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Research Article

Global Well-Posedness for Certain Density-Dependent Modified-Leray-α Models

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Global well-posedness result is established for both a 3D density-dependent modified-Leray-α model and a 3D density-dependent modified-Leray-α-MHD model.

1. Introduction

A density-dependent Leray-α model can be written as

$$\begin{gathered}
\rho_t + \operatorname{div}(\rho u) = 0, \\
\rho v_t + \rho u \cdot \nabla v + \nabla \pi - \Delta v = 0, \\
v = \left(1 - \alpha^2 \Delta\right) u, \quad \text{in } (0, \infty) \times \Omega, \\
\operatorname{div} v = \operatorname{div} u = 0, \quad \text{in } (0, \infty) \times \Omega, \\
v = u = 0 \quad \text{on } (0, \infty) \times \partial\Omega, \\
(\rho, \rho v)\big|_{t=0} = (\rho_0, \rho_0 v_0) \quad \text{in } \Omega \subseteq \mathbb{R}^3,
\end{gathered} \tag{1.1}$$

where ρ is the fluid density, v is the fluid velocity field, u is the "filtered" fluid velocity, and π is the pressure, which are unknowns. α is the lengthscale parameter that represents the width of the filter, and for simplicity, we will take $\alpha \equiv 1$. $\Omega \subseteq \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$.

When $\rho \equiv 1$, the above system reduces to the well-known Leray-α model and has been studied in [1, 2]. When $\alpha \to 0$, the above system reduces to the classical density-dependent Navier-Stokes equation, which has received many studies [3–6]. Specifically, it is proved in [3, 4] that the density-dependent Navier-Stokes equations has a unique locally smooth solution (ρ, v) if the following two hypotheses (H1) and (H2) are satisfied:

(H1) $\rho_0 \in W^{1,q}$ for some $q \in (3,6]$, $v_0 \in H_0^1 \cap H^2$, and $\operatorname{div} v_0 = 0$ in $\mathbb{R}^3$,

(H2) $\exists \tilde{\pi}$ and $g \in L^2$ such that $-\Delta v_0 + \nabla\tilde{\pi} = \rho_0^{1/2} g$ in Ω.

One of the aims of this paper is to prove a global well-posedness result for the density-dependent Leray-α model (1.1).

Theorem 1.1. *Let (H1) and (H2) be satisfied. Then the problem* (1.1) *has a unique smooth solution* (ρ, π, v) *satisfying*

$$\begin{gathered}
\rho \in L^\infty\left(0,T;W^{1,q}\right), \qquad \rho_t \in L^\infty(0,T;L^q), \\
\pi \in L^\infty\left(0,T;H^1\right) \cap L^2\left(0,T;W^{1,6}\right), \\
v \in L^\infty\left(0,T;H^2\right) \cap L^2\left(0,T;W^{2,6}\right), \\
\sqrt{\rho} v_t \in L^\infty\left(0,T;L^2\right), \qquad v_t \in L^2\left(0,T;H_0^1\right),
\end{gathered} \tag{1.2}$$

for any $T > 0$.

Next, we consider the following density-dependent modified-Leray-α-MHD model:

$$\rho_t + \operatorname{div}(\rho u) = 0, \tag{1.3}$$

$$\rho v_t + \rho u \cdot \nabla v + \nabla\pi - \Delta v = (B_s \cdot \nabla)B, \tag{1.4}$$

$$\partial_t B_s + u \cdot \nabla B - B_s \cdot \nabla v = \Delta B, \tag{1.5}$$

$$v = \left(1 - \alpha^2\Delta\right)u, \qquad B = \left(1 - \alpha_M^2\Delta\right)B_s, \tag{1.6}$$

$$\operatorname{div} v = \operatorname{div} u = \operatorname{div} B = \operatorname{div} B_s = 0, \quad \text{in } (0,\infty) \times \Omega, \tag{1.7}$$

$$v = u = 0, \quad B \cdot n = B_s \cdot n = \operatorname{curl} B \times n = \operatorname{curl} B_s \times n = 0, \quad \text{on } \partial\Omega, \tag{1.8}$$

$$(\rho, v, B_s)|_{t=0} = (\rho_0, v_0, B_{s0}) \quad \textit{in } \Omega \subseteq R^3, \tag{1.9}$$

where B and B_s represent the unknown magnetic field and the "filtered" magnetic field, respectively. $\alpha_M > 0$ is the lengthscale parameter representing the width of the filter and we will take $\alpha_M = 1$ for simplicity. n is the unit outward vector to $\partial\Omega$. When $\alpha \to 0$ and $\alpha_M \to 0$, the above system (1.3)–(1.9) reduces to the well-known density-dependent MHD equations, which have been studied by many authors (see [7–9] and referees therein). When

$\rho = 1$ and $\alpha_M = 0$, the above system has been studied in [10] recently, and also modified models were analyzed in [11]. In this paper, we will prove the following theorem.

