STABILITY FOR DELAYED GENERALIZED 2D DISCRETE LOGISTIC SYSTEMS

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This paper is concerned with delayed generalized 2D discrete logistic systems of the form $x_{m+1,n} = f(m,n,x_{m,n},x_{m,n+1},x_{m-\sigma,n-\tau})$, where σ and τ are positive integers, $f:\mathbb{N}_0^2\times\mathbb{R}^3\to\mathbb{R}$ is a real function, which contains the logistic map as a special case, and m and n are nonnegative integers, where $\mathbb{N}_0=\{0,1,\dots\}$ and $\mathbb{R}=(-\infty,\infty)$. Some sufficient conditions for this system to be stable and exponentially stable are derived.

1. Introduction

In engineering applications, particularly in the fields of digital filtering, imaging, and spatial dynamical systems, 2D discrete systems have been a subject of focus for investigation (see, e.g., [1, 2, 3, 4, 5, 6] and the references cited therein). In this paper, we consider the delayed generalized 2D discrete systems of the form

$$x_{m+1,n} = f(m,n,x_{m,n},x_{m,n+1},x_{m-\sigma,n-\tau}), \tag{1.1}$$

where σ and τ are positive integers, m and n are nonnegative integers, and $f:\mathbb{N}_0^2\times\mathbb{R}^3\to\mathbb{R}$ is a real function containing the logistic map as a special case, where $\mathbb{R}=(-\infty,\infty)$, $\mathbb{R}^3=\mathbb{R}\times\mathbb{R}\times\mathbb{R}$, $\mathbb{N}_0=\{0,1,\dots\}$, and $\mathbb{N}_0^2=\mathbb{N}_0\times\mathbb{N}_0=\{(m,n)\mid m,n=0,1,\dots\}$.

Obviously, if

$$f(m,n,x,y,z)\equiv\mu_{m,n}x(1-x)-a_{m,n}y, \tag{1.2}$$

$$f(m,n,x,y,z)\equiv\mu_{m,n}x(1-z)-a_{m,n}y, \tag{1.3}$$

$$f(m,n,x,y,z)\equiv 1-\mu x^2-ay, \tag{1.4}$$

or

$$f(m,n,x,y,z)\equiv b_{m,n}x-a_{m,n}y-p_{m,n}z, \tag{1.5}$$

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then system (1.1) becomes, respectively,

$$x_{m+1,n} + a_{m,n}x_{m,n+1} = \mu_{m,n}x_{m,n}(1 - x_{m,n}), \tag{1.6}$$

$$x_{m+1,n} + a_{m,n}x_{m,n+1} = \mu_{m,n}x_{m,n}(1 - x_{m-\sigma,n-\tau}), \tag{1.7}$$

$$x_{m+1,n} + ax_{m,n+1} = 1 - \mu(x_{m,n})^2, \tag{1.8}$$

or

$$x_{m+1,n} + a_{m,n}x_{m,n+1} - b_{m,n}x_{m,n} + p_{m,n}x_{m-\sigma,n-\tau} = 0. \tag{1.9}$$

Systems (1.6), (1.7), and (1.8) are regular 2D discrete logistic systems of different forms, and particularly system (1.9) has been studied in the literature [2, 4, 5, 6].

If $a_{m,n} = 0$, $\mu_{m,n} = \mu$, and $n = n_0$ is fixed, then system (1.6) becomes the 1D logistic system

$$x_{m+1,n_0} = \mu x_{m,n_0}(1 - x_{m,n_0}), \tag{1.10}$$

where μ is a parameter. System (1.10) has been intensively investigated in the literature. Hence, system (1.1) is quite general.

This paper is concerned with the stability of solutions of system (1.1), in which some sufficient conditions for the stability and exponential stability of system (1.1) will be derived.

Let $\mathbb{N}_t = \{t, t+1, t+2, \ldots\}$ for any $t \in \mathbb{Z}$, and $\Omega = \mathbb{N}_{-\sigma} \times \mathbb{N}_{-\tau} \setminus \mathbb{N}_1 \times \mathbb{N}_0$. It is obvious that for any given function $\phi = \{\phi_{m,n}\}$ defined on Ω, it is easy to construct by induction a double sequence $\{x_{m,n}\}$ that equals the initial condition ϕ on Ω and satisfies (1.1) on $\mathbb{N}_1 \times \mathbb{N}_0$. In fact, from (1.1), one can calculate successively a solution sequence: $x_{1,0}, x_{1,1}, x_{2,0}, x_{1,2}, x_{2,1}, x_{3,0}, \ldots$, by using the initial conditions, which is said to be a solution of system (1.1) with the initial condition ϕ.

Definition 1.1. Let $x^* \in \mathbb{R}$ be a constant. If x^* is a root of the equation

$$x - f(m,n,x,x,x) = 0 \quad \text{for any } (m,n) \in \mathbb{N}_0^2, \tag{1.11}$$

then x^* is said to be a fixed point or equilibrium point of system (1.1). The set of all fixed points of system (1.1) is called a fixed plane or equilibrium plane of the system.

It is easy to see that $x^* = 0$ is a fixed point of systems (1.6), (1.7), and (1.9), and $x^* = (-(a+1) \pm \sqrt{(a+1)^2 + 4\mu})/2\mu$ are two fixed points of system (1.8).

