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*Research Article*

# Iterative Methods for Family of Strictly Pseudocontractive Mappings and System of Generalized Mixed Equilibrium Problems and Variational Inequality Problems

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We introduce a new iterative scheme by hybrid method for finding a common element of the set of common fixed points of infinite family of $k$-strictly pseudocontractive mappings and the set of common solutions to a system of generalized mixed equilibrium problems and the set of solutions to a variational inequality problem in a real Hilbert space. We then prove strong convergence of the scheme to a common element of the three above described sets. We give an application of our results. Our results extend important recent results from the current literature.

## 1. Introduction

Let $K$ be a nonempty closed and convex subset of a real Hilbert space $H$. A mapping $A : K \to H$ is called *monotone* if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in K. \tag{1.1}$$

A mapping $A : K \to H$ is called *inverse-strongly monotone* (see, e.g., [1, 2]) if there exists a positive real number $\lambda$ such that $\langle Ax - Ay, x - y \rangle \geq \lambda \|Ax - Ay\|^2$, for all $x, y \in K$. For such a case, $A$ is called $\lambda$-inverse-strongly monotone. A $\lambda$-inverse-strongly monotone is sometime called *$\lambda$-cocoercive*. A mapping $A$ is said to be *relaxed $\lambda$-cocoercive* if there exists $\lambda > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq -\lambda \|Ax - Ay\|^2, \quad \forall x, y \in K. \tag{1.2}$$

$A$ is said to be *relaxed* $(\lambda,\gamma)$*-cocoercive* if there exist $\lambda,\gamma > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq -\lambda\|Ax - Ay\|^2 + \gamma\|x - y\|^2, \quad \forall x, y \in K. \tag{1.3}$$

A mapping $A : H \to H$ is said to be $\mu$-Lipschitzian if there exists $\mu \geq 0$ such that

$$\|Ax - Ay\| \leq \mu\|x - y\|, \quad x, y \in H. \tag{1.4}$$

Let $A : K \to H$ be a nonlinear mapping. The variational inequality problem is to find an $x^* \in K$ such that

$$\langle Ax^*, y - x^*\rangle \geq 0, \quad \forall y \in K. \tag{1.5}$$

(See, e.g., [3, 4].) We will denote the set of solutions of the variational inequality problem (1.5) by $\mathrm{VI}(K,A)$.

A monotone mapping $A$ is said to be *maximal* if the graph $G(A)$ is not properly contained in the graph of any other monotone map, where $G(A) := \{(x,y) \in H \times H : y \in Ax\}$ for a multivalued mapping $A$. It is also known that $A$ is maximal if and only if for $(x,f) \in H \times H, \langle x - y, f - g\rangle \geq 0$ for every $(y,g) \in G(A)$ implies $f \in Ax$. Let $A$ be a monotone mapping defined from $K$ into $H$ and $N_K q$ a normal cone to $K$ at $q \in K$, that is, $N_K q = \{p \in H : \langle q - u, p\rangle \geq 0, \text{for all } u \in K\}$. Define a mapping $M$ by

$$Mq = \begin{cases} Aq + N_K q, & q \in K, \\ \emptyset, & q \notin K. \end{cases} \tag{1.6}$$

Then, $M$ is maximal monotone and $x^* \in M^{-1}(0) \Leftrightarrow x^* \in \mathrm{VI}(K,A)$ (see, e.g., [5]).

A mapping $T : K \to K$ is said to be *k-strictly pseudocontractive* if there exists a constant $k \in [0,1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \tag{1.7}$$

for all $x, y \in K$. If $k = 0$, then the mapping $T$ is *nonexpansive.* A point $x \in K$ is called *a fixed point* of $T$ if $Tx = x$. The fixed points set of $T$ is the set $F(T) := \{x \in K : Tx = x\}$. Iterative approximation of fixed points of $k$-strictly pseudocontractive mappings have been studied extensively by many authors (see, e.g., [1, 6–9] and the references contained therein).

