TRICHOTOMY, STABILITY, AND OSCILLATION OF A FUZZY DIFFERENCE EQUATION

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Received 10 November 2003

We study the trichotomy character, the stability, and the oscillatory behavior of the positive solutions of a fuzzy difference equation.

1. Introduction

Difference equations have already been successfully applied in a number of sciences (for a detailed study of the theory of difference equations and their applications, see [1, 2, 7, 8, 11].

The problem of identifying, modeling, and solving a nonlinear difference equation concerning a real-world phenomenon from experimental input-output data, which is uncertain, incomplete, imprecise, or vague, has been attracting increasing attention in recent years. In addition, nowadays, there is an increasing recognition that for understanding vagueness, a fuzzy approach is required. The effect is the introdution and the study of the fuzzy difference equations (see [3, 4, 13, 14, 15]).

In this paper, we study the trichotomy character, the stability, and the oscillatory behavior of the positive solutions of the fuzzy difference equation

$$x_{n+1} = A + \frac{\sum_{i=1}^{k} c_i x_{n-p_i}}{\sum_{j=1}^{m} d_j x_{n-q_j}}, \tag{1.1}$$

where $k,m \in \{1,2,\dots\}$, A, c_i, d_j, $i \in \{1,2,\dots,k\}$, $j \in \{1,2,\dots,m\}$, are positive fuzzy numbers, p_i, $i \in \{1,2,\dots,k\}$, q_j, $j \in \{1,2,\dots,m\}$, are positive integers such that $p_1 < p_2 < \cdots < p_k$, $q_1 < q_2 < \cdots < q_m$, and the initial values x_i, $i \in \{-\pi, -\pi+1,\dots,0\}$, where

$$\pi = \max\{p_k, q_m\}, \tag{1.2}$$

are positive fuzzy numbers.

Studying a fuzzy difference equation results concerning the behavior of a related family of systems of parametric ordinary difference equations is required. Some necessary results

Advances in Difference Equations 2004:4 (2004) 337–357
2000 Mathematics Subject Classification: 39A10
URL: http://dx.doi.org/10.1155/S1687183904311015

concerning the corresponding family of systems of ordinary difference equations of (1.1) have been proved in [16] and others are given in this paper.

2. Preliminaries

We need the following definitions.

For a set B, we denote by $\overline{B}$ the closure of B. We say that a fuzzy set A, from $\mathbb{R}^+ = (0,\infty)$ into the interval $[0,1]$, is a fuzzy number, if A is normal, convex, upper semicontinuous (see [14]), and the support $\operatorname{supp} A = \overline{\bigcup_{a\in(0,1]}[A]_a} = \overline{\{x : A(x) > 0\}}$ is compact. Then from [12, Theorems 3.1.5 and 3.1.8], the a-cuts of the fuzzy number A, $[A]_a = \{x \in \mathbb{R}^+ : A(x) \geq a\}$, are closed intervals.

We say that a fuzzy number A is positive if $\operatorname{supp} A \subset (0,\infty)$.

It is obvious that if A is a positive real number, then A is a positive fuzzy number and $[A]_a = [A,A]$, $a \in (0,1]$. In this case, we say that A is a trivial fuzzy number.

We say that x_n is a positive solution of (1.1) if x_n is a sequence of positive fuzzy numbers which satisfies (1.1).

A positive fuzzy number x is a positive equilibrium for (1.1) if

$$x = A + \frac{\sum_{i=1}^{k} c_i x}{\sum_{j=1}^{m} d_j x}. \tag{2.1}$$

Let E, H be fuzzy numbers with

$$[E]_a = [E_{l,a}, E_{r,a}], \quad [H]_a = [H_{l,a}, H_{r,a}], \quad a \in (0,1]. \tag{2.2}$$

According to [10] and [13, Lemma 2.3], we have that $\mathrm{MIN}\{E,H\} = E$ if

$$E_{l,a} \leq H_{l,a}, \quad E_{r,a} \leq H_{r,a}, \quad a \in (0,1]. \tag{2.3}$$

Moreover, let c_i, f_i, d_j, g_j, $i = 1,2,\dots,k$, $j = 1,2,\dots,m$, be positive fuzzy numbers such that for $a \in (0,1]$,

$$\begin{gathered}
[c_i]_a = [c_{i,l,a}, c_{i,r,a}], \qquad [f_i]_a = [f_{i,l,a}, f_{i,r,a}], \\
[d_j]_a = [d_{j,l,a}, d_{j,r,a}], \qquad [g_j]_a = [g_{j,l,a}, g_{j,r,a}],
\end{gathered} \tag{2.4}$$

$$E = \frac{\sum_{i=1}^{k} c_i}{\sum_{j=1}^{m} d_j}, \qquad H = \frac{\sum_{i=1}^{k} f_i}{\sum_{j=1}^{m} g_j}. \tag{2.5}$$

We will say that E is less than H and we will write

$$E \prec H \tag{2.6}$$

if

$$\frac{\sum_{i=1}^{k} \sup_{a\in(0,1]} c_{i,r,a}}{\sum_{j=1}^{m} \inf_{a\in(0,1]} d_{j,l,a}} < \frac{\sum_{i=1}^{k} \inf_{a\in(0,1]} f_{i,l,a}}{\sum_{j=1}^{m} \sup_{a\in(0,1]} g_{j,r,a}}. \tag{2.7}$$

In addition, we will say that E is equal to H and we will write

$$E \doteq H \quad \text{if } E \prec H,\ H \prec E, \tag{2.8}$$

which means that for $i = 1,2,\dots,k$, $j = 1,2,\dots,m$, and $a \in (0,1]$,

$$c_{i,l,a} = c_{i,r,a}, \qquad f_{i,l,a} = f_{i,r,a}, \qquad d_{j,l,a} = d_{j,r,a}, \qquad g_{j,l,a} = g_{j,r,a}, \tag{2.9}$$

and so

$$E_{l,a} = E_{r,a} = H_{l,a} = H_{r,a}, \quad a \in (0,1], \tag{2.10}$$

which imples that E, H are equal real numbers.

For the fuzzy numbers E, H, we give the metric (see [9, 17, 18])

$$D(E,H) = \sup \max\{|E_{l,a} - H_{l,a}|, |E_{r,a} - H_{r,a}|\}, \tag{2.11}$$

where sup is taken for all $a \in (0,1]$.

The fuzzy analog of boundedness and persistence (see [5, 6]) is given as follows: we say that a sequence of positive fuzzy numbers x_n persists (resp., is bounded) if there exists a positive number M (resp., N) such that

$$\operatorname{supp} x_n \subset [M,\infty) \quad (\text{resp., } \operatorname{supp} x_n \subset (0,N]), \quad n = 1,2,\dots. \tag{2.12}$$

In addition, we say that x_n is bounded and persists if there exist numbers $M,N \in (0,\infty)$ such that

$$\operatorname{supp} x_n \subset [M,N], \quad n = 1,2,\dots. \tag{2.13}$$

Let x_n be a sequence of positive fuzzy numbers such that

$$[x_n]_a = [L_{n,a}, R_{n,a}], \quad a \in (0,1],\ n = 0,1,\dots, \tag{2.14}$$

and let x be a positive fuzzy number such that

$$[x]_a = [L_a, R_a], \quad a \in (0,1]. \tag{2.15}$$

We say that x_n nearly converges to x with respect to D as $n \to \infty$ if for every $\delta > 0$, there exists a measurable set T, $T \subset (0,1]$, of measure less than δ such that

$$\lim D_T(x_n, x) = 0, \quad \text{as } n \longrightarrow \infty, \tag{2.16}$$

where

$$D_T(x_n, x) = \sup_{a \in (0,1]-T} \{\max\{|L_{n,a} - L_a|, |R_{n,a} - R_a|\}\}. \tag{2.17}$$

If $T = \varnothing$, we say that x_n converges to x with respect to D as $n \to \infty$.

Let X be the set of positive fuzzy numbers. Let $E,H \in X$. From [18, Theorem 2.1], we have that $E_{l,a}$, $H_{l,a}$ (resp., $E_{r,a}$, $H_{r,a}$) are increasing (resp., decreasing) functions on $(0,1]$. Therefore, using the definition of the fuzzy numbers, there exist the Lebesque integrals

$$\int_J \left|E_{l,a} - H_{l,a}\right| da, \qquad \int_J \left|E_{r,a} - H_{r,a}\right| da, \tag{2.18}$$

where $J = (0,1]$. We define the function $D_1 : X \times X \to R^+$ such that

$$D_1(E,H) = \max\left\{\int_J \left|E_{l,a} - H_{l,a}\right| da,\ \int_J \left|E_{r,a} - H_{r,a}\right| da\right\}. \tag{2.19}$$

If $D_1(E,H) = 0$, we have that there exists a measurable set T of measure zero such that

$$E_{l,a} = H_{l,a}, \quad E_{r,a} = H_{r,a} \quad \forall a \in (0,1] - T. \tag{2.20}$$

We consider, however, two fuzzy numbers E, H to be equivalent if there exists a measurable set T of measure zero such that (2.20) hold and if we do not distinguish between equivalence of fuzzy numbers, then X becomes a metric space with metric D_1.