Let x^* be a fixed point of system (1.1), let $\phi = \{\phi_{m,n}\}$ be a function defined on Ω, and let

$$\|\phi\|_{x^*} = \sup\{|\phi_{m,n} - x^*| : (m,n) \in \Omega\}. \tag{1.12}$$

For any positive number $\delta > 0$, let $S_\delta(x^*) = \{\phi : \|\phi\|_{x^*} < \delta\}$.

Definition 1.2. Let $x^* \in \mathbb{R}$ be a fixed point of system (1.1). If, for any $\varepsilon > 0$, there exists a positive constant $\delta > 0$ such that for any given bounded function $\phi = \{\phi_{m,n}\}$ defined on Ω, $\phi \in S_\delta(x^*)$ implies that the solution $x = \{x_{m,n}\}$ of system (1.1) with the initial condition ϕ satisfies

$$|x_{m,n} - x^*| < \varepsilon, \quad \forall (m,n) \in \mathbb{N}_1 \times \mathbb{N}_0, \tag{1.13}$$

then system (1.1) is said to be stable about the fixed point x^*.

Definition 1.3. Let $x^* \in \mathbb{R}$ be a fixed point of system (1.1). If there exist positive constants $M > 0$ and $\xi \in (0,1)$ such that for any given constant $\delta \in (0,M)$ and any given bounded function $\phi = \{\phi_{m,n}\}$ defined on Ω, $\phi \in S_\delta(x^*)$ implies that the solution $\{x_{m,n}\}$ of system (1.1) with the initial condition ϕ satisfies

$$|x_{m,n} - x^*| < M\xi^{m+n}, \quad (m,n) \in \mathbb{N}_1 \times \mathbb{N}_0, \tag{1.14}$$

then system (1.1) is said to be DB-exponentially stable about the fixed point x^*, where D means double variables and B means bounded initial condition.

Definition 1.4. Let $x^* \in \mathbb{R}$ be a fixed point of system (1.1). If there exist positive constants $M > 0$ and $\xi \in (0,1)$ such that for any given bounded number $\delta \in (0,M)$ and any given bounded function $\phi = \{\phi_{m,n}\}$ defined on Ω, $\phi \in S_\delta(x^*)$ implies that the solution $\{x_{m,n}\}$ of system (1.1) with the initial condition ϕ satisfies

$$|x_{m,n} - x^*| < M\xi^{m}, \quad (m,n) \in \mathbb{N}_1 \times \mathbb{N}_0, \tag{1.15}$$

then system (1.1) is said to be SB-exponentially stable about the fixed point x^*, where S means single variable and B means bounded initial condition.

Obviously, if system (1.1) is DB-exponentially stable, then it is SB-exponentially stable.

Definition 1.5. Let $f(m,n,x,y,z)$ be a function defined on $\mathbb{N}_0^2 \times D$ and let $(x_0,y_0,z_0) \in D$ be a fixed inner point, where $D \subset \mathbb{R}^3$. If, for any positive constant $\varepsilon > 0$, there exists a constant $\delta > 0$ such that for any $|x - x_0| < \delta$, $|y - y_0| < \delta$, and $|z - z_0| < \delta$,

$$|f(m,n,x,y,z) - f(m,n,x_0,y_0,z_0)| < \varepsilon \quad \text{for any } (m,n) \in \mathbb{N}_0^2, \tag{1.16}$$

then $f(m,n,x,y,z)$ is said to be uniformly continuous at the point (x_0,y_0,z_0) (over m and n). If the partial derivative functions $f_x'(m,n,x,y,z)$, $f_y'(m,n,x,y,z)$, and $f_z'(m,n,x,y,z)$ are all uniformly continuous at (x_0,y_0,z_0), then $f(m,n,x,y,z)$ is said to be uniformly continuously differentiable at (x_0,y_0,z_0).

Let D be an open subset of $\mathbb{R}^3$. If $f(m,n,x,y,z)$ is uniformly continuous at any point $(x,y,z) \in D$, then it is said to be uniformly continuous on D.

Obviously, if $f(m,n,x,y,z)$ and $g(m,n,x,y,z)$ are uniformly continuous at (x,y,z), then $af(m,n,x,y,z)$, $|f(m,n,x,y,z)|$, $f(m,n,x,y,z)+g(m,n,x,y,z)$ are also uniformly continuous at (x,y,z) for any constant $a \in \mathbb{R}$.

2. Stability

Lemma 2.1. *Let $D \subset \mathbb{R}^3$ be an open convex domain and $(x_0, y_0, z_0) \in D$. Assume that the function $f(m,n,x,y,z)$ is continuously differentiable on D for any fixed m and n. Then for any $(\tilde{x}, \tilde{y}, \tilde{z}) \in D$ and any $(m,n) \in \mathbb{N}_0^2$, there exists a constant $t_0 = t(m,n,\tilde{x},\tilde{y},\tilde{z}) \in (0,1)$ such that*

$$\begin{aligned} f(m,n,\tilde{x},\tilde{y},\tilde{z}) &- f(m,n,x_0,y_0,z_0) \\ &= f_x'(m,n,x_0+t_0(\tilde{x}-x_0),y_0+t_0(\tilde{y}-y_0),z_0+t_0(\tilde{z}-z_0))(\tilde{x}-x_0) \\ &\quad + f_y'(m,n,x_0+t_0(\tilde{x}-x_0),y_0+t_0(\tilde{y}-y_0),z_0+t_0(\tilde{z}-z_0))(\tilde{y}-y_0) \\ &\quad + f_z'(m,n,x_0+t_0(\tilde{x}-x_0),y_0+t_0(\tilde{y}-y_0),z_0+t_0(\tilde{z}-z_0))(\tilde{z}-z_0). \end{aligned} \tag{2.1}$$