Let $\varphi : K \to \mathbb{R}$ be a real-valued function and $A : K \to H$ a nonlinear mapping. Suppose $F : K \times K$ into $\mathbb{R}$ is an equilibrium bifunction. That is, $F(u,u) = 0$, forall $u \in K$. The generalized mixed equilibrium problem is to find $x \in K$ (see, e.g., [10–12]) such that

$$F(x,y) + \varphi(y) - \varphi(x) + \langle Ax, y - x\rangle \geq 0, \tag{1.8}$$

for all $y \in K$. We shall denote the set of solutions of this generalized mixed equilibrium problem by $\mathrm{GMEP}(F,A,\varphi)$. Thus,

$$\mathrm{GMEP}(F,A,\varphi) := \{x^* \in K : F(x^*,y) + \varphi(y) - \varphi(x^*) + \langle Ax^*, y - x^*\rangle \geq 0, \forall y \in K\}. \tag{1.9}$$

If $\varphi = 0$, $A = 0$, then problem (1.8) reduces to equilibrium problem studied by many authors (see, e.g., [8, 13–17]), which is to find $x^* \in K$ such that

$$F(x^*, y) \geq 0, \tag{1.10}$$

for all $y \in K$. The set of solutions of (1.10) is denoted by EP$(F)$.

If $\varphi = 0$, then problem (1.8) reduces to generalized equilibrium problem studied by many authors (see, e.g., [18–20]), which is to find $x^* \in K$ such that

$$F(x^*, y) + \langle Ax^*, y - x^* \rangle \geq 0, \tag{1.11}$$

for all $y \in K$. The set of solutions of (1.11) is denoted by EP.

If $A = 0$, then problem (1.8) reduces to mixed equilibrium problem considered by many authors (see, e.g., [21–23]), which is to find $x^* \in K$ such that

$$F(x^*, y) + \varphi(y) - \varphi(x^*) \geq 0, \tag{1.12}$$

for all $y \in K$. The set of solutions of (1.12) is denoted by MEP.

The generalized mixed equilibrium problems include fixed-point problems, optimization problems, variational inequality problems, Nash equilibrium problems, and equilibrium problems as special cases (see, e.g., [24]). Numerous problems in Physics, optimization, and economics reduce to find a solution of problem (1.8). Several methods have been proposed to solve the fixed-point problems, variational inequality problems and equilibrium problems in the literature (see, e.g., [5, 11, 12, 20, 25–30]).

Recently, Ceng and Yao [25] introduced a new iterative scheme of approximating a common element of the set of solutions to mixed equilibrium problem and set of common fixed points of finite family of nonexpansive mappings in a real Hilbert space $H$. In their results, they imposed the following condition on a nonempty closed and convex subset $K$ of $H$:

(E) $A : K \to \mathbb{R}$ is $\eta$-strongly convex and its derivative $A'$ is sequentially continuous from weak topology to the strong topology.

We remark here that this condition (E) has been used by many authors for approximation of solution to mixed equilibrium problem in a real Hilbert space (see, e.g., [31, 32]). However, it is observed that the condition (E) does not include the case $A(x) = \|x\|^2/2$ and $\eta(x, y) = x - y$. Furthermore, Peng and Yao [21], R. Wangkeeree and R. Wangkeeree [30], and many other authors replaced condition (E) with the following conditions:

(B1) for each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subseteq K$ and $y_x \in K$ such that for any $z \in K \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x \rangle < 0, \tag{1.13}$$

or

(B2) $K$ is a bounded set.

Consequently, conditions (B1) and (B2) have been used by many authors in approximating solution to generalized mixed equilibrium (mixed equilibrium) problems in a real Hilbert space (see, e.g., [21, 30]).

Recently, Takahashi et al. [33] proved the following convergence theorem using hybrid method.

**Theorem 1.1** (Takahashi et al. [33]). *Let $K$ be a nonempty closed and convex subset of a real Hilbert space $H$. Let $T$ be a nonexpansive mapping of $K$ into itself such that $F(T) \neq \emptyset$. For $C_1 = K, x_1 = P_{C_1}x_0$, define sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ of $K$ as follows:*

$$\begin{aligned} y_n &= \alpha_n x_n + (1-\alpha_n)Tx_n, \quad n \geq 1, \\ C_{n+1} &= \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \quad n \geq 1, \\ x_{n+1} &= P_{C_{n+1}}x_0, \quad n \geq 1. \end{aligned} \tag{1.14}$$

*Assume that $\{\alpha_n\}_{n=1}^{\infty} \subset [0,1)$ satisfies $0 \leq \alpha_n < \alpha < 1$. Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to $P_{F(T)}x_0$.*

Motivated by the results of Takahashi et al. [33], Kumam [28] studied the problem of approximating a common element of set of solutions to an equilibrium problem, set of solutions to variational inequality problem and the set of fixed points of a nonexpansive mapping in a real Hilbert space. In particular, he proved the following theorem.