We say that a sequence of positive fuzzy numbers x_n converges to a positive fuzzy number x with respect to D_1 as $n \to \infty$ if

$$\lim D_1(x_n, x) = 0, \quad \text{as } n \longrightarrow \infty. \tag{2.21}$$

We define the fuzzy analog for periodicity (see [11]) as follows.

A sequence $\{x_n\}$ of positive fuzzy numbers x_n is said to be periodic of period p if

$$D(x_{n+p}, x_n) = 0, \quad n = 0, 1, \ldots. \tag{2.22}$$

Suppose that (1.1) has a unique positive equilibrium x. We say that the positive equilibrium x of (1.1) is stable if for every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon)$ such that for every positive solution x_n of (1.1) which satisfies $D(x_{-i}, x) \leq \delta$, $i = 0, 1, \ldots, \pi$, we have $D(x_n, x) \leq \epsilon$ for all $n \geq 0$.

Moreover, we say that the positive equilibrium x of (1.1) is nearly asymptotically stable if it is stable and every positive solution of (1.1) nearly tends to the positive equilibrium of (1.1) with respect to D as $n \to \infty$.

Finally, we give the fuzzy analog of the concept of oscillation (see [11]). Let x_n be a sequence of positive fuzzy numbers and let x be a positive fuzzy number. We say that x_n oscillates about x if for every $n_0 \in \mathbb{N}$, there exist $s, m \in \mathbb{N}$, $s, m \geq n_0$, such that

$$\text{MIN}\{x_m, x\} = x_m, \qquad \text{MIN}\{x_s, x\} = x \tag{2.23}$$

or

$$\text{MIN}\{x_m, x\} = x, \qquad \text{MIN}\{x_s, x\} = x_s. \tag{2.24}$$

3. Main results

Arguing as in [13, 14, 15], we can easily prove the following proposition which concerns the existence and the uniqueness of the positive solutions of (1.1).

Proposition 3.1. *Consider (1.1), where* $k,m \in \{1,2,\dots\}$, A, c_i, d_j, $i \in \{1,2,\dots,k\}$, $j \in \{1,2,\dots,m\}$, *are positive fuzzy numbers, and* p_i, q_j, $i \in \{1,2,\dots,k\}$, $j \in \{1,2,\dots,m\}$, *are positive integers. Then for any positive fuzzy numbers* $x_{-\pi}, x_{-\pi+1},\dots,x_0$, *there exists a unique positive solution* x_n *of (1.1) with initial values* $x_{-\pi}, x_{-\pi+1},\dots,x_0$.

Now, we present conditions so that (1.1) has unbounded solutions.

Proposition 3.2. *Consider (1.1), where* $k,m \in \{1,2,\dots\}$, A, c_i, d_j, $i \in \{1,2,\dots,k\}$, $j \in \{1,2,\dots,m\}$, *are positive fuzzy numbers, and* p_i, $i \in \{1,2,\dots,k\}$, q_j, $j \in \{1,2,\dots,m\}$, *are positive integers. If*

$$A \prec G, \quad G = \frac{\sum_{i=1}^{k} c_i}{\sum_{j=1}^{m} d_j}, \tag{3.1}$$

then (1.1) has unbounded solutions.

Proof. Let

$$[A]_a = [A_{l,a}, A_{r,a}], \quad a \in (0,1]. \tag{3.2}$$

From (2.4) and (3.2) and since A, c_i, d_j, $i = 1,2,\dots,k$, $j = 1,2,\dots,m$, are positive fuzzy numbers, there exist positive real numbers B, C, a_i, e_i, h_j, b_j, $i = 1,2,\dots,k$, $j = 1,2,\dots,m$, such that

$$\begin{aligned} B &= \inf_{a\in(0,1]} A_{l,a}, \qquad C = \sup_{a\in(0,1]} A_{r,a}, \qquad a_i = \inf_{a\in(0,1]} c_{i,l,a}, \\ e_i &= \sup_{a\in(0,1]} c_{i,r,a}, \qquad h_j = \inf_{a\in(0,1]} d_{j,l,a}, \qquad b_j = \sup_{a\in(0,1]} d_{j,r,a}. \end{aligned} \tag{3.3}$$

Let x_n be a positive solution of (1.1) such that (2.14) hold and the initial values x_i, $i = -\pi, -\pi+1,\dots,0$, are positive fuzzy numbers which satisfy

$$[x_i]_a = [L_{i,a}, R_{i,a}], \quad i = -\pi, -\pi+1,\dots,0,\ a \in (0,1] \tag{3.4}$$

and for a fixed $\bar{a} \in (0,1]$, the relations

$$R_{i,\bar{a}} > \frac{Z^2}{W - C}, \quad L_{i,\bar{a}} < W, \quad i = -\pi, -\pi+1,\dots,0, \tag{3.5}$$

are satisfied, where

$$Z = \frac{\sum_{i=1}^{k} e_i}{\sum_{j=1}^{m} h_j}, \qquad W = \frac{\sum_{i=1}^{k} a_i}{\sum_{j=1}^{m} b_j}. \tag{3.6}$$

Using [15, Lemma 1], we can easily prove that $L_{n,a}$, $R_{n,a}$ satisfy the family of systems of parametric ordinary difference equations

$$\begin{aligned} L_{n+1,a} &= A_{l,a} + \frac{\sum_{i=1}^{k} c_{i,l,a} L_{n-p_i,a}}{\sum_{j=1}^{m} d_{j,r,a} R_{n-q_j,a}}, \\ R_{n+1,a} &= A_{r,a} + \frac{\sum_{i=1}^{k} c_{i,r,a} R_{n-p_i,a}}{\sum_{j=1}^{m} d_{j,l,a} L_{n-q_j,a}}, \end{aligned} \qquad n = 0,1,\ldots. \tag{3.7}$$

Since (3.1) holds, it is obvious that

$$A_{l,\bar{a}} < \frac{\sum_{i=1}^{k} c_{i,r,\bar{a}}}{\sum_{j=1}^{m} d_{j,l,\bar{a}}}. \tag{3.8}$$

Using (3.8) and applying [16, Proposition 1] to the system (3.7) for $a = \bar{a}$, we have that

$$\lim_{n\to\infty} L_{n,\bar{a}=A_{l,\bar{a}}}, \qquad \lim_{n\to\infty} R_{n,\bar{a}} = \infty. \tag{3.9}$$

Therefore, from (3.9), the solution x_n of (1.1) which satisfies (3.4) and (3.5) is unbounded. □

Remark 3.3. From the proof of Proposition 3.2, it is obvious that (1.1) has unbounded solutions if there exists at least one $a \in (0,1]$ such that (3.8) holds.

In the following proposition, we study the boundedness and persistence of the positive solutions of (1.1).

Proposition 3.4. *Consider (1.1), where $k,m \in \{1,2,\ldots\}$, A, c_i, d_j, $i \in \{1,2,\ldots,k\}$, $j \in \{1,2,\ldots,m\}$, are positive fuzzy numbers, and p_i, $i \in \{1,2,\ldots,k\}$, q_j, $j \in \{1,2,\ldots,m\}$, are positive integers. If either*

$$A \doteq G \tag{3.10}$$

or

$$G \prec A \tag{3.11}$$

holds, then every positive solution of (1.1) is bounded and persists.