Proof. Let $g(t) = f(m,n,x_0+t(\tilde{x}-x_0),y_0+t(\tilde{y}-y_0),z_0+t(\tilde{z}-z_0))$. Then, from the given conditions, the function $g(t)$ is continuously differentiable on $[0,1]$. Hence, from the mean value theorem, there exists a constant $t_0 \in (0,1)$ such that $g(1)-g(0) = g'(t_0)$, that is, Lemma 2.1 holds. The proof is completed. □

Theorem 2.2. *Assume that x^* is a fixed point of system (1.1), the function $f(m,n,x,y,z)$ is both continuously differentiable on $\mathbb{R}^3$ for any fixed $(m,n) \in \mathbb{N}_0^2$ and uniformly continuously differentiable at the point $(x^*,x^*,x^*) \in \mathbb{R}^3$, and there exists a constant $r \in (0,1)$ such that for any $(m,n) \in \mathbb{N}_0^2$,*

$$\left|f_x'(m,n,x^*,x^*,x^*)\right| + \left|f_y'(m,n,x^*,x^*,x^*)\right| + \left|f_z'(m,n,x^*,x^*,x^*)\right| \le r. \tag{2.2}$$

Then system (1.1) is stable.

Proof. Using relation (2.2) and since the function $f(m,n,x,y,z)$ is uniformly continuously differentiable at the point (x^*,x^*,x^*), there exists a positive number $M > 0$ such that for any $(m,n) \in \mathbb{N}_0^2$ and any $(x,y,z) \in \mathbb{R}^3$ satisfying $|x-x^*| < M$, $|y-x^*| < M$, and $|z-x^*| < M$,

$$\left|f_x'(m,n,x,y,z)\right| + \left|f_y'(m,n,x,y,z)\right| + \left|f_z'(m,n,x,y,z)\right| \le 1. \tag{2.3}$$

In view of the given conditions and Lemma 2.1, for any $m \ge 0$ and $n \ge 0$, and any point $(x,y,z) \in \mathbb{R}^3$ which satisfies $|x-x^*| < M$, $|y-x^*| < M$, and $|z-x^*| < M$, there exists a constant $t_0 = t(m,n,x,y,z) \in (0,1)$ such that

$$\begin{aligned} f(m,n,x,y,z) &- f(m,n,x^*,x^*,x^*) \\ &= f_x'(m,n,\lambda,\eta,\theta)(x-x^*) + f_y'(m,n,\lambda,\eta,\theta)(y-x^*) \\ &\quad + f_z'(m,n,\lambda,\eta,\theta)(z-x^*), \end{aligned} \tag{2.4}$$

where $\lambda = x^* + t_0(x-x^*)$, $\eta = x^* + t_0(y-x^*)$, and $\theta = x^* + t_0(z-x^*)$. Obviously,

$$\begin{gathered} \left|\lambda - x^*\right| \le \left|x - x^*\right|, \qquad \left|\eta - x^*\right| \le \left|y - x^*\right|, \\ \left|\theta - x^*\right| \le \left|z - x^*\right|. \end{gathered} \tag{2.5}$$

For any sufficiently small number $\varepsilon > 0$, without loss of generality, let $\varepsilon < M$ and $\delta = \varepsilon$, and let $\phi = \{\phi_{m,n}\}$ be a given bounded function defined on Ω which satisfies $|\phi_{m,n} - x^*| < \delta$ for all $(m,n) \in \Omega$. Let the sequence $\{x_{m,n}\}$ be a solution of system (1.1) with the initial condition ϕ. In view of (1.1) and the inequalities

$$\begin{gathered} \left|x_{0,0} - x^*\right| \leq \delta < M, \qquad \left|x_{0,1} - x^*\right| \leq \delta < M, \\ \left|x_{-\sigma,-\tau} - x^*\right| \leq \delta < M, \end{gathered} \tag{2.6}$$

it follows from (2.3), (2.4), and Lemma 2.1 that there exists a constant

$$t_0 = t(0,0,x_{0,0},x_{0,1},x_{-\sigma,-\tau}) \in (0,1), \tag{2.7}$$

such that

$$\begin{aligned} \left|x_{1,0} - x^*\right| &= \left|f(0,0,x_{0,0},x_{0,1},x_{-\sigma,-\tau}) - f(0,0,x^*,x^*,x^*)\right| \\ &\leq \left|f'_x(0,0,\lambda,\eta,\theta)\right|\left|x_{0,0} - x^*\right| + \left|f'_y(0,0,\lambda,\eta,\theta)\right|\left|x_{0,1} - x^*\right| \\ &\quad + \left|f'_z(0,0,\lambda,\eta,\theta)\right|\left|x_{-\sigma,-\tau} - x^*\right| \\ &\leq \delta \leq \varepsilon < M, \end{aligned} \tag{2.8}$$

where $\lambda = x^* + t_0(x_{0,0} - x^*)$, $\eta = x^* + t_0(x_{0,1} - x^*)$, and $\theta = x^* + t_0(x_{-\sigma,-\tau} - x^*)$. Similarly, from (1.1), (2.3), and (2.4), one has

$$\left|x_{1.1} - x^*\right| = \left|f(0,1,x_{0,1},x_{0,2},x_{-\sigma,1-\tau}) - f(0,1,x^*,x^*,x^*)\right| \leq \varepsilon < M. \tag{2.9}$$

In general, for any integer $n \geq 0$, $|x_{1,n} - x^*| \leq \varepsilon < M$.