**Theorem 1.2** (Kumam, [28]). *Let $K$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $F$ be a bifunction from $K \times K$ satisfying (A1)–(A4) and let $B$ be a $\beta$-inverse-strongly monotone mapping of $K$ into $H$. Let $T$ be a nonexpansive mapping of $K$ into $H$ such that $F(T) \bigcap \mathrm{EP}(F) \bigcap \mathrm{VI}(K,B) \neq \emptyset$. For $C_1 = K, x_1 = P_{C_1}x_0$, define sequences $\{x_n\}_{n=0}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ of $K$ as follows:*

$$\begin{aligned} &F(z_n, y) + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \quad \forall y \in K, \\ &y_n = \alpha_n x_n + (1-\alpha_n)TP_K(z_n - \lambda_n Bz_n), \quad n \geq 1, \\ &C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \quad n \geq 1, \\ &x_{n+1} = P_{C_{n+1}}x_0, \quad n \geq 1. \end{aligned} \tag{1.15}$$

*Assume that $\{\alpha_n\}_{n=1}^{\infty} \subset [0,1), \{r_n\}_{n=1}^{\infty} \subset (0,\infty)$ and $\{\lambda_n\}_{n=1}^{\infty} \subset [0,2\beta]$ satisfy*

$$\liminf_{n\to\infty} r_n > 0, \quad 0 < c \leq \lambda_n \leq f < 2\beta, \quad \lim_{n\to\infty}\alpha_n = 0. \tag{1.16}$$

*Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to $P_{F(T)\bigcap \mathrm{EP}(F)\bigcap \mathrm{VI}(K,B)}x_0$.*

Motivated by the ongoing research and the above-mentioned results, we introduce *a new iterative scheme for finding a common element of the set of fixed points of an infinite family of $k$-strictly pseudocontractive mappings, the set of common solutions to a system of generalized mixed equilibrium problems and the set of solutions to a variational inequality problem in a real Hilbert space*. Furthermore, we show that our new iterative scheme converges strongly to a common element of the three afore mentioned sets. In our results, we use conditions (B1) and (B2) mentioned above. Our result extends many important recent results. Finally, we give some applications of our results.

## 2. Preliminaries

Let $H$ be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ and let $K$ be a nonempty closed and convex subset of $H$. The strong convergence of $\{x_n\}_{n=0}^{\infty}$ to $x$ is denoted by $x_n \longrightarrow x$ as $n \longrightarrow \infty$.

For any point $u \in H$, there exists a unique point $P_K u \in K$ such that

$$\|u - P_K u\| \leq \|u - y\|, \quad \forall y \in K. \tag{2.1}$$

$P_K$ is called the *metric projection* of $H$ onto $K$. We know that $P_K$ is a nonexpansive mapping of $H$ onto $K$. It is also known that $P_K$ satisfies

$$\langle x - y, P_K x - P_K y\rangle \geq \|P_K x - P_K y\|^2, \tag{2.2}$$

for all $x, y \in H$. Furthermore, $P_K x$ is characterized by the properties $P_K x \in K$ and

$$\langle x - P_K x, P_K x - y\rangle \geq 0, \tag{2.3}$$

for all $y \in K$ and

$$\|x - P_K x\|^2 \leq \|x - y\|^2 - \|y - P_K x\|^2, \quad \forall x \in H,\ y \in K. \tag{2.4}$$

In the context of the variational inequality problem, (2.3) implies that

$$x^* \in \mathrm{VI}(A, K) \Longleftrightarrow x^* = P_K(x^* - \lambda A x^*), \quad \forall \lambda > 0. \tag{2.5}$$