Theorem 1.2. *Let* $0 < m \le \rho_0 \le M < \infty$, $\rho_0 \in W^{1,q}$ *with* $q \in (3,6]$, $v_0 \in H_0^1 \cap H^2$, $B_0 \in H^3$, *and* $\operatorname{div} v_0 = \operatorname{div} u_0 = \operatorname{div} B_0 = \operatorname{div} B_{s0} = 0$ *in* Ω. *Then the problem* (1.3)–(1.9) *has a unique smooth solution* (ρ, π, v, B, B_s) *satisfying*

$$\begin{gathered} 0 < m \le \rho \le M < \infty, \quad \rho \in L^\infty\left(0,T;W^{1,q}\right), \quad \rho_t \in L^\infty(0,T;L^q), \\ \pi \in L^\infty\left(0,T;H^1\right) \cap L^2\left(0,T;W^{1,6}\right), \\ v \in L^\infty\left(0,T;H^2\right) \cap L^2\left(0,T;W^{2,6}\right), \qquad v_t \in L^\infty\left(0,T;L^2\right) \cap L^2\left(0,T;H_0^1\right), \\ B \in L^\infty\left(0,T;H^3\right), \qquad \partial_t B_s \in L^\infty\left(0,T;H^1\right), \qquad \partial_t B \in L^2\left(0,T;H^1\right), \end{gathered} \tag{1.10}$$

for any $T > 0$.

For other related models, we refer to [12–16].

Since the proof of Theorem 1.1 is similar to and simpler than that of Theorem 1.2, we only prove Theorem 1.2 for concision.

2. Proof of Theorem 1.2

By similar argument as that in [3, 4], it is easy to prove that there are $T_0 > 0$ and a unique smooth solution (ρ, v, B, B_s) to the problem (1.3)–(1.9) in $[0, T_0]$, and we only need to establish some a priori estimates for any time. Therefore, in the following estimates, we assume that the solution (ρ, v, B, B_s) is sufficiently smooth.

First, it follows from (1.3), (1.7), and the maximum principle that

$$0 < m \le \rho(x,t) \le M < +\infty. \tag{2.1}$$

Testing (1.4) and (1.5) by v and B, respectively, using (1.3), (1.6), and (1.7), summing up them, we see that

$$\frac{1}{2}\frac{d}{dt}\int \rho v^2 + |B_s|^2 + |\nabla B_s|^2 dx + \int |\nabla v|^2 + |\nabla B|^2 dx = 0. \tag{2.2}$$

Hence

$$\|u\|_{L^\infty(0,T;H^2)} + \|u\|_{L^2(0,T;H^3)} \le C, \tag{2.3}$$

$$\|v\|_{L^\infty(0,T;L^2)} + \|v\|_{L^2(0,T;H^1)} \le C, \tag{2.4}$$

$$\|B_s\|_{L^\infty(0,T;H^1)} + \|B_s\|_{L^2(0,T;H^3)} \le C, \tag{2.5}$$

$$\|B\|_{L^2(0,T;H^1)} \le C. \tag{2.6}$$

Taking ∂_i to (1.3), multiplying it by $|\partial_i\rho|^{q-2}\partial_i\rho$, summing over i, using (1.7) and (2.3), we have

$$\frac{d}{dt}\int |\nabla\rho|^q dx \leq C\|\nabla u\|_{L^\infty}\|\nabla\rho\|_{L^q}^q \leq C\|u\|_{H^3}\|\nabla\rho\|_{L^q}^q, \tag{2.7}$$

which yields

$$\|\rho\|_{L^\infty(0,T;W^{1,q})} \leq C. \tag{2.8}$$

Using (1.3), (2.3) and (2.8), we find that

$$\|\rho_t\|_{L^\infty(0,T;L^q)} \leq \|u\nabla\rho\|_{L^\infty(0,T;L^q)} \leq \|u\|_{L^\infty}\|\nabla\rho\|_{L^\infty(0,T;L^q)} \leq C\|\nabla\rho\|_{L^\infty(0,T;L^q)} \leq C. \tag{2.9}$$