Assume that for a certain integer $k \geq 1$,

$$|x_{i,n} - x^*| \leq \varepsilon < M \quad \text{for any } i \in \{1,2,\dots,k\},\ n \geq 0. \tag{2.10}$$

Then, it follows from (1.1), (2.3), and (2.4) that there exists a constant

$$t_0 = t(k,n,x_{k,n},x_{k,n+1},x_{k-\sigma,n-\tau}) \in (0,1), \tag{2.11}$$

such that

$$\begin{aligned} \left|x_{k+1,n} - x^*\right| &= \left|f(k,n,x_{k,n},x_{k,n+1},x_{k-\sigma,n-\tau}) - f(k,n,x^*,x^*,x^*)\right| \\ &\leq \left|f'_x(k,n,\lambda,\eta,\theta)\right|\left|x_{k,n} - x^*\right| + \left|f'_y(k,n,\lambda,\eta,\theta)\right|\left|x_{k,n+1} - x^*\right| \\ &\quad + \left|f'_z(k,n,\lambda,\eta,\theta)\right|\left|x_{k-\sigma,n-\tau} - x^*\right| \\ &\leq \left(\left|f'_x(k,n,\lambda,\eta,\theta)\right| + \left|f'_y(k,n,\lambda,\eta,\theta)\right| + \left|f'_z(k,n,\lambda,\eta,\theta)\right|\right) \cdot \varepsilon \leq \varepsilon, \end{aligned} \tag{2.12}$$

where $\lambda = x^* + t_0(x_{k,n} - x^*)$, $\eta = x^* + t_0(x_{k,n+1} - x^*)$, and $\theta = x^* + t_0(x_{k-\sigma,n-\tau} - x^*)$. Hence, by induction, $|x_{m,n} - x^*| \leq \varepsilon$ for any $(m,n) \in \mathbb{N}_1 \times \mathbb{N}_0$, that is, system (1.1) is stable. The proof is completed. □

Similar to the above proof of Theorem 2.2, it is easy to obtain the following result.

Theorem 2.3. *Assume that x^* is a fixed point of system (1.1), and the function $f(m,n,x,y,z)$ is continuously differentiable on $\mathbb{R}^3$ for any fixed m and n. Further, assume that there exists an open subset $D \subset \mathbb{R}^3$ such that $(x^*,x^*,x^*) \in D$ and, for any $(m,n) \in \mathbb{N}_0^2$ and any $(x,y,z) \in D$,*

$$\left| f_x'(m,n,x,y,z) \right| + \left| f_y'(m,n,x,y,z) \right| + \left| f_z'(m,n,x,y,z) \right| \le 1. \tag{2.13}$$

Then system (1.1) is stable.

From Theorems 2.2 and 2.3, one obtains the following results.

Corollary 2.4. *Assume that there exists a constant $r \in (0,1)$ such that*

$$\left| \mu_{m,n} \right| + \left| a_{m,n} \right| \le r \quad \forall m \ge 0,\ n \ge 0. \tag{2.14}$$

Then systems (1.6) and (1.7) are both stable.

In fact, system (1.6) is a special case of system (1.1) when

$$f(m,n,x,y,z) \equiv \mu_{m,n} x(1-x) - a_{m,n} y. \tag{2.15}$$

In view of (2.14), it is obvious that the function $f(m,n,x,y,z)$ is both continuously differentiable on $\mathbb{R}^3$ for any fixed $(m,n) \in \mathbb{N}_0^2$ and uniformly continuously differentiable at the point $(0,0,0)$. Since $x^* = 0$ is a fixed point of systems (1.6) and (1.7),

$$\begin{gathered} f_x'(m,n,x^*,x^*,x^*) = \mu_{m,n}, \qquad f_y'(m,n,x^*,x^*,x^*) = -a_{m,n}, \\ f_z'(m,n,x^*,x^*,x^*) = 0. \end{gathered} \tag{2.16}$$

Hence (2.14) implies (2.2). By Theorem 2.2, systems (1.6) and (1.7) are both stable.

Corollary 2.5. *System (1.8) has fixed points $x^* = (-(a+1) \pm \sqrt{(a+1)^2 + 4\mu})/2\mu$. Assume that there exists a constant $r \in (0,1)$ such that*

$$2\left| \mu \cdot x^* \right| + |a| \le r. \tag{2.17}$$

Then system (1.8) is stable.

Corollary 2.6. *Assume that*

$$\left| a_{m,n} \right| + \left| b_{m,n} \right| + \left| p_{m,n} \right| \le 1 \quad \forall m \ge 0,\ n \ge 0. \tag{2.18}$$

Then system (1.9) is stable.