Proof. Firstly, suppose that (3.10) is satisfied; then A, c_i, d_j, $i = 1,2,\ldots,k$, $j = 1,2,\ldots,m$, are positive real numbers. Hence, for $i = 1,2,\ldots,k$, $j = 1,2,\ldots,m$, we get

$$A = A_{l,a} = A_{r,a}, \quad c_i = c_{i,l,a} = c_{i,r,a}, \quad d_j = d_{j,l,a} = d_{j,r,a}, \quad a \in (0,1], \tag{3.12}$$

$$A = \frac{\sum_{i=1}^{k} c_i}{\sum_{j=1}^{m} d_j}. \tag{3.13}$$

Let x_n be a positive solution of (1.1) such that (2.14) hold and let x_i, $i = -\pi, -\pi + 1, \dots, 0$, be the positive initial values of x_n such that (3.4) hold. Then there exist positive numbers T_i, S_i, $i = -\pi, -\pi+1, \dots, 0$, such that

$$T_i \leq L_{i,a}, R_{i,a} \leq S_i, \quad i = -\pi, -\pi+1, \dots, 0. \tag{3.14}$$

Let (y_n, z_n) be the positive solution of the system of ordinary difference equations

$$y_{n+1} = A + \frac{\sum_{i=1}^{k} c_i y_{n-p_i}}{\sum_{j=1}^{m} d_j z_{n-q_j}}, \qquad z_{n+1} = A + \frac{\sum_{i=1}^{k} c_i z_{n-p_i}}{\sum_{j=1}^{m} d_j y_{n-q_j}}, \tag{3.15}$$

with initial values (y_i, z_i), $i = -\pi, -\pi+1, \dots, 0$, such that $y_i = T_i$, $z_i = S_i$, $i = -\pi, -\pi + 1, \dots, 0$. Then from (3.14) and (3.15), we can easily prove that

$$y_1 \leq L_{1,a}, \quad R_{1,a} \leq z_1, \quad a \in (0,1], \tag{3.16}$$

and working inductively, we take

$$y_n \leq L_{n,a}, \quad R_{n,a} \leq z_n, \quad n = 1,2,\dots,\ a \in (0,1]. \tag{3.17}$$

Since from (3.13) and [16, Proposition 3], (y_n, z_n) is bounded and persists, from (3.17), it is obvious that x_n is also bounded and persists.

Now, suppose that (3.11) holds; then

$$B > Z, \qquad C > W. \tag{3.18}$$

We concider the system of ordinary difference equations

$$y_{n+1} = B + \frac{\sum_{i=1}^{k} a_i y_{n-p_i}}{\sum_{j=1}^{m} b_j z_{n-q_j}}, \qquad z_{n+1} = C + \frac{\sum_{i=1}^{k} e_i z_{n-p_i}}{\sum_{j=1}^{m} h_j y_{n-q_j}}, \tag{3.19}$$

where B, C, a_i, e_i, b_j, h_j, $i = 1,2,\dots,k$, $j = 1,2,\dots,m$, are defined in (3.3).

Let (y_n, z_n) be a solution of (3.19) with initial values (y_i, z_i), $i = -\pi, -\pi+1, \dots, 0$, such that $y_i = T_i$, $z_i = S_i$, $i = -\pi, -\pi+1, \dots, 0$, where T_i, S_i, $i = -\pi, -\pi+1, \dots, 0$, are defined in (3.14). Arguing as above, we can prove that (3.17) holds. Since from (3.18) and [16, Proposition 3], (y_n, z_n) is bounded and persists, then from (3.17), it is obvious that, x_n is also bounded and persists. This completes the proof of the proposition. □

In what follows, we need the following lemmas.

LEMMA 3.5. *Let r_i, s_j, $i = 1,2,\dots,k$, $j = 1,2,\dots,m$, be positive integers such that*

$$(r_1, r_2, \dots, r_k, s_1, s_2, \dots, s_m) = 1, \tag{3.20}$$

where $(r_1, r_2, \dots, r_k, s_1, s_2, \dots, s_m)$ is the greatest common divisor of the integers r_i, s_j, $i = 1,2, \dots, k$, $j = 1,2,\dots,m$. Then the following statements are true.

(I) *There exists an even positive integer w_1 such that for any nonnegative integer p, there exist nonnegative integers α_{ip}, β_{jp}, $i = 1,2,\dots,k$, $j = 1,2,\dots,m$, such that*

$$\sum_{i=1}^{k} \alpha_{ip} r_i + \sum_{j=1}^{m} \beta_{jp} s_j = w_1 + 2p, \quad p = 0,1,\dots, \tag{3.21}$$

where $\sum_{j=1}^{m} \beta_{jp}$ is an even integer.

(II) *Suppose that all r_i, $i = 1,2,\dots,k$, are not even and all s_j, $j = 1,2,\dots,m$, are not odd integers. Then there exists an odd positive integer w_2 such that for any nonnegative integer p, there exist nonnegative integers γ_{ip}, δ_{jp}, $i = 1,2,\dots,k$, $j = 1,2,\dots,m$, such that*

$$\sum_{i=1}^{k} \gamma_{ip} r_i + \sum_{j=1}^{m} \delta_{jp} s_j = w_2 + 2p, \quad p = 0,1,\dots, \tag{3.22}$$

where $\sum_{j=1}^{m} \delta_{jp}$ is an even integer.

(III) *Suppose that all r_i, $i = 1,2,\dots,k$, are not even and all s_j, $j = 1,2,\dots,m$, are not odd integers. Then there exists an even positive integer w_3 such that for any nonnegative integer p, there exist nonnegative integers ϵ_{ip}, ξ_{jp}, $i = 1,2,\dots,k$, $j = 1,2,\dots,m$, such that*

$$\sum_{i=1}^{k} \epsilon_{ip} r_i + \sum_{j=1}^{m} \xi_{jp} s_j = w_3 + 2p, \quad p = 0,1,\dots, \tag{3.23}$$

where $\sum_{j=1}^{m} \xi_{jp}$ is an odd integer.

(IV) *There exists an odd positive integer w_4 such that for any nonnegative integer p, there exist nonnegative integers λ_{ip}, μ_{jp}, $i = 1,2,\dots,k$, $j = 1,2,\dots,m$, such that*

$$\sum_{i=1}^{k} \lambda_{ip} r_i + \sum_{j=1}^{m} \mu_{jp} s_j = w_4 + 2p, \quad p = 0,1,\dots, \tag{3.24}$$

where $\sum_{j=1}^{m} \mu_{jp}$ is an odd integer.

Proof. (I) Since(3.20) holds, there exist integers η_i, ι_j, $i = 1,2,\dots,k$, $j = 1,2,\dots,m$, such that

$$\sum_{i=1}^{k} \eta_i r_i + \sum_{j=1}^{m} \iota_j s_j = 1. \tag{3.25}$$

If for any real number a, we denote by $[a]$ the integral part of a, we set for $i = 2,3,\dots,k$, $j = 1,2,\dots,m$,

$$\begin{aligned} \alpha_{1p} &= 2p\eta_1 + 2\sum_{i=2}^{k} r_i + 2\sum_{j=1}^{m} s_j - 2\sum_{i=2}^{k} g_{ip} r_i - 2\sum_{j=1}^{m} h_{jp} s_j, \\ \alpha_{ip} &= 2p\eta_i + 2g_{ip} r_1, \qquad \beta_{jp} = 2p\iota_j + 2h_{jp} r_1, \end{aligned} \tag{3.26}$$

where

$$g_{ip} = \left[\frac{-p\eta_i}{r_1}\right] + 1, \qquad h_{jp} = \left[\frac{-p\iota_j}{r_1}\right] + 1, \quad i = 2,3,\ldots,k,\ j = 1,2,\ldots,m. \tag{3.27}$$

Therefore, from (3.25) and (3.26), we can easily prove that α_{ip}, β_{jp}, $i = 1,2,\ldots,k$, $j = 1,2,\ldots,m$, which are defined in (3.26), are positive integers satisfying (3.21) for

$$w_1 = 2r_1\left(\sum_{i=2}^{k} r_i + \sum_{j=1}^{m} s_j\right) \tag{3.28}$$

and $\sum_{j=1}^{m} \beta_{jp}$ is an even number.

(II) Firstly, suppose that one of r_i, $i = 1,2,\ldots,k$, is an odd positive integer and without loss of generality, let r_1 be an odd positive integer. Relation (3.22) follows immediately if we set for $i = 2,\ldots,k$ and for $j = 1,2,\ldots,m$,

$$\gamma_{1p} = \alpha_{1p} + 1, \qquad \gamma_{ip} = \alpha_{ip}, \qquad \delta_{jp} = \beta_{jp}, \qquad w_2 = w_1 + r_1. \tag{3.29}$$

Now, suppose that r_i, $i = 1,2,\ldots,k$, are even positive integers; then from (3.20), one of s_j, $j = 1,2,\ldots,m$, is an odd positive integer and from the hypothesis, one of s_j, $j = 1,2,\ldots,m$, is an even positive integer. Without loss of generality, let s_1 be an odd positive integer and s_2 be an even positive integer. Relation (3.22) follows immediately if we set for $i = 1,2,\ldots,k$ and for $j = 3,\ldots,m$,

$$\gamma_{ip} = \alpha_{ip}, \qquad \delta_{1p} = \beta_{1p} + 1, \qquad \delta_{2p} = \beta_{2p} + 1, \qquad \delta_{jp} = \beta_{jp}, \qquad w_2 = w_1 + s_1 + s_2. \tag{3.30}$$

(III) Firstly, suppose that one of s_j, $j = 1,2,\ldots,m$, is an even positive integer and without loss of generality, let s_1 be an even positive integer. Relation (3.23) follows immediately if we set for $i = 1,2,\ldots,k$ and $j = 2,\ldots,m$,

$$\epsilon_{ip} = \alpha_{ip}, \qquad \xi_{1p} = \beta_{1p} + 1, \qquad \xi_{jp} = \beta_{jp}, \qquad w_3 = w_1 + s_1. \tag{3.31}$$