Multiplying (1.5) by $-\Delta B$, using (1.6), (1.7), (2.3), and (2.4), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\nabla B_s|^2 + |\Delta B_s|^2 dx + \int |\Delta B|^2 dx \\
&\quad = \int [(u\cdot\nabla)B - (B_s\cdot\nabla)v]\Delta B\, dx \\
&\quad \leq (\|u\|_{L^\infty}\|\nabla B\|_{L^2} + \|B_s\|_{L^\infty}\|\nabla v\|_{L^2})\|\Delta B\|_{L^2} \\
&\quad \leq C(\|\nabla B\|_{L^2} + \|B_s\|_{H^2}\|\nabla v\|_{L^2})\|\Delta B\|_{L^2} \\
&\quad \leq C\left(\|B\|_{L^2}^{1/2}\|\Delta B\|_{L^2}^{1/2} + \|B_s\|_{H^2}\|\nabla v\|_{L^2}\right) \\
&\quad \leq \frac{1}{2}\|\Delta B\|_{L^2}^2 + C\|B\|_{L^2}^2 + C\|\nabla v\|_{L^2}^2\|B_s\|_{H^2}^2,
\end{aligned} \tag{2.10}$$

which yields

$$\|B_s\|_{L^\infty(0,T;H^2)} + \|B_s\|_{L^2(0,T;H^4)} \leq C, \tag{2.11}$$

$$\|B\|_{L^\infty(0,T;L^2)} + \|B\|_{L^2(0,T;H^2)} \leq C. \tag{2.12}$$

Multiplying (1.4) by v_t, using (1.3), (2.11), (2.12), (2.1), (2.3), and (2.4), we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int |\nabla v|^2 dx + \int \rho v_t^2 dx &= \int (B_s\cdot\nabla)B\cdot v_t dx - \int \rho u\cdot\nabla v\cdot v_t dx \\
&\leq \|B_s\|_{L^\infty}\|\nabla B\|_{L^2}\|v_t\|_{L^2} + \|\sqrt{\rho}\|_{L^\infty}\cdot\|u\|_{L^\infty}\cdot\|\nabla v\|_{L^2}\cdot\|\sqrt{\rho}v_t\|_{L^2} \\
&\leq C\|\nabla B\|_{L^2}\cdot\|\sqrt{\rho}v_t\|_{L^2} + C\|\nabla v\|_{L^2}\|\sqrt{\rho}v_t\|_{L^2} \\
&\leq \frac{1}{2}\|\sqrt{\rho}v_t\|_{L^2}^2 + C\|\nabla B\|_{L^2}^2 + C\|\nabla v\|_{L^2}^2,
\end{aligned} \tag{2.13}$$

which implies

$$\|v\|_{L^\infty(0,T;H^1)} + \|u\|_{L^\infty(0,T;H^3)} \le C, \tag{2.14}$$

$$\|v_t\|_{L^2(0,T;L^2)} \le C. \tag{2.15}$$

It follows from (1.4), (2.14), (2.15), (2.11), (2.12), and the H^2-theory for Stokes system that [17]

$$\|v\|_{L^2(0,T;H^2)} + \|u\|_{L^2(0,T;H^4)} \le C. \tag{2.16}$$

Similarly, it follows from (1.5), (2.11), (2.12), and (2.16) that

$$\|\partial_t B_s\|_{L^2(0,T;L^2)} \le C. \tag{2.17}$$

Taking ∂_t to (1.5), multiplying it by $\partial_t B$, using (1.7), (1.8), (2.12), (2.11), (2.14), and (2.15), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\partial_t B_s|^2 + |\nabla \partial_t B_s|^2 dx + \int |\nabla B_t|^2 dx \\
&\quad = -\int u_t \cdot \nabla B \cdot B_t dx + \int \partial_t B_s \cdot \nabla v \cdot B_t dx + \int B_s \cdot \nabla v_t \cdot B_t dx \\
&\quad = \int u_t \nabla B_t \cdot B dx + \int \partial_t B_s \cdot \nabla v \cdot B_t dx - \int B_s \cdot \nabla B_t \cdot v_t dx \\
&\quad \le \|u_t\|_{L^\infty} \|\nabla B_t\|_{L^2} \|B\|_{L^2} + \|\partial_t B_s\|_{L^3} \cdot \|\nabla v\|_{L^2} \cdot \|B_t\|_{L^6} + \|B_s\|_{L^\infty} \|\nabla B_t\|_{L^2} \|v_t\|_{L^2} \\
&\quad \le C\|v_t\|_{L^2} \|\nabla B_t\|_{L^2} + C\|\partial_t B_s\|_{H^1} \|\nabla B_t\|_{L^2} \\
&\quad \le \frac{1}{2}\|\nabla B_t\|_{L^2}^2 + C\|v_t\|_{L^2}^2 + C\|\partial_t B_s\|_{H^1}^2,
\end{aligned} \tag{2.18}$$