Define four subsets of $\mathbb{N}_0 \times \mathbb{N}_0$ as follows:

$$\begin{aligned} B_1 &= \{(i,j) \mid 0 \le i \le \sigma,\ 0 \le j < \tau\}, \qquad & B_2 &= \{(i,j) \mid 0 \le i \le \sigma,\ j \ge \tau\}, \\ B_3 &= \{(i,j) \mid i > \sigma,\ 0 \le j < \tau\}, & B_4 &= \{(i,j) \mid i > \sigma,\ j \ge \tau\}. \end{aligned} \tag{2.19}$$

Obviously, B_1 is a finite set, B_2, B_3, and B_4 are infinite sets, B_1, B_2, B_3, and B_4 are mutually disjoint, and $\mathbb{N}_0^2 = B_1 \cup B_2 \cup B_3 \cup B_4$.

Theorem 2.7. *Assume that x^* is a fixed point of system (1.1), the function $f(m,n,x,y,z)$ is both continuously differentiable on $\mathbb{R}^3$ for any $(m,n) \in \mathbb{N}_0^2$ and uniformly continuously differentiable at the point $(x^*,x^*,x^*) \in \mathbb{R}^3$, and there exists a constant $r \in (0,1)$ such that for any $(m,n) \in B_3$,*

$$\left| f_x'(m,n,x^*,x^*,x^*) \right| + \left| f_y'(m,n,x^*,x^*,x^*) \right| + r^{-m} \left| f_z'(m,n,x^*,x^*,x^*) \right| \le r, \quad (2.20)$$

and for $(m,n) \in B_1 \cup B_2 \cup B_4$,

$$\left| f_x'(m,n,x^*,x^*,x^*) \right| + \left| f_y'(m,n,x^*,x^*,x^*) \right| + r^{-\sigma} \left| f_z'(m,n,x^*,x^*,x^*) \right| \le r. \quad (2.21)$$

Then, system (1.1) is SB-exponentially stable.

Proof. From the given conditions, there exist two positive constants, $M > 0$ and $\xi \in (r,1)$, such that (2.4) holds and, for any $(m,n) \in B_3$,

$$\left| f_x'(m,n,x,y,z) \right| + \left| f_y'(m,n,x,y,z) \right| + \xi^{-m} \left| f_z'(m,n,x,y,z) \right| \le \xi, \quad (2.22)$$

and for $(m,n) \in B_1 \cup B_2 \cup B_4$,

$$\left| f_x'(m,n,x,y,z) \right| + \left| f_y'(m,n,x,y,z) \right| + \xi^{-\sigma} \left| f_z'(m,n,x,y,z) \right| \le \xi, \quad (2.23)$$

for $|x - x^*| < M$, $|y - x^*| < M$, and $|z - x^*| < M$.

Let $\delta \in (0,M)$ be a given constant and let $\phi = \{\phi_{m,n}\}$ be a given bounded function defined on Ω which satisfies $|\phi_{m,n} - x^*| < \delta$ for all $(m,n) \in \Omega$. Let the sequence $\{x_{m,n}\}$ be a solution of system (1.1) with the initial condition ϕ. In view of (1.1) and the following inequalities:

$$\begin{gathered} \left| x_{0,0} - x^* \right| \le \delta < M, \qquad \left| x_{0,1} - x^* \right| \le \delta < M, \\ \left| x_{-\sigma,-\tau} - x^* \right| \le \delta < M, \end{gathered} \quad (2.24)$$

it follows from (2.4), (2.23), and Lemma 2.1 that there exists a constant

$$t_0 = t(0,0,x_{0,0},x_{0,1},x_{-\sigma,-\tau}) \in (0,1), \quad (2.25)$$

such that

$$\begin{aligned} \left| x_{1,0} - x^* \right| &= \left| f(0,0,x_{0,0},x_{0,1},x_{-\sigma,-\tau}) - f(0,0,x^*,x^*,x^*) \right| \\ &\le \left| f_x'(0,0,\lambda,\eta,\theta) \right| \left| x_{0,0} - x^* \right| + \left| f_y'(0,0,\lambda,\eta,\theta) \right| \left| x_{0,1} - x^* \right| \\ &\quad + \left| f_z'(0,0,\lambda,\eta,\theta) \right| \left| x_{-\sigma,-\tau} - x^* \right| \\ &\le M\xi, \end{aligned} \quad (2.26)$$

where $\lambda = x^* + t_0(x_{0,0} - x^*)$, $\eta = x^* + t_0(x_{0,1} - x^*)$, and $\theta = x^* + t_0(x_{-\sigma,-\tau} - x^*)$. Similarly, from (1.1), (2.4), and (2.23), one has

$$|x_{1.1} - x^*| = |f(0,1,x_{0,1},x_{0,2},x_{-\sigma,1-\tau}) - f(0,1,x^*,x^*,x^*)| \le M\xi. \tag{2.27}$$

In general, for any integer $n \ge 0$, $|x_{1,n} - x^*| \le M\xi$.