Now, suppose that s_j, $j = 1,2,\ldots,m$, are odd positive integers; then from the hypothesis, at least one of r_i, $i = 1,2,\ldots,k$, is an odd positive integer, and without loss of generality, let r_1 be an odd integer. Relation (3.23) follows immediately if we set for $i = 2,\ldots,k$, $j = 2,3,\ldots,m$,

$$\epsilon_{1p} = \alpha_{1p} + 1, \qquad \epsilon_{ip} = \alpha_{ip}, \qquad \delta_{1p} = \beta_{1p} + 1, \qquad \delta_{jp} = \beta_{jp}, \qquad w_3 = w_1 + s_1 + r_1. \tag{3.32}$$

(IV) Firstly, suppose that at least one of s_j, $j = 1,2,\ldots,m$, is an odd positive integer and without loss of generality, let s_1 be an odd positive integer. Relation (3.24) follows immediately if we set for $i = 1,2,\ldots,k$, $j = 2,3,\ldots,m$,

$$\lambda_{ip} = \alpha_{ip}, \qquad \mu_{1p} = \beta_{1p} + 1, \qquad \mu_{jp} = \beta_{jp}, \qquad w_4 = w_1 + s_1. \tag{3.33}$$

Now, suppose that s_j, $j = 1,2,\dots,m$, are even positive integers; then from (3.20), at least one of r_i, $i = 1,2,\dots,k$, is an odd positive integer, and without loss of generality, let r_1 be an odd positive integer. Relation (3.24) follows immediately if we set for $i = 2,3,\dots,k$, $j = 2,3,\dots,m$,

$$\lambda_{1p} = \alpha_{1p} + 1, \qquad \lambda_{ip} = \alpha_{ip}, \qquad \mu_{1p} = \beta_{1p} + r_1, \qquad \mu_{jp} = \beta_{jp}, \qquad w_4 = w_1 + r_1(s_1 + 1). \tag{3.34}$$

This completes the proof of the lemma. □

Lemma 3.6. *Consider system (3.19), where B, C are positive constants such that*

$$B = \frac{\sum_{i=1}^{k} e_i}{\sum_{j=1}^{m} h_j}, \qquad C = \frac{\sum_{i=1}^{k} a_i}{\sum_{j=1}^{m} b_j}. \tag{3.35}$$

Then the following statements are true.

(I) *Let r be a common divisor of the integers $p_i + 1$, $q_j + 1$, $i = 1,2,\dots,k$, $j = 1,2,\dots,m$, such that*

$$p_i + 1 = r r_i, \quad i = 1,2,\dots,k, \qquad q_j + 1 = r s_j, \quad j = 1,2,\dots,m; \tag{3.36}$$

then system (3.19) has periodic solutions of prime period r. Moreover, if all r_i, $i = 1,2,\dots,k$, (resp., s_j, $j = 1,2,\dots,m$) are even (resp., odd) positive integers, then system (3.19) has periodic solutions of prime period $2r$.

(II) *Let r be the greatest common divisor of the integers $p_i + 1$, $q_j + 1$, $i = 1,2,\dots,k$, $j = 1,2,\dots,m$, such that (3.36) hold; then if all r_i, $i = 1,2,\dots,k$, (resp., s_j, $j = 1,2,\dots,m$) are even (resp., odd) positive integers, every positive solution of (3.19) tends to a periodic solution of period $2r$; otherwise, every positive solution of (3.19) tends to a periodic solution of period r.*

Proof. (I) From relations (3.35), (3.36), and [16, Proposition 2], system (3.19) has periodic solutions of prime period r.

Now, we prove that system (3.19) has periodic solutions of prime period $2r$, if all r_i, $i = 1,2,\dots,k$, (resp., s_j, $j = 1,2,\dots,m$) are even (resp., odd) positive integers.

Suppose first that $p_k < q_m$. Let (y_n, z_n) be a positive solution of (3.19) with initial values satisfying

$$\begin{gathered} y_{-rs_m+2r\lambda+\zeta} = y_{-r+\zeta}, \qquad z_{-rs_m+2r\lambda+\zeta} = z_{-r+\zeta}, \\ y_{-rs_m+2r\nu+r+\zeta} = y_{-2r+\zeta}, \qquad z_{-rs_m+2r\nu+r+\zeta} = z_{-2r+\zeta}, \\ \lambda = 0,1,\dots,\frac{s_m - 1}{2}, \qquad \nu = 0,1,\dots,\frac{s_m - 3}{2}, \qquad \zeta = 1,2,\dots,r, \end{gathered} \tag{3.37}$$

and, in addition, for $\zeta = 1,2,\dots,r$,

$$y_{-2r+\zeta} > B, \qquad y_{-r+\zeta} > B, \qquad z_{-r+\zeta} = \frac{C y_{-2r+\zeta}}{y_{-2r+\zeta} - B}, \qquad z_{-2r+\zeta} = \frac{C y_{-r+\zeta}}{y_{-r+\zeta} - B}. \tag{3.38}$$

From (3.19), (3.35), (3.36), (3.37), and (3.38), we get for $\zeta = 1,2,\dots,r$,

$$\begin{aligned} y_\zeta &= B + C\frac{y_{-2r+\zeta}}{z_{-r+\zeta}} = y_{-2r+\zeta}, \qquad z_\zeta = C + B\frac{z_{-2r+\zeta}}{y_{-r+\zeta}} = z_{-2r+\zeta}, \\ y_{r+\zeta} &= B + C\frac{y_{-r+\zeta}}{z_{-2r+\zeta}} = y_{-r+\zeta}, \qquad z_{r+\zeta} = C + B\frac{z_{-r+\zeta}}{y_{-2r+\zeta}} = z_{-r+\zeta}. \end{aligned} \tag{3.39}$$

Let a $\nu \in \{2,3,\dots\}$. Suppose that for all $u = 1,2,\dots,\nu-1$ and $\zeta = 1,2,\dots,r$, we have

$$y_{2ur+\zeta} = y_{-2r+\zeta}, \qquad z_{2ur+\zeta} = z_{-2r+\zeta}, \qquad y_{2ur+r+\zeta} = y_{-r+\zeta}, \qquad z_{2ur+r+\zeta} = z_{-r+\zeta}. \tag{3.40}$$

Then from (3.19), (3.35)–(3.40), we get for $\zeta = 1,2,\dots,r$,

$$y_{2\nu r+\zeta} = B + C\frac{y_{-2r+\zeta}}{z_{-r+\zeta}} = y_{-2r+\zeta}. \tag{3.41}$$

Similarly, we can prove that for $\zeta = 1,2,\dots,r$,

$$z_{2\nu r+\zeta} = z_{-2r+\zeta}, \qquad y_{2\nu r+r+\zeta} = y_{-r+\zeta}, \qquad z_{2\nu r+r+\zeta} = z_{-r+\zeta}. \tag{3.42}$$

Therefore, from (3.39)–(3.42), we have that system (3.19) has periodic solutions of period $2r$.

Now, suppose that $q_m < p_k$. Let (y_n, z_n) be a positive solution of (3.19) such that the initial values satisfy relations (3.38) and for $\omega = 0,1,\dots,r_k/2-1$, $\theta = 1,2,\dots,2r$,

$$y_{-rr_k+2r\omega+\theta} = y_{-2r+\theta}, \qquad z_{-rr_k+2r\omega+\theta} = z_{-2r+\theta}. \tag{3.43}$$

Then arguing as above, we can easily prove that (y_n, z_n) is a periodic solution of period $2r$. This completes the proof of statement (I).

