**1**	**1**
–	–	–	**–**	**–**	–	**–**	**–**	**–**	**–**	**–**	**–**	**–**
33	**1**	**1**	**1**	**1**	0.999994	1	**1**	**1**	**1**	**1**	**1**	**1**
–	**–**	**–**	**–**	**–**	–	**–**	**–**	**–**	**–**	**–**	**–**	**–**
68	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
69	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**

Table 3 Superlinear functions with multiple roots

Number of iterations	Noor iteration		Picard iteration		Mann iteration		Ishikawa iteration		Agarwal *et al.* iteration		*S*-iteration	
n	fx_n	x_{n+1}	fx_n	x_{n+1}	fx_n	x_{n+1}	fx_n	x_{n+1}	fx_n	x_{n+1}	fx_n	x_{n+1}
0	0.944	0.999988	0.944	0.944	0.944	0.944	0.944	0.996513	0.944	0.996513	0.944	0.996513
1	1	0.999996	0.996513	0.996513	0.996513	0.981132	0.999988	0.998978	0.999988	0.999996	0.999988	0.999999
2	1	0.999998	0.999988	0.999988	0.999631	0.991812	0.999999	0.999568	**1**	**1**	**1**	**1**
3	1	0.999999	**1**	**1**	0.999932	0.995872	1	0.999784	**1**	**1**	**1**	**1**
4	**1**	**1**	**1**	**1**	0.999983	0.99771	1	0.999881	**1**	**1**	**1**	**1**
5	**1**	**1**	**1**	**1**	0.999995	0.998643	1	0.999929	**1**	**1**	**1**	**1**
6	**1**	**1**	**1**	**1**	0.999998	0.999155	1	0.999956	**1**	**1**	**1**	**1**
7	**1**	**1**	**1**	**1**	0.999999	0.999454	1	0.999972	**1**	**1**	**1**	**1**
8	**1**	**1**	**1**	**1**	1	0.999636	1	0.999981	**1**	**1**	**1**	**1**
9	**1**	**1**	**1**	**1**	1	0.999751	1	0.999987	**1**	**1**	**1**	**1**
–	–	–	–	–	–	–	–	–	–	–	–	–
19	**1**	**1**	**1**	**1**	1	0.999987	1	0.999999	**1**	**1**	**1**	**1**
20	**1**	**1**	**1**	**1**	1	0.99999	1	0.999999	**1**	**1**	**1**	**1**
21	**1**	**1**	**1**	**1**	1	0.999992	**1**	**1**	**1**	**1**	**1**	**1**
22	**1**	**1**	**1**	**1**	1	0.999994	**1**	**1**	**1**	**1**	**1**	**1**
23	**1**	**1**	**1**	**1**	1	0.999995	**1**	**1**	**1**	**1**	**1**	**1**
–	–	–	–	–	–	–	–	–	–	–	–	–
32	**1**	**1**	**1**	**1**	1	0.999999	**1**	**1**	**1**	**1**	**1**	**1**
33	**1**	**1**	**1**	**1**	1	0.999999	**1**	**1**	**1**	**1**	**1**	**1**
34	**1**	**1**	**1**	**1**	1	0.999999	**1**	**1**	**1**	**1**	**1**	**1**
35	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
36	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**