Assume that for a certain integer $k \in \{1,\dots,\sigma\}$,

$$|x_{i,n} - x^*| \le M\xi^i \quad \text{for any } i \in \{1,2,\dots,k\},\ n \ge 0. \tag{2.28}$$

Then $(k,n) \in B_1 \cup B_2 \cup B_4$ and $(k-\sigma,n-\tau) \in \Omega$. From (2.4) and (2.23), one obtains

$$\begin{aligned} |x_{k+1,n} - x^*| &= |f(k,n,x_{k,n},x_{k,n+1},x_{k-\sigma,n-\tau}) - f(k,n,x^*,x^*,x^*)| \\ &\le |f'_x(k,n,\lambda,\eta,\theta)|\,|x_{k,n} - x^*| + |f'_y(k,n,\lambda,\eta,\theta)|\,|x_{k,n+1} - x^*| \\ &\quad + |f'_z(k,n,\lambda,\eta,\theta)|\,|x_{k-\sigma,n-\tau} - x^*| \\ &\le |f'_x(k,n,\lambda,\eta,\theta)| \cdot M\xi^k + |f'_y(k,n,\lambda,\eta,\theta)| \cdot M\xi^k \\ &\quad + |f'_z(k,n,\lambda,\eta,\theta)| \cdot M \\ &\le M\xi^{k+1}. \end{aligned} \tag{2.29}$$

By induction, $|x_{m,n} - x^*| \le M\xi^m$ for any $m \in \{1,2,\dots,\sigma+1\}$ and $n \ge 0$.

Assume that for a certain integer $k \ge \sigma + 1$,

$$|x_{i,n} - x^*| \le M\xi^i \quad \text{for any } i \in \{1,2,\dots,k\},\ n \ge 0. \tag{2.30}$$

If $n \in \{0,1,\dots,\tau-1\}$, then $(k,n) \in B_3$ and $(k-\sigma,n-\tau) \in \Omega$. Hence, from (1.1), (2.4), (2.22), and Lemma 2.1, there exists a constant $t_0 = t(k,n,x_{k,n},x_{k,n+1},x_{k-\sigma,n-\tau}) \in (0,1)$ such that

$$\begin{aligned} |x_{k+1,n} - x^*| &\le |f'_x(k,n,\lambda,\eta,\theta)| \cdot M\xi^k + |f'_y(k,n,\lambda,\eta,\theta)| \cdot M\xi^k \\ &\quad + |f'_z(k,n,\lambda,\eta,\theta)| \cdot M \\ &\le M\xi^{k+1}, \end{aligned} \tag{2.31}$$

where $\lambda = x^* + t_0(x_{k,n} - x^*)$, $\eta = x^* + t_0(x_{k,n+1} - x^*)$, and $\theta = x^* + t_0(x_{k-\sigma,n-\tau} - x^*)$.

If $n \ge \tau$, then $(k,n) \in B_1 \cup B_2 \cup B_4$ and $(k-\sigma,n-\tau) \notin \Omega$. Hence, from (2.4), (2.23), and the assumption, one has

$$\begin{aligned} |x_{k+1,n} - x^*| &\le |f'_x(k,n,\lambda,\eta,\theta)| \cdot M\xi^k + |f'_y(k,n,\lambda,\eta,\theta)| \cdot M\xi^k \\ &\quad + |f'_z(k,n,\lambda,\eta,\theta)| \cdot M\xi^{k-\sigma} \\ &\le M\xi^{k+1}. \end{aligned} \tag{2.32}$$

By induction, $|x_{m,n} - x^*| \le M\xi^m$ for any $(m,n) \in \mathbb{N}_1 \times \mathbb{N}_0$, that is, system (1.1) is SB-exponentially stable. The proof is completed. □

From Theorem 2.7, it is easy to obtain the following corollaries.

Corollary 2.8. *Assume that there exists a constant $r \in (0,1)$ such that*

$$|\mu_{m,n}| + |a_{m,n}| \le r \quad \forall m \ge 0,\ n \ge 0. \tag{2.33}$$

Then systems (1.6) and (1.7) are both SB-exponentially stable.

Corollary 2.9. *System (1.8) has fixed points $x^* = (-(a+1) \pm \sqrt{(a+1)^2 + 4\mu})/2\mu$. Assume that there exists a constant $r \in (0,1)$ such that*

$$2|\mu \cdot x^*| + |a| \le r. \tag{2.34}$$

Then system (1.8) is SB-exponentially stable.

Corollary 2.10. *Assume that there exists a constant $r \in (0,1)$ such that for $(m,n) \in B_3$,*

$$|a_{m,n}| + |b_{m,n}| + r^{-m}|p_{m,n}| \le r, \tag{2.35}$$

and for $(m,n) \in B_1 \cup B_2 \cup B_4$,

$$|a_{m,n}| + |b_{m,n}| + r^{-\sigma}|p_{m,n}| \le r. \tag{2.36}$$

Then system (1.9) is SB-exponentially stable.

Let

$$\begin{aligned} D_1 &= \{(m,n) : 1 \le m \le \sigma,\ 0 \le n < \tau\}, & D_2 &= \{(m,n) : m > \sigma,\ 0 \le n < \tau\}, \\ D_3 &= \{(m,n) : 1 \le m \le \sigma,\ n \ge \tau\}, & D_4 &= \{(m,n) : m > \sigma,\ n \ge \tau\}. \end{aligned} \tag{2.37}$$

Obviously, D_1, D_2, D_3, and D_4 are mutually disjoint, and $\mathbb{N}_1 \times \mathbb{N}_0 = D_1 \cup D_2 \cup D_3 \cup D_4$.