(II) Now, we prove that every positive solution of system (3.19) tends to a periodic solution of period κr, where

$$\kappa = \begin{cases} 2 & \text{if } r_i,\ i = 1,2,\dots,k, \text{ are even, } s_j,\ j = 1,2,\dots,m, \text{ are odd,} \\ 1 & \text{otherwise.} \end{cases} \tag{3.44}$$

Let (y_n, z_n) be an arbitrary positive solution of (3.19). We prove that there exist the

$$\lim_{n\to\infty} y_{\kappa nr+i} = \epsilon_i, \quad i = 0,1,\dots,\kappa r-1. \tag{3.45}$$

We fix a $\tau \in \{0,1,\dots,\kappa r-1\}$. Since from [16, Proposition 3], the solution (y_n, z_n) is bounded and persists, we have

$$\begin{aligned} \liminf_{n\to\infty} y_{\kappa nr+\tau} &= l_\tau \geq B, \qquad \liminf_{n\to\infty} z_{\kappa nr+\tau} = m_\tau \geq C, \\ \limsup_{n\to\infty} y_{\kappa nr+\tau} &= L_\tau < \infty, \qquad \limsup_{n\to\infty} z_{\kappa nr+\tau} = M_\tau < \infty. \end{aligned} \tag{3.46}$$

Therefore, from relations (3.19), (3.35), and (3.46), we take

$$m_\tau = \frac{CL_\tau}{L_\tau - B}, \qquad l_\tau = \frac{BM_\tau}{M_\tau - C}. \tag{3.47}$$

We prove that (3.45) is true for $i = \tau$. Suppose on the contrary that $l_\tau < L_\tau$. Then from (3.46), there exists an $\epsilon > 0$ such that

$$L_\tau > l_\tau + \epsilon > B + \epsilon. \tag{3.48}$$

In view of (3.46), there exists a sequence n_μ, $\mu = 1,2,\ldots$, such that

$$\begin{gathered}\lim_{\mu\to\infty} y_{\kappa r n_\mu+\tau} = L_\tau, \qquad \lim_{\mu\to\infty} y_{r(\kappa n_\mu - r_i)+\tau} = T_{r_i,\tau} \le L_\tau,\\ \lim_{\mu\to\infty} z_{r(\kappa n_\mu - s_j)+\tau} = S_{s_j,\tau} \ge m_\tau.\end{gathered} \tag{3.49}$$

In view of (3.19), (3.35), (3.46), (3.47), and (3.49), we take

$$L_\tau = B + \frac{\sum_{i=1}^{k} a_i T_{r_i,\tau}}{\sum_{j=1}^{m} b_j S_{s_j,\tau}} \le B + \frac{CL_\tau}{m_\tau} = L_\tau \tag{3.50}$$

and obviously, we have that

$$\begin{aligned} T_{r_i,\tau} &= L_\tau, \quad i = 1,2,\ldots,k,\\ S_{s_j,\tau} &= m_\tau, \quad j = 1,2,\ldots,m.\end{aligned} \tag{3.51}$$

In addition, using (3.19), (3.35), (3.46), (3.47), and (3.51), for $\kappa = 2$, from statements (I) and (IV) of Lemma 3.5 and arguing as above, we take for $\gamma = 0,1,\ldots,$

$$\lim_{\mu\to\infty} y_{r(2n_\mu - w_1 - 2\gamma)+\tau} = L_\tau, \qquad \lim_{\mu\to\infty} z_{r(2n_\mu - w_1 - s_1 - 2\gamma)+\tau} = m_\tau, \tag{3.52}$$

and for $\kappa = 1$ and from all the statements of Lemma 3.5,

$$\begin{aligned} &\lim_{\mu\to\infty} y_{r(n_\mu - w_1 - 2\gamma)+\tau} = L_\tau, \qquad \lim_{\mu\to\infty} y_{r(n_\mu - w_2 - 2\gamma)+\tau} = L_\tau,\\ &\lim_{\mu\to\infty} z_{r(n_\mu - w_3 - 2\gamma)+\tau} = m_\tau, \qquad \lim_{\mu\to\infty} z_{r(n_\mu - w_4 - 2\gamma)+\tau} = m_\tau,\end{aligned} \tag{3.53}$$

w_1, w_2, w_3, w_4 are defined in Lemma 3.5.

Let a $\sigma_\kappa \in \{0,1,\ldots,(3-\kappa)\phi\}$, $\phi = \max\{r_k, s_m\}$. Suppose first that $\kappa = 2$. Then in view of (3.19), there exist positive integers p, q and a continuous function $F_{\sigma_2} : \mathbb{R}\times\mathbb{R}\times\cdots\times\mathbb{R}\to\mathbb{R}$ such that

$$y_{r(2n_\mu + 2\sigma_2)+\tau} = B + F_{\sigma_2}(\zeta_{n_\mu,0},\ldots,\zeta_{n_\mu,p},\xi_{n_\mu,0},\ldots,\xi_{n_\mu,q}), \tag{3.54}$$

where for $i = 0,1,\ldots,p$, $j = 0,1,\ldots,q$,

$$\zeta_{n_\mu,i} = y_{r(2n_\mu - w_1 - 2i)+\tau}, \qquad \xi_{n_\mu,j} = z_{r(2n_\mu - w_1 - s_1 - 2j)+\tau}. \tag{3.55}$$

If $\kappa = 1$, there exist positive integers v_1, v_2, v_3, v_4 and a continuous function $G_{\sigma_1} : \mathbb{R}\times\mathbb{R}\times\cdots\times\mathbb{R}\to\mathbb{R}$ such that

$$y_{r(n_\mu+\sigma_1)+\tau} = B + G_{\sigma_1}(\zeta_{n_\mu,0},\ldots,\zeta_{n_\mu,v_1},\bar{\zeta}_{n_\mu,0},\ldots,\bar{\zeta}_{n_\mu,v_2},\xi_{n_\mu,0},\ldots,\xi_{n_\mu,v_3},\bar{\xi}_{n_\mu,0},\ldots,\bar{\xi}_{n_\mu,v_4}), \tag{3.56}$$

where for $i = 0,1,\dots,v_1$, $\bar{i} = 0,1,\dots,v_2$, $j = 0,1,\dots,v_3$, and $\bar{j} = 0,1,\dots,v_4$,

$$\begin{aligned} \zeta_{n_\mu,i} &= y_{r(n_\mu - w_1 - 2i)+\tau}, & \bar{\zeta}_{n_\mu,\bar{i}} &= y_{r(n_\mu - w_2 - 2\bar{i})+\tau}, \\ \xi_{n_\mu,j} &= z_{r(n_\mu - w_3 - 2j)+\tau}, & \bar{\xi}_{n_\mu,\bar{j}} &= z_{r(n_\mu - w_4 - 2\bar{j})+\tau}. \end{aligned} \tag{3.57}$$

Therefore, from (3.47), (3.52), (3.53), (3.54), and (3.56), it follows that

$$\lim_{\mu\to\infty} y_{r(\kappa n_\mu + \kappa\sigma_\kappa)+\tau} = B + \frac{CL_\tau}{m_\tau} = L_\tau. \tag{3.58}$$

Using the same argument to prove (3.58) and using (3.19), we can easily prove that for $i = 1,2,\dots,k$, $j = 1,2,\dots,m$,

$$\lim_{\mu\to\infty} y_{r(\kappa n_\mu + \kappa\sigma_\kappa - r_i)+\tau} = L_\tau, \qquad \lim_{\mu\to\infty} z_{r(\kappa n_\mu + \kappa\sigma_\kappa - s_j)+\tau} = m_\tau. \tag{3.59}$$

Therefore, if $\delta = \epsilon(m_\tau - C)/(L_\tau - \epsilon - B)$, then in view of (3.19), (3.47), (3.58), and (3.59), there exists a $\mu_0 \in \{1,2,\dots\}$ such that for $j = 1,2,\dots,m$,

$$z_{r(\kappa n_{\mu_0} + 2\phi + \kappa - s_j)+\tau} \le C + \frac{B(m_\tau + \delta)}{L_\tau - \epsilon} = m_\tau + \delta \tag{3.60}$$

and so from (3.19), (3.47), (3.48), (3.58), (3.59), and (3.60), we get

$$y_{r(\kappa n_{\mu_0} + 2\phi + \kappa)+\tau} \ge B + \frac{C(L_\tau - \epsilon)}{m_\tau + \delta} = L_\tau - \epsilon > l_\tau. \tag{3.61}$$

Using (3.19), (3.47), (3.48), (3.58), (3.59), and (3.61) and working inductively, we can easily prove that

$$y_{r(\kappa n_{\mu_0} + 2\phi + \kappa\omega)+\tau} \ge L_\tau - \epsilon > l_\tau, \quad \omega = 2,3,\dots, \tag{3.62}$$

which is a contradiction since $\liminf_{n\to\infty} y_{\kappa rn+\tau} = l_\tau$. Therefore, since τ is an arbitrary number such that $\tau \in \{0,1,\dots,\kappa r - 1\}$, relations (3.45) are satisfied.

Moreover, from (3.19) and (3.47), we have that

$$\lim_{n\to\infty} z_{\kappa nr+i} = \xi_i, \quad i = 0,1,\dots,\kappa r - 1. \tag{3.63}$$

This completes the proof of the lemma. □

In the next proposition, we study the periodicity of the positive solutions of (1.1).