Table 4 Oscillatory functions

Number of iterations	Noor iteration		Picard iteration		Mann iteration		Ishikawa iteration		Agarwal *et al.* iteration		*S*-iteration	
n	fx_n	x_{n+1}	fx_n	x_{n+1}	fx_n	x_{n+1}	fx_n	x_{n+1}	fx_n	x_{n+1}	fx_n	x_{n+1}
0	1.25	1.25	1.25	1.25	1.25	1.25	1.25	0.8	1.25	0.8	1.25	0.8
1	0.8	0.921512	0.8	0.8	0.8	0.871597	1.25	0.840872	1.25	0.912469	1.25	0.848606
2	1.085173	1.022975	1.25	1.25	1.14732	1.081101	1.189241	0.889224	1.095928	0.985571	1.178403	0.909754
3	0.977541	0.997476	0.8	0.8	0.924983	0.970709	1.124576	0.930283	1.01464	0.999983	1.099198	0.95812
4	1.002531	1.000107	1.25	1.25	1.030175	1.010476	1.074942	0.95931	1.000017	1.000002	1.043711	0.984874
5	0.999893	1.000002	0.8	0.8	0.989632	0.997158	1.042416	0.977477	0.999998	1	1.015358	0.995667
6	0.999998	1	1.25	1.25	1.00285	1.000657	1.023042	0.987967	**1**	**1**	1.004352	0.998995
7	**1**	**1**	0.8	0.8	0.999343	0.999876	1.01218	0.99372	**1**	**1**	1.001006	0.999809
8	**1**	**1**	0.8	0.8	1.000124	1.000019	1.00632	0.996772	**1**	**1**	1.000191	0.99997
9	**1**	**1**	1.25	1.25	0.999981	0.999998	1.003238	0.998358	**1**	**1**	1.00003	0.999996
10	**1**	**1**	0.8	0.8	1.000002	1	1.001645	0.99917	**1**	**1**	1.000004	1
11	**1**	**1**	0.8	0.8	**1**	**1**	1.000831	0.999583	**1**	**1**	**1**	**1**
12	**1**	**1**	1.25	1.25	**1**	**1**	1.000418	0.999791	**1**	**1**	**1**	**1**
13	**1**	**1**	0.8	0.8	**1**	**1**	1.000209	0.999895	**1**	**1**	**1**	**1**
14	**1**	**1**	0.8	0.8	**1**	**1**	1.000105	0.999948	**1**	**1**	**1**	**1**
15	**1**	**1**	1.25	1.25	**1**	**1**	1.000052	0.999974	**1**	**1**	**1**	**1**
16	**1**	**1**	0.8	0.8	**1**	**1**	1.000026	0.999987	**1**	**1**	**1**	**1**
17	**1**	**1**	1.25	1.25	**1**	**1**	1.000013	0.999993	**1**	**1**	**1**	**1**
18	**1**	**1**	0.8	0.8	**1**	**1**	1.000007	0.999997	**1**	**1**	**1**	**1**
19	**1**	**1**	1.25	1.25	**1**	**1**	1.000003	0.999998	**1**	**1**	**1**	**1**
20	**1**	**1**	0.8	0.8	**1**	**1**	1.000002	0.999999	**1**	**1**	**1**	**1**
21	**1**	**1**	1.25	1.25	**1**	**1**	1.000001	1	**1**	**1**	**1**	**1**
22	**1**	**1**	0.8	0.8	**1**	**1**	**1**	**1**	**1**	**1**		

2. Taking initial guess $x_0 = 0.6$ (away from the fixed point), the Picard process converges to a fixed point in 8 iterations, the Mann process converges in 75 iterations, the Ishikawa process converges in 38 iterations, the Noor process converges in 27 iterations, the Agarwal *et al.* process converges in 6 iterations and the S-iterative process converges in 6 iterations.

3. Taking $\alpha_n = \beta_n = \gamma_n = \frac{1}{(1+n)^{\frac{1}{4}}}$ and $x_0 = 0.8$, we obtain that the Mann process converges in 23 iterations, the Ishikawa process converges in 12 iterations, the Noor process converges in 9 iterations, the Agarwal *et al.* process converges in 6 iterations and the S-iterative process converges in 6 iterations.

Superlinear functions with multiple roots

1. For $x_0 = 0.8$, the Picard process converges to a fixed point in 4 iterations, the Mann process converges in 36 iterations, the Ishikawa process converges in 22 iterations, the Noor process converges in 5 iterations and the Agarwal *et al.* as well as the S-iterative processes converge in 3 iterations.