Theorem 2.11. *Assume that x^* is a fixed point of system (1.1), and $f(m,n,x,y,z)$ is both continuously differentiable on $\mathbb{R}^3$ for any $(m,n) \in \mathbb{N}_0^2$ and uniformly continuously differentiable at $(x^*,x^*,x^*) \in \mathbb{R}^3$. Further, assume that there exist a constant $r \in (0,1)$ and an open subset $D \subset \mathbb{R}^3$ with $(x^*,x^*,x^*) \in D$ such that for any $(x,y,z) \in D$ and any $n \ge 0$,*

$$|f'_x(0,n,x,y,z)| + |f'_y(0,n,x,y,z)| + |f'_z(0,n,x,y,z)| \le r^{n+1}, \tag{2.38}$$

and for all $(m,n) \in D_1 \cup D_2 \cup D_3$,

$$|f'_x(m,n,x^*,x^*,x^*)| + r|f'_y(m,n,x^*,x^*,x^*)| + r^{-m-n}|f'_z(m,n,x^*,x^*,x^*)| \le r, \tag{2.39}$$

and for $(m,n) \in D_4$,

$$|f'_x(m,n,x^*,x^*,x^*)| + r|f'_y(m,n,x^*,x^*,x^*)| + r^{-\sigma-\tau}|f'_z(m,n,x^*,x^*,x^*)| \le r. \tag{2.40}$$

Then, system (1.1) is DB-exponentially stable.

Proof. From the given conditions and from (2.38), (2.39), and (2.40), there exist positive constants $M>0$ and $\xi\in(r,1)$ such that (2.4) holds and, for all $n\geq 0$,

$$\left|f_x'(0,n,x,y,z)\right|+\left|f_y'(0,n,x,y,z)\right|+\left|f_z'(0,n,x,y,z)\right|\leq\xi^{n+1},\tag{2.41}$$

and for all $(m,n)\in D_1\cup D_2\cup D_3$,

$$\left|f_x'(m,n,x,y,z)\right|+\xi\left|f_y'(m,n,x,y,z)\right|+\xi^{-m-n}\left|f_z'(m,n,x,y,z)\right|\leq\xi,\tag{2.42}$$

and for all $(m,n)\in D_4$,

$$\left|f_x'(m,n,x,y,z)\right|+\xi\left|f_y'(m,n,x,y,z)\right|+\xi^{-\sigma-\tau}\left|f_z'(m,n,x,y,z)\right|\leq\xi,\tag{2.43}$$

for $|x-x^*|<M$, $|y-x^*|<M$, and $|z-x^*|<M$.

Let $\delta\in(0,M)$ be a constant and let $\phi=\{\phi_{m,n}\}$ be a given bounded function defined on Ω which satisfies $|\phi_{m,n}-x^*|<\delta$ for all $(m,n)\in\Omega$. Let the sequence $\{x_{m,n}\}$ be a solution of system (1.1) with the initial condition ϕ. In view of (1.1) and the inequalities

$$\begin{gathered}\left|x_{0,n}-x^*\right|\leq\delta<M,\qquad\left|x_{0,n+1}-x^*\right|\leq\delta<M,\\ \left|x_{-\sigma,n-\tau}-x^*\right|\leq\delta<M,\end{gathered}\tag{2.44}$$

it follows from (2.4) and (2.41) that there exists a constant $t_{0,n}=t(0,n,x_{0,n},x_{0,n+1},x_{-\sigma,n-\tau})\in(0,1)$ such that for any $n\in\mathbb{N}_0$,

$$\begin{aligned}\left|x_{1,n}-x^*\right|&=\left|f(0,n,x_{0,n},x_{0,n+1},x_{-\sigma,n-\tau})-f(0,n,x^*,x^*,x^*)\right|\\&\leq\left|f_x'(0,n,\lambda_{0,n},\eta_{0,n},\theta_{0,n})\right|\left|x_{0,n}-x^*\right|\\&\quad+\left|f_y'(0,n,\lambda_{0,n},\eta_{0,n},\theta_{0,n})\right|\left|x_{0,n+1}-x^*\right|\\&\quad+\left|f_z'(0,n,\lambda_{0,n},\eta_{0,n},\theta_{0,n})\right|\left|x_{-\sigma,n-\tau}-x^*\right|\\&\leq M\xi^{n+1},\end{aligned}\tag{2.45}$$

where

$$\begin{gathered}\lambda_{0,n}=x^*+t_{0,n}(x_{0,n}-x^*),\qquad\eta_{0,n}=x^*+t_{0,n}(x_{0,n+1}-x^*),\\ \theta_{0,n}=x^*+t_{0,n}(x_{-\sigma,n-\tau}-x^*).\end{gathered}\tag{2.46}$$

Assume that for some $m\in\{1,2,\dots,\sigma\}$,

$$\left|x_{i,j}-x^*\right|\leq M\xi^{i+j},\quad\forall 1\leq i\leq m\text{ and all }j\in\mathbb{N}_0.\tag{2.47}$$