PROPOSITION 3.7. *Consider (1.1), where $k,m \in \{1,2,\dots\}$, A, c_i, d_j, $i \in \{1,2,\dots,k\}$, $j \in \{1,2,\dots,m\}$, are positive fuzzy numbers, and p_i, $i \in \{1,2,\dots,k\}$, q_j, $j \in \{1,2,\dots,m\}$, are positive integers. If (3.10) holds and r is a common divisor of the integers $p_i + 1$, $q_j + 1$, $i = 1,2,\dots,k$, $j = 1,2,\dots,m$, then (1.1) has periodic solutions of prime period r. Moreover, if r_i, $i = 1,2,\dots,k$, (resp., s_j, $j = 1,2,\dots,m$)—r_i, s_j are defined in (3.36)—are even (resp., odd) integers, then (1.1) has periodic solutions of prime period $2r$.*

Proof. From (3.10), we have that A, c_i, $i = 1,2,\dots,k$, $d_j, j = 1,2,\dots,m$, are positive real numbers such that (3.12) and (3.13) hold. We consider functions $L_{i,a}, R_{i,a}$, $i = -\pi, -\pi + 1,\dots,0$, such that for $\lambda = 0,1,\dots,\phi - 1$, $\theta = 1,2,\dots,r$, and $a \in (0,1]$,

$$L_{-r\phi+r\lambda+\theta,a} = L_{-r+\theta,a}, \qquad R_{-r\phi+r\lambda+\theta,a} = R_{-r+\theta,a}, \tag{3.64}$$

the functions $L_{w,a}$, $w = -r+1, -r+2,\dots,0$, are increasing, left continuous, and for all $w = -r+1, -r+2,\dots,0$, we have

$$A + \epsilon < L_{w,a} < 2A, \qquad R_{w,a} = \frac{AL_{w,a}}{L_{w,a} - A}, \tag{3.65}$$

where ϵ is a positive number such that $\epsilon < A$. Using (3.65) and since the functions $L_{w,a}$, $w = -r+1, -r+2,\dots,0$, are increasing, if $a_1, a_2 \in (0,1]$, $a_1 \le a_2$, we get

$$AL_{w,a_1}L_{w,a_2} - A^2 L_{w,a_1} \ge AL_{w,a_1}L_{w,a_2} - A^2 L_{w,a_2} \tag{3.66}$$

which implies that $R_{w,a}$, $w = -r+1, -r+2,\dots,0$, are decreasing functions. Moreover, from (3.65), we get

$$L_{w,a} \le R_{w,a}, \qquad A + \epsilon \le L_{w,a}, R_{w,a} \le \frac{2A^2}{\epsilon}, \tag{3.67}$$

and so from [18, Theorem 2.1], $(L_{w,a}, R_{w,a})$, $w = -r+1, -r+2,\dots,0$, determine the fuzzy numbers x_w, $w = -r+1, -r+2,\dots,0$, such that $[x_w]_a = [L_{w,a}, R_{w,a}]$, $w = -r+1, -r+2,\dots,0$. Let x_n be a positive solution of (1.1) which satisfies (2.14) and let the initial values be positive fuzzy numbers such that (3.4) hold and the functions $L_{i,a}, R_{i,a}$, $i = -\pi, -\pi+1,\dots,0$, $a \in (0,1]$, are defined in (3.64), (3.65); $L_{i,a}$, $i = -\pi, -\pi+1,\dots,0$, $a \in (0,1]$, are increasing and left continuous. Then from [16, Proposition 2], we have that for any $a \in (0,1]$, the system given by (3.7), (3.12), and (3.13) has periodic solutions of prime period r, which means that there exists solution $(L_{n,a}, R_{n,a})$, $a \in (0,1]$, of the system such that

$$L_{n+r,a} = L_{n,a}, \quad R_{n+r,a} = R_{n,a}, \quad a \in (0,1]. \tag{3.68}$$

Therefore, from (2.22) and (3.68), we have that (1.1) has periodic solutions of prime period r.

Now, suppose that r_i, $i = 1,2,\dots,k$, (resp., s_i, $j = 1,2,\dots,m$) are even (resp., odd) integers. We consider the functions $L_{i,a}, R_{i,a}$, $i = -\pi, -\pi+1,\dots,0$, such that analogous relations (3.37), (3.38), and (3.43) hold, $L_{w,a}$, $w = -r+1,\dots,0$, are increasing, left continuous functions, and the first relation of (3.65) holds. Arguing as above, the solution x_n of (1.1) with initial values x_i, $i = -\pi, -\pi+1,\dots,0$, satisfying (3.4), where $L_{i,a}, R_{i,a}$, $i = -\pi, -\pi+1,\dots,0$, are defined above, is a periodic solution of prime period $2r$. □

In the following proposition, we study the convergence of the positive solutions of (1.1).

PROPOSITION 3.8. *Consider (1.1), where $k,m \in \{1,2,\dots\}$, A,c_i,d_j, $i \in \{1,2,\dots,k\}$, $j \in \{1,2,\dots,m\}$, are positive fuzzy numbers, and p_i, $i \in \{1,2,\dots,k\}$, q_j, $j \in \{1,2,\dots,m\}$, are positive integers. Then the following statements are true.*

(i) *If (3.11), holds, then (1.1) has a unique positive equilibrium x and every positive solution of (1.1) nearly converges to the unique positive equilibrium x with respect to D as $n \to \infty$ and converges to x with respect to D_1 as $n \to \infty$.*

(ii) *If (3.10) is satisfied and r is the greatest common divisor of the integers p_i+1, q_j+1, $i=1,2,\dots,k$, $j=1,2,\dots,m$, such that (3.36) holds, then every positive solution of (1.1) nearly converges to a period κr solution of (1.1) with respect to D as $n \to \infty$ and converges to a period κr solution of (1.1) with respect to D_1 as $n \to \infty$; κ is defined in (3.44).*

Proof. (i) Let x_n be a positive solution of (1.1) which satisfies (2.14). Since (3.7) and (3.11) hold, we can apply [16, Proposition 4] and we have that for any $a \in (0,1]$, there exist the $\lim_{n\to\infty} L_{n,a}$, $\lim_{n\to\infty} R_{n,a}$, and

$$\lim_{n\to\infty} L_{n,a} = L_a, \quad \lim_{n\to\infty} R_{n,a} = R_a, \quad a \in (0,1], \tag{3.69}$$

where

$$\begin{aligned} L_a &= \frac{A_{l,a}A_{r,a} - C_a D_a}{A_{r,a} - C_a}, & R_a &= \frac{A_{l,a}A_{r,a} - C_a D_a}{A_{l,a} - D_a}, \\ C_a &= \frac{\sum_{i=1}^{k} c_{i,l,a}}{\sum_{j=1}^{m} d_{j,r,a}}, & D_a &= \frac{\sum_{i=1}^{k} c_{i,r,a}}{\sum_{j=1}^{m} d_{j,l,a}}. \end{aligned} \tag{3.70}$$

In addition, from (3.3) and (3.70), we get

$$L_a \geq \frac{B^2 - Z^2}{C - W} = \lambda, \qquad R_a \leq \frac{C^2 - W^2}{B - Z} = \mu, \tag{3.71}$$

where B,C (resp., Z,W) are defined in (3.3) (resp., (3.5)). Then from (3.69), (3.71), and arguing as in [13, 14, 15], we can easily prove that L_a, R_a determine a fuzzy number x such that $[x]_a = [L_a, R_a]$. Finally, using (3.70), we take that x is the unique positive equilibrium of (1.1). Using relations (3.11), (3.69), and arguing as in [15, Proposition 2], we can prove that every positive solution of (1.1) nearly converges to the unique positive equilibrium x with respect to D as $n \to \infty$ and converges to x with respect to D_1 as $n \to \infty$.

(ii) Suppose that (3.10) holds. Let x_n be a positive solution of (1.1) such that (2.14) holds. Since $(L_{n,a}, R_{n,a})$ is a positive solution of the system which is defined by (3.7), (3.12), and (3.13), from Lemma 3.6, we have that

$$\lim_{n\to\infty} L_{\kappa nr+l,a} = \epsilon_{l,a}, \quad \lim_{n\to\infty} R_{\kappa nr+l,a} = \xi_{l,a}, \quad a \in (0,1],\ l = 0,1,\dots,\kappa r - 1, \tag{3.72}$$

where κ is defined in (3.44). Using (3.72) and arguing as in [15, Proposition 2], we can prove that every positive solution of (1.1) nearly converges to a period κr solution of (1.1) with respect to D as $n \to \infty$ and converges to a period κr solution of (1.1) with respect to D_1 as $n \to \infty$. Thus, the proof of the proposition is completed. □

From Propositions 3.2–3.8, it is obvious that (1.1) exhibits the trichotomy character described concentratively by the following proposition.

Proposition 3.9. *Consider the fuzzy difference equation (1.1), where* $k,m \in \{1,2,\dots\}$, *and* A,c_i,d_j, $i \in \{1,2,\dots,k\}$, $j \in \{1,2,\dots,m\}$, *are positive fuzzy numbers. Then (1.1) possesses the following trichotomy.*

(i) *If relation (3.1) is satisfied, then (1.1) has unbounded solutions.*

(ii) *If (3.10) holds and* r *is the greatest common divisor of the integers* p_i+1, q_j+1, $i=1,2,\dots,k$, $j=1,2,\dots,m$, *such that (3.36) holds, then every positive solution of (1.1) nearly converges to a period* κr *solution of (1.1) with respect to* D *as* $n \to \infty$ *and converges to a period* κr *solution of (1.1) with respect to* D_1 *as* $n \to \infty$.