2. Taking initial guess $x_0 = 0.6$ (away from the fixed point), the Picard process converges to a fixed point in 5 iterations, the Mann process converges in 52 iterations, the Ishikawa process converges in 40 iterations, the Noor process converges in 57 iterations and the Agarwal *et al.* as well as the S-iterative processes converge in 4 iterations.

3. Taking $\alpha_n = \beta_n = \gamma_n = \frac{1}{(1+n)^{\frac{1}{4}}}$ and $x_0 = 0.8$, we obtain that the Mann process converges in 13 iterations, the Ishikawa process converges in 9 iterations, the Noor process converges in 14 iterations and the Agarwal *et al.* as well as the S-iterative processes converge in 3 iterations.

Oscillatory functions

1. For $x_0 = 0.8$, the Picard process never converges to a fixed point, the Mann process converges in 12 iterations, the Ishikawa process converges in 23 iterations, the Noor process converges in 8 iterations, the Agarwal *et al.* process converges in 7 iterations and the S-iterative process converges in 12 iterations.

2. Taking initial guess $x_0 = 0.6$ (away from the fixed point), the Picard process converges to a fixed point in 5 iterations, the Mann process converges in 12 iterations, the Ishikawa process converges in 25 iterations, the Noor process converges in 8 iterations, the Agarwal *et al.* process converges in 8 iterations and the S-iterative process converges in 13 iterations.

3. Taking $\alpha_n = \beta_n = \gamma_n = \frac{1}{(1+n)^{\frac{1}{6}}}$ and $x_0 = 0.8$, we obtain that the Mann process converges in 21 iterations, the Ishikawa process converges in 30 iterations, the Noor process converges in 14 iterations, the Agarwal *et al.* process converges in 12 iterations and the S-iterative process converges in 22 iterations.

5 Conclusions

Decreasing *cum* sublinear functions

1. The Picard process does not converge while the decreasing order of convergence rate of other iterative processes is Agarwal *et al.*, Noor, Mann, S and Ishikawa processes.

2. On increasing the value of m, all the above-mentioned processes require more number of iterations to converge except Picard and S-iterative processes which do not converge.

3. For initial guess nearer to the fixed point, Mann and Noor processes show an increase, while Ishikawa, S and Agarwal *et al*. processes show no change in the number of iterations to converge.

4. The speed of iterative processes depends on α_n and β_n. If we increase the values of α_n and β_n, the fixed point is obtained in more number of iterations for all processes.

Increasing functions

1. The decreasing order of rate of convergence for iterative processes is S, Agarwal *et al*., Picard, Noor, Ishikawa and Mann processes.

2. For initial guess away from the fixed point, the number of iterations increases in each iterative process except the S-iterative process which shows no change. Hence, the closer the initial guess to the fixed point, the quicker the result is achieved.

3. If we increase the values of α_n and β_n, the fixed point is obtained in less number of iterations for all processes except the S-iterative process which shows no change.

Superlinear functions with multiple roots

1. The decreasing order of rate of convergence for iterative processes is Agarwal *et al*., Picard, Noor, Ishikawa and Mann processes.

2. For initial guess away from the fixed point, the number of iterations increases in each iterative process. Hence, the closer the initial guess to the fixed point, the quicker the result is achieved.

3. If we increase the values of α_n and β_n, the fixed point is obtained in less number of iterations for Noor, Ishikawa and Mann processes, while Agarwal *et al*. and S-iterative processes show no change.

Oscillatory functions

1. The Picard process does not converge, Mann and S-iterative processes show equivalence, while the decreasing order of convergence rate of other iterative processes is Agarwal *et al*., Noor, Mann and Ishikawa processes.

2. For initial guess away from the fixed point, Ishikawa, Agarwal *et al*. and S-iterative processes show an increase, while Mann and Noor processes show no change in the number of iterations to converge.

3. The speed of iterative processes depends on α_n and β_n. If we increase the values of α_n and β_n, the fixed point is obtained in more number of iterations for all processes.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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