Then, for all $n \geq 0$, one has $(m-\sigma, n-\tau) \in \Omega$ and $(m,n) \in D_1 \cup D_2 \cup D_3$. Hence, it follows from (2.4) and (2.42) that there exists a constant $t_{m,n} = t(m,n,x_{m,n},x_{m,n+1},x_{m-\sigma,n-\tau}) \in (0,1)$ such that

$$\begin{aligned} |x_{m+1,n}-x^*| &= |f(m,n,x_{m,n},x_{m,n+1},x_{m-\sigma,n-\tau}) - f(m,n,x^*,x^*,x^*)| \\ &\leq |f'_x(m,n,\lambda_{m,n},\eta_{m,n},\theta_{m,n})|\,|x_{m,n}-x^*| \\ &\quad + |f'_y(m,n,\lambda_{m,n},\eta_{m,n},\theta_{m,n})|\,|x_{m,n+1}-x^*| \\ &\quad + |f'_z(m,n,\lambda_{m,n},\eta_{m,n},\theta_{m,n})|\,|x_{m-\sigma,n-\tau}-x^*| \\ &\leq \{|f'_x(m,n,\lambda_{m,n},\eta_{m,n},\theta_{m,n})| + \xi|f'_y(m,n,\lambda_{m,n},\eta_{m,n},\theta_{m,n})| \\ &\qquad + \xi^{-m-n}|f'_z(m,n,\lambda_{m,n},\eta_{m,n},\theta_{m,n})|\} \times M\xi^{m+n} \\ &\leq M\xi^{m+n+1}, \end{aligned} \tag{2.48}$$

where

$$\begin{gathered} \lambda_{m,n} = x^* + t_{m,n}(x_{m,n}-x^*), \qquad \eta_{m,n} = x^* + t_{m,n}(x_{m,n+1}-x^*), \\ \theta_{m,n} = x^* + t_{m,n}(x_{m-\sigma,n-\tau}-x^*). \end{gathered} \tag{2.49}$$

By induction, $|x_{m,n}-x^*| \leq M\xi^{m+n}$ for all $m \in \{1,2,\ldots,\sigma+1\}$ and all $n \geq 0$.

Assume that for some $m \geq \sigma+1$,

$$|x_{i,n}-x^*| \leq M\xi^{i+n}, \quad \forall 1 \leq i \leq m \text{ and all } n \in \mathbb{N}_0. \tag{2.50}$$

Then, if $n \in \{0,1,\ldots,\tau-1\}$, then $(m,n) \in D_1 \cup D_2 \cup D_3$ and $(m-\sigma,n-\tau) \in \Omega$. From (2.4) and (2.42), one has

$$\begin{aligned} |x_{m+1,n}-x^*| &\leq |f'_x(m,n,\lambda_{m,n},\eta_{m,n},\theta_{m,n})|\,|x_{m,n}-x^*| \\ &\quad + |f'_y(m,n,\lambda_{m,n},\eta_{m,n},\theta_{m,n})|\,|x_{m,n+1}-x^*| \\ &\quad + |f'_z(m,n,\lambda_{m,n},\eta_{m,n},\theta_{m,n})|\,|x_{m-\sigma,n-\tau}-x^*| \\ &\leq \{|f'_x(m,n,\lambda_{m,n},\eta_{m,n},\theta_{m,n})| + \xi|f'_y(m,n,\lambda_{m,n},\eta_{m,n},\theta_{m,n})| \\ &\qquad + \xi^{-m-n}|f'_z(m,n,\lambda_{m,n},\eta_{m,n},\theta_{m,n})|\} \times M\xi^{m+n} \\ &\leq M\xi^{m+n+1}. \end{aligned} \tag{2.51}$$

If $n \geq \tau$, then $(m,n) \in D_4$ and $(m-\sigma,n-\tau) \in \mathbb{N}_1 \times \mathbb{N}_0$. From (2.4), (2.43), and the assumption,

$$\begin{aligned} |x_{m+1,n}-x^*| &\leq \{|f'_x(m,n,\lambda_{m,n},\eta_{m,n},\theta_{m,n})| + \xi|f'_y(m,n,\lambda_{m,n},\eta_{m,n},\theta_{m,n})| \\ &\qquad + \xi^{-\sigma-\tau}|f'_x(m,n,\lambda_{m,n},\eta_{m,n},\theta_{m,n})|\} \times M\xi^{m+n} \\ &\leq M\xi^{m+n+1}. \end{aligned} \tag{2.52}$$

By induction, $|x_{m,n}-x^*| \leq M\xi^{m+n}$ for any $(m,n) \in \mathbb{N}_1 \times \mathbb{N}_0$, that is, system (1.1) is DB-exponentially stable. The proof is completed. □

From Theorem 2.11, it is easy to obtain the following corollaries.

Corollary 2.12. *Assume that there exist two constants $r \in (0,1)$ and $C \in (0,1)$ such that*

$$\begin{aligned} |\mu_{0,n}| + |a_{0,n}| &\leq Cr^{n+1}, \quad \forall n \geq 0, \\ |\mu_{m,n}| + r|a_{m,n}| &\leq r, \quad \text{for any } (m,n) \in \mathbb{N}_1 \times \mathbb{N}_0. \end{aligned} \tag{2.53}$$

Then, systems (1.6) and (1.7) are both DB-exponentially stable.

Corollary 2.13. *Assume that there exists a constant $r \in (0,1)$ such that*

$$\begin{aligned} |a_{0,n}| + |b_{0,n}| + |p_{0,n}| &\leq r^{n+1}, \quad \text{for any } n \geq 0, \\ r|a_{m,n}| + |b_{m,n}| + r^{-m-n}|p_{m,n}| &\leq r, \quad \text{for any } (m,n) \in D_1 \cup D_2 \cup D_3, \\ r|a_{m,n}| + |b_{m,n}| + r^{-\sigma-\tau}|p_{m,n}| &\leq r, \quad \text{for any } (m,n) \in D_4. \end{aligned} \tag{2.54}$$

Then, system (1.9) is DB-exponentially stable.

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