(iii) *If (3.11) holds, then every positive solution of (1.1) nearly converges to the unique positive equilibrium* x *with respect to* D *as* $n \to \infty$ *and converges to* x *with respect to* D_1 *as* $n \to \infty$.

In the next proposition, we study the asymptotic stability of the unique positive equilibrium of (1.1).

Proposition 3.10. *Consider the fuzzy difference equation (1.1), where* $k,m \in \{1,2,\dots\}$, A,c_i,d_j, $i \in \{1,2,\dots,k\}$, $j \in \{1,2,\dots,m\}$, *are positive fuzzy numbers, and* p_i, $i \in \{1,2,\dots,k\}$, q_j, $j \in \{1,2,\dots,m\}$, *are positive integers such that (3.11) holds. Suppose that there exists a positive number* θ *such that*

$$\theta < B, \qquad Z < \frac{2B + C - \theta - \sqrt{(C-\theta)^2 + 4BC}}{2}, \tag{3.73}$$

where B,C *are defined in (3.3) and* Z *is defined in (3.5). Then the unique positive equilibrium* x *of (1.1) is nearly asymptotically stable.*

Proof. Since (3.11) holds, from Proposition 3.8, equation (1.1) has a unique positive equilibrium x which satisfies (2.15).

Let ϵ be a positive real number. Since (3.18) holds, we can define the positive real number δ as follows:

$$\delta < \min\{\epsilon, \lambda, \theta, B - Z\}. \tag{3.74}$$

Let x_n be a positive solution of (1.1) such that

$$D(x_{-i}, x) \le \delta \le \epsilon, \quad i = 0,1,\dots,\pi. \tag{3.75}$$

From (3.75), we have

$$|L_{-i,a} - L_a| \le \delta, \quad |R_{-i,a} - R_a| \le \delta, \quad i = 0,1,\dots,\pi,\ a \in (0,1]. \tag{3.76}$$

In addition, from (3.3), (3.7), (3.74), and (3.76) and since (L_a, R_a) satisfies (3.7), we get

$$\begin{aligned} L_{1,a} - L_a &= A_{l,a} + \frac{\sum_{i=1}^{k} c_{i,l,a} L_{-p_i,a}}{\sum_{j=1}^{m} d_{j,r,a} R_{-q_j,a}} - L_a \le A_{l,a} + \frac{\sum_{i=1}^{k} c_{i,l,a}(L_a + \delta)}{\sum_{j=1}^{m} d_{j,r,a}(R_a - \delta)} - L_a \\ &= \delta \frac{C_a - A_{l,a} + L_a}{R_a - \delta} \le \delta \frac{R_a - (B - Z)}{R_a - \delta}. \end{aligned} \tag{3.77}$$

From (3.74) and (3.77), it is obvious that

$$|L_{1,a} - L_a| < \delta < \epsilon. \tag{3.78}$$

Moreover, arguing as above, we can easily prove that

$$R_{1,a} - R_a \le \delta \frac{D_a - A_{r,a} + R_a}{L_a - \delta}. \tag{3.79}$$

We claim that

$$\theta < L_a - R_a + A_{r,a} - D_a, \quad a \in (0,1]. \tag{3.80}$$

We fix an $a \in (0,1]$ and we concider the function

$$g(h) = \frac{A_{l,a}A_{r,a} - D_a h}{A_{r,a} - h} - \frac{A_{l,a}A_{r,a} - D_a h}{A_{l,a} - D_a} + A_{r,a} - D_a, \tag{3.81}$$

where h is a nonnegative real variable. Moreover, we consider the function

$$f(x,y,z) = \frac{x^2 - (2x+y)z + z^2}{x - z} - \theta, \tag{3.82}$$

where $B \le x \le y \le C$ and $W \le z \le Z$, B,C (resp., W,Z) are defined in (3.3) (resp., (3.5)). Using (3.82), we can easily prove that the function f is increasing (resp., decreasing) (resp., decreasing) with respect to x (resp., y) (resp., z) for all y,z (resp., x,z) (resp., x,y) and so from (3.73),

$$f(x,y,z) > f(B,C,Z) = \frac{B^2 - (2B+C)Z + Z^2}{B - Z} - \theta > 0. \tag{3.83}$$

Therefore, from (3.3), (3.81), (3.82), and (3.83), we have

$$g(0) = f(A_{l,a}, A_{r,a}, D_a) + \theta > \theta. \tag{3.84}$$

In addition, from (3.81), we can prove that g is an increasing function with respect to h and so we have $g(0) < g(C_a)$, $a \in (0,1]$. Therefore, from (3.70), (3.81), and (3.84), relation (3.80) is true. Hence, from (3.74), (3.79), and (3.80), we get

$$|R_{1,a} - R_a| < \delta < \epsilon. \tag{3.85}$$

From (3.7), (3.76), (3.78), and (3.85) and working inductively, we can easily prove that

$$|L_{n,a} - L_a| \le \epsilon, \quad |R_{n,a} - R_a| \le \epsilon, \quad a \in (0,1],\ n = 0,1,\ldots, \tag{3.86}$$

and so

$$D(x_n, x) \leq \epsilon, \quad n \geq 0. \tag{3.87}$$

Therefore, the positive equilibrium x is stable. Moreover, from Proposition 3.8, we have that every positive solution of (1.1) nearly tends to x with respect to D as $n \to \infty$. So, x is nearly asymptotically stable. So, the proof of the proposition is completed. □

Finally, we study the oscillatory behavior of the positive solutions of the fuzzy difference equation

$$x_{n+1} = A + \frac{\sum_{s=0}^{k} c_{2s+1} x_{n-2s-1}}{\sum_{s=0}^{k} d_{2s+2} x_{n-2s}}, \tag{3.88}$$

where k is a positive integer, and A, c_{2s+1}, d_{2s+2}, $s \in \{0, 1, \ldots, k\}$, are positive fuzzy numbers. Obviously, (3.88) is a special case of (1.1).

In what follows, we need to study the oscillatory behavior of the positive solutions of the system of ordinary difference equations

$$\begin{aligned} y_{n+1} &= B + \frac{\sum_{s=0}^{k} a_{2s+1} y_{n-2s-1}}{\sum_{s=0}^{k} b_{2s+2} z_{n-2s}}, \\ z_{n+1} &= C + \frac{\sum_{s=0}^{k} e_{2s+1} z_{n-2s-1}}{\sum_{s=0}^{k} h_{2s+2} y_{n-2s}}, \end{aligned} \quad n = 0, 1, \ldots, \tag{3.89}$$

where k is a positive integer, $B, C, a_{2s+1}, b_{2s+2}, e_{2s+1}, h_{2s+2}$, $s \in \{0, 1, \ldots, k\}$, are positive real constants, and the initial values y_j, z_j, $j = -2k-1, -2k, \ldots, 0$, are positive real numbers.

Let (y_n, z_n) be a positive solution of (3.89). We say that the solution (y_n, z_n) oscillates about (y, z), $y, z \in \mathbb{R}^+$, if for every $n_0 \in \mathbb{N}$, there exist s, $m \in \mathbb{N}$, s, $m \geq n_0$, such that

$$\begin{aligned} (y_s - y)(y_m - y) &\leq 0, \qquad (z_s - z)(z_m - z) \leq 0, \\ (y_s - y)(z_s - z) &\geq 0, \qquad (y_m - y)(z_m - z) \geq 0. \end{aligned} \tag{3.90}$$