If $A$ is $\alpha$-inverse-strongly monotone mapping of $K$ into $H$, then it is obvious that $A$ is $(1/\alpha)$-Lipschitz continuous. We also have that for all $x, y \in K$ and $r > 0$,

$$\begin{aligned}\|(I - rA)x - (I - rA)y\|^2 &= \|x - y - r(Ax - Ay)\|^2 \\ &= \|x - y\|^2 - 2r\langle Ax - Ay, x - y\rangle + r^2\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 + r(r - 2\alpha)\|Ax - Ay\|^2.\end{aligned} \tag{2.6}$$

So, if $r \leq 2\alpha$, then $I - rA$ is a nonexpansive mapping of $K$ into $H$.

For solving the generalized mixed equilibrium problem for a bifunction $F : K \times K \longrightarrow \mathbb{R}$, let us assume that $F$ satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in K$,

(A2) $F$ is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in K$,

(A3) for each $x, y, z \in K, \lim_{t \to 0} F(tz + (1-t)x, y) \leq F(x, y)$,

(A4) for each $x \in K, y \mapsto F(x, y)$ is convex and lower semicontinuous.

We need the following technical result.

**Lemma 2.1** (R. Wangkeeree and R. Wangkeeree [30]). *Assume that $F : K \times K \to \mathbb{R}$ satisfies (A1)–(A4) and let $\varphi : K \to \mathbb{R}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, define a mapping $T_r^{(F,\varphi)} : H \to K$ as follows:*

$$T_r^{(F,\varphi)}(x) = \left\{ z \in K : F(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \ \forall y \in K \right\} \tag{2.7}$$

*for all $z \in H$. Then, the following hold:*

(1) *for each $x \in H, T_r^{(F,\varphi)} \neq \emptyset$,*

(2) *$T_r^{(F,\varphi)}$ is single-valued,*

(3) *$T_r^{(F,\varphi)}$ is firmly nonexpansive, that is, for any $x, y \in H$,*

$$\left\| T_r^{(F,\varphi)}x - T_r^{(F,\varphi)}y \right\| \leq \left\langle T_r^{(F,\varphi)}x - T_r^{(F,\varphi)}y, x - y \right\rangle, \tag{2.8}$$

(4) *$F(T_r^{(F,\varphi)}) = MEP(F)$,*

(5) *$MEP(F)$ is closed and convex.*

## 3. Main Results

**Theorem 3.1.** *Let $K$ be a nonempty closed and convex subset of a real Hilbert space $H$. For each $m = 1, 2$, let $F_m$ be a bifunction from $K \times K$ satisfying (A1)–(A4), $\varphi_m : K \to \mathbb{R} \cup \{+\infty\}$ a proper lower semicontinuous and convex function with assumption (B1) or (B2), $A$ an $\alpha$-inverse-strongly monotone mapping of $K$ into $H$, $B$ a $\beta$-inverse-strongly monotone mapping of $K$ into $H$ and for each $i = 1, 2, \ldots$, let $T_i : K \to K$ be a $k_i$-strictly pseudocontractive mapping for some $0 \leq k_i < 1$ such that $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $D$ be a $\mu$-Lipschitzian, relaxed $(\lambda, \gamma)$-cocoercive mapping of $K$ into $H$. Suppose $\Omega := \cap_{i=1}^{\infty} F(T_i) \cap \mathrm{GMEP}(F_1, A, \varphi_1) \cap \mathrm{GMEP}(F_2, B, \varphi_2) \cap \mathrm{VI}(K, D) \neq \emptyset$. Let $\{z_n\}_{n=1}^{\infty}$, $\{u_n\}_{n=1}^{\infty}$,*