which implies

$$\|\partial_t B_s\|_{L^\infty(0,T;H^1)} + \|\partial_t B_s\|_{L^2(0,T;H^3)} \le C, \tag{2.19}$$

$$\|B_t\|_{L^2(0,T;H^1)} \le C. \tag{2.20}$$

Due to (1.5), (2.3), (2.11), (2.12), (2.14), (2.19), (2.16), and the H^2-theory of the elliptic equations, we have

$$\|B\|_{L^\infty(0,T;H^2)} + \|B\|_{L^2(0,T;H^3)} \le C, \tag{2.21}$$

$$\|B_s\|_{L^\infty(0,T;H^4)} + \|B_s\|_{L^2(0,T;H^5)} \le C. \tag{2.22}$$

Taking ∂_t to (1.4), we see that

$$\rho v_{tt} + \rho u \cdot \nabla v_t + \nabla \pi_t - \Delta v_t = \partial_t B_s \cdot \nabla B + B_s \cdot \nabla \partial_t B - \rho_t v_t - \left(\rho_t u + \rho u_t\right) \cdot \nabla v. \tag{2.23}$$

Multiplying the above equation by v_t, using (1.3), (2.19), (2.21), (2.22), (2.9), and (2.14), we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \rho v_t^2 dx + \int |\nabla v_t|^2 dx \\
&\quad\le \|\partial_t B_s\|_{L^6} \cdot \|\nabla B\|_{L^2} \cdot \|v_t\|_{L^3} \\
&\qquad + \|B_s\|_{L^\infty} \cdot \|\nabla \partial_t B\|_{L^2} \cdot \|v_t\|_{L^2} + \left\|\rho_t\right\|_{L^q} \cdot \|v_t\|_{L^{2q/(q-2)}} \cdot \|v_t\|_{L^2} \\
&\qquad + \left\|\rho_t\right\|_{L^q} \cdot \|u\|_{L^\infty} \cdot \|\nabla v\|_{L^2} \cdot \|v_t\|_{L^{2q/(q-2)}} + \left\|\rho\right\|_{L^\infty} \|u_t\|_{L^\infty} \cdot \|\nabla v\|_{L^2} \cdot \|v_t\|_{L^2} \\
&\quad\le C\|v_t\|_{L^3} + C\|\nabla \partial_t B\|_{L^2}\|v_t\|_{L^2} + C\|v_t\|_{L^{2q/(q-2)}}\|v_t\|_{L^2} + C\|v_t\|_{L^{2q/(q-2)}} + C\|v_t\|_{L^2}^2 \\
&\quad\le \frac{1}{2}\|\nabla v_t\|_{L^2}^2 + C\|v_t\|_{L^2}^2 + C\|\nabla \partial_t B\|_{L^2}^2 + C,
\end{aligned} \tag{2.24}$$

which gives

$$\|v_t\|_{L^\infty(0,T;L^2)} + \|v_t\|_{L^2(0,T;H_0^1)} \le C. \tag{2.25}$$

Combining (1.4), (2.21), (2.22), (2.25), (2.14), and the regularity theory of the Stokes system [17], we obtain

$$\begin{aligned}
\|v\|_{L^\infty(0,T;H^2)} + \|v\|_{L^2(0,T;W^{2,6})} &\le C, \\
\|\pi\|_{L^\infty(0,T;H^1)} + \|\pi\|_{L^2(0,T;W^{1,6})} &\le C, \\
\|u\|_{L^\infty(0,T;H^4)} + \|u\|_{L^2(0,T;W^{4,6})} &\le C.
\end{aligned} \tag{2.26}$$

Similarly, one can prove that

$$\|B\|_{L^\infty(0,T;H^3)} \le C. \tag{2.27}$$

This completes the proof.

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