Lemma 3.11. *Consider system (3.89), where k is a positive integer, B, C, a_{2s+1}, b_{2s+2}, e_{2s+1}, h_{2s+2}, $s \in \{0, 1, \ldots, k\}$, are positive real constants, and the initial values y_j, z_j, $j = -2k-1, -2k, \ldots, 0$, are positive real numbers. A positive solution (y_n, z_n) of system (3.89) oscillates about the unique positive equilibrium $(\bar{x}, \bar{y})$ of system (3.89) if either the relations*

$$\Lambda \geq \max\{\Lambda_{1,s}, \Lambda_{2,s}\}, \quad \Delta \geq \max\{\Delta_{1,s}, \Delta_{2,s}\}, \quad s = 0, 1, \ldots, k, \tag{3.91}$$

or the relations

$$\Lambda \leq \min\{\Lambda_{1,s}, \Lambda_{2,s}\}, \quad \Delta \leq \min\{\Delta_{1,s}, \Delta_{2,s}\}, \quad s = 0, 1, \ldots, k, \tag{3.92}$$

hold, where for $s = 0, 1, \ldots, k$,

$$\Lambda = \frac{\sum_{s=0}^{k} e_{2s+1} z_{-2s-1}}{\sum_{s=0}^{k} h_{2s+2} y_{-2s}}, \qquad \Delta = \frac{\sum_{s=0}^{k} a_{2s+1} y_{-2s-1}}{\sum_{s=0}^{k} b_{2s+2} z_{-2s}},$$

$$\Delta_{1,s} = \frac{1}{a_{2s+1}} \left[\mu \frac{\bar{y}}{\bar{z}} \left(\sum_{j=0}^{s} b_{2j+2} \bar{z} + \sum_{j=s+1}^{k} b_{2j+2} z_{-2j+2+2s} \right) - \left(\sum_{j=0}^{s-1} a_{2j+1} \bar{y} + \sum_{j=s+1}^{k} a_{2j+1} y_{-2j+1+2s} \right) \right] - B,$$

$$\Delta_{2,s} = \frac{1}{h_{2s+2}} \left[\frac{\bar{y}}{\lambda \bar{z}} \left(\sum_{j=0}^{s-1} e_{2j+1} \bar{z} + \sum_{j=s}^{k} e_{2j+1} z_{-2j+2s} \right) - \left(\sum_{j=0}^{s-1} h_{2j+2} \bar{y} + \sum_{j=s+1}^{k} h_{2j+2} y_{-2j+1+2s} \right) \right] - B,$$

$$\Lambda_{1,s} = \frac{1}{e_{2s+1}} \left[\lambda \frac{\bar{z}}{\bar{y}} \left(\sum_{j=0}^{s} h_{2j+2} \bar{y} + \sum_{j=s+1}^{k} h_{2j+2} y_{-2j+2+2s} \right) - \left(\sum_{j=0}^{s-1} e_{2j+1} \bar{z} + \sum_{j=s+1}^{k} e_{2j+1} z_{-2j+1+2s} \right) \right] - C,$$

$$\Lambda_{2,s} = \frac{1}{b_{2s+2}} \left[\frac{\bar{z}}{\mu \bar{y}} \left(\sum_{j=0}^{s-1} a_{2j+1} \bar{y} + \sum_{j=s}^{k} a_{2j+1} y_{-2j+2s} \right) - \left(\sum_{j=0}^{s-1} b_{2j+2} \bar{z} + \sum_{j=s+1}^{k} b_{2j+2} z_{-2j+1+2s} \right) \right] - C,$$

$$\lambda = \frac{\sum_{s=0}^{k} e_{2s+1}}{\sum_{s=0}^{k} h_{2s+2}}, \qquad \mu = \frac{\sum_{s=0}^{k} a_{2s+1}}{\sum_{s=0}^{k} b_{2s+2}}. \tag{3.93}$$

Proof. Suppose that (3.91) hold. We prove that for $\rho = 0, 1, \ldots, k$,

$$y_{2\rho+1} \geq \bar{y}, \qquad z_{2\rho+1} \geq \bar{z}, \qquad y_{2\rho+2} \leq \bar{y}, \qquad z_{2\rho+2} \leq \bar{z}. \tag{3.94}$$

From (3.89) and (3.91), we have

$$\begin{gathered} y_1 = B + \frac{\sum_{s=0}^{k} a_{2s+1} y_{-2s-1}}{\sum_{s=0}^{k} b_{2s+2} z_{-2s}} = B + \Delta \geq B + \Delta_{1,k} = \bar{y}, \\ z_1 = C + \Lambda \geq C + \Lambda_{1,k} = \bar{z}. \end{gathered} \tag{3.95}$$

Since from (3.91), $\Lambda \geq \Lambda_{2,0}$ and $\Delta \geq \Delta_{2,0}$, then from (3.89), we have

$$\begin{gathered} y_2 = B + \frac{\sum_{s=0}^{k} a_{2s+1} y_{-2s}}{b_2 z_1 + \sum_{s=1}^{k} b_{2s+2} z_{1-2s}} \leq B + \frac{(C+\Lambda) b_2 + \sum_{s=1}^{k} b_{2s+2} z_{1-2s}}{b_2 z_1 + \sum_{s=1}^{k} b_{2s+2} z_{1-2s}} \frac{\mu \bar{y}}{\bar{z}} = B + \frac{\mu \bar{y}}{\bar{z}} = \bar{y}, \\ z_2 \leq C + \frac{\lambda \bar{z}}{\bar{y}} = \bar{z}. \end{gathered} \tag{3.96}$$

Using (3.89), (3.91), (3.95), and (3.96), relations $\Delta \geq \Delta_{1,\rho-1}$, $\Lambda \geq \Lambda_{1,\rho-1}$ (resp., $\Delta \geq \Delta_{2,\rho}$, $\Lambda \geq \Lambda_{2,\rho}$), $\rho = 1, 2, \ldots, k$, and working inductively, we can easily prove (3.94) for $\rho = 1, 2, \ldots, k$:

$$y_{2\rho+1} \geq \bar{y}, \qquad z_{2\rho+1} \geq \bar{z} \quad (\text{resp., } y_{2\rho+2} \leq \bar{y},\ z_{2\rho+2} \leq \bar{z}). \tag{3.97}$$

Therefore, (3.94) hold for $\rho = 0,1,\dots,k$. Then since (3.94) hold for $\rho = 0,1,\dots,k$, using (3.89) and working inductively, we can easily prove that(3.94) hold for any $\rho = k+1, k+2,\dots$, and so if (3.91) hold, the proof of the lemma is completed. □

Similarly, if (3.92) are satisfied, then we can easily prove that

$$y_{2\rho+1} \leq \bar{y}, \quad z_{2\rho+1} \leq \bar{z}, \quad y_{2\rho+2} \geq \bar{y}, \quad z_{2\rho+2} \geq \bar{z}, \quad \rho = 0,1,\dots. \tag{3.98}$$

This completes the proof of the lemma.

Using Lemma 3.11 and arguing as in [13, Proposition 2.4], we can easily prove the following proposition which concerns the oscillatory behavior of the positive solutions of the fuzzy difference equation (3.88).

Proposition 3.12. *Consider (3.88), where k is a positive integer, and A, c_{2s+1}, d_{2s+2}, $s \in \{0, 1,\dots,k\}$, are positive fuzzy numbers. Then a positive solution x_n of (3.88) satisfying (2.14) oscillates about the positive equilibrium x, which satisfies (2.15) if, for any $s = 0,1,\dots,k$ and $a \in (0,1]$, either the relations*

$$\bar{\Lambda}_a \geq \max\{\bar{\Lambda}_{1,s,a}, \bar{\Lambda}_{2,s,a}\}, \qquad \bar{\Delta}_a \geq \max\{\bar{\Delta}_{1,s,a}, \bar{\Delta}_{2,s,a}\} \tag{3.99}$$

or the relations

$$\bar{\Lambda}_a \leq \min\{\bar{\Lambda}_{1,s,a}, \bar{\Lambda}_{2,s,a}\}, \qquad \bar{\Delta}_a \leq \min\{\bar{\Delta}_{1,s,a}, \bar{\Delta}_{2,s,a}\} \tag{3.100}$$

hold, where $\bar{\Lambda}_a, \bar{\Delta}_a$, $\bar{\Lambda}_{1,s,a}, \bar{\Lambda}_{2,s,a}$, $\bar{\Delta}_{1,s,a}, \bar{\Delta}_{2,s,a}$ are defined for the analogous system (3.7) in the same way as Λ, Δ, $\Lambda_{1,s}, \Lambda_{2,s}$, $\Delta_{1,s}, \Delta_{2,s}$ were defined in Lemma 3.11 for system (3.89).

Using Proposition 3.12, we take the following corollary.

Corollary 3.13. *Consider (3.88), where k is a positive integer, and A, c_{2s+1}, d_{2s+2}, $s \in \{0,1,\dots,k\}$, are positive fuzzy numbers. Then a positive solution x_n of (3.88) satisfying (2.14) oscillates about the positive equilibrium x, which satisfies (2.15) if, for any $p = 0,1,\dots,k$ and $a \in (0,1]$, either the relations*

$$\begin{aligned} L_{-2k-1+2p,a} &\geq L_a, \qquad R_{-2k-1+2p,a} \geq R_a, \\ L_{-2k+2p,a} &\leq L_a, \qquad R_{-2k+2p,a} \leq R_a \end{aligned} \tag{3.101}$$

or the relations

$$\begin{aligned} L_{-2k-1+2p,a} &\leq L_a, \qquad R_{-2k-1+2p,a} \leq R_a, \\ L_{-2k+2p,a} &\geq L_a, \qquad R_{-2k+2p,a} \geq R_a \end{aligned} \tag{3.102}$$

hold.

Acknowledgment

This work is a part of the first author Doctoral thesis.

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