*$\{w_n\}_{n=1}^{\infty}$, $\{y_{n,i}\}_{n=1}^{\infty}$ $(i = 1,2,\ldots)$ and $\{x_n\}_{n=0}^{\infty}$ be generated by $x_0 \in K$, $C_{1,i} = K$, $C_1 = \cap_{i=1}^{\infty} C_{1,i}$, $x_1 = P_{C_1}x_0$*

$$\begin{aligned}
z_n &= T_{r_n}^{(F_1,\varphi_1)}(x_n - r_n A x_n),\\
u_n &= T_{\lambda_n}^{(F_2,\varphi_2)}(z_n - \lambda_n B z_n),\\
w_n &= P_K(u_n - s_n D u_n),\\
y_{n,i} &= \alpha_{n,i} w_n + (1-\alpha_{n,i}) T_i w_n,\\
C_{n+1,i} &= \left\{z \in C_{n,i} : \left\|y_{n,i} - z\right\| \le \|x_n - z\|\right\},\\
C_{n+1} &= \cap_{i=1}^{\infty} C_{n+1,i},\\
x_{n+1} &= P_{C_{n+1}} x_0, \quad n \ge 1.
\end{aligned} \tag{3.1}$$

*Assume that $\{\alpha_{n,i}\}_{n=1}^{\infty} \subset [0,1)$ $(i = 1,2,\ldots)$, $\{r_n\}_{n=1}^{\infty} \subset [0,2\alpha]$ and $\{\lambda_n\}_{n=1}^{\infty} \subset [0,2\beta]$ satisfy*

(i) $0 < a \le r_n \le b < 2\alpha$,

(ii) $0 < c \le \lambda_n \le f < 2\beta$,

(iii) $0 \le k_i \le \alpha_{n,i} \le d_i < 1$,

(iv) $0 < h \le s_n \le j < 2(\gamma - \lambda\mu^2)/\mu^2$.

*Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to $P_\Omega x_0$.*

*Proof.* For all $x, y \in K$ and $s_n \in (0, 2(\gamma - \lambda\mu^2)/\mu^2]$, we obtain

$$\begin{aligned}
\left\|(I - s_n D)x - (I - s_n D)y\right\|^2 &= \left\|x - y - s_n(Dx - Dy)\right\|^2\\
&= \left\|x - y\right\|^2 - 2s_n\langle x - y, Dx - Dy\rangle + s_n^2\left\|Dx - Dy\right\|^2\\
&\le \left\|x - y\right\|^2 - 2s_n\left[-\lambda\left\|Dx - Dy\right\|^2 + \gamma\left\|x - y\right\|^2\right] + s_n^2\left\|Dx - Dy\right\|^2\\
&\le \left\|x - y\right\|^2 + 2s_n\mu^2\lambda\left\|x - y\right\|^2 - 2s_n\gamma\left\|x - y\right\|^2 + \mu^2 s_n^2\left\|x - y\right\|^2\\
&= \left(1 + 2s_n\mu^2\lambda - 2s_n\gamma + \mu^2 s_n^2\right)\left\|x - y\right\|^2\\
&\le \left\|x - y\right\|^2.
\end{aligned} \tag{3.2}$$

This shows that $I - s_n D$ is nonexpansive for each $n \ge 1$. Let $x^* \in \Omega$. Then

$$\begin{aligned}
\|w_n - x^*\|^2 &= \|P_K(u_n - s_n D u_n) - P_K(x^* - s_n D x^*)\|^2\\
&\le \|(u_n - s_n D u_n) - (x^* - s_n D x^*)\|^2\\
&\le \|u_n - x^*\|^2.
\end{aligned} \tag{3.3}$$

Since both $I-r_nA$ and $I-\lambda_nB$ are nonexpansive for each $n \geq 1$ and $x^* = T_{r_n}^{(F_1,\varphi_1)}(x^*-r_nAx^*)$, $x^* = T_{\lambda_n}^{(F_2,\varphi_2)}(x^* - \lambda_nBx^*)$, from (2.6), we have

$$\begin{aligned}
\|u_n - x^*\|^2 &= \left\|T_{\lambda_n}^{(F_2,\varphi_2)}(z_n - \lambda_nBz_n) - x^*\right\|^2 \\
&= \left\|T_{\lambda_n}^{(F_2,\varphi_2)}(z_n - \lambda_nBz_n) - T_{\lambda_n}^{(F_2,\varphi_2)}(x^* - \lambda_nBx^*)\right\|^2 \\
&\leq \|(I - \lambda_nB)z_n - (I - \lambda_nB)x^*\|^2 \\
&\leq \|z_n - x^*\|^2 + \lambda_n(\lambda_n - 2\beta)\|Bz_n - Bx^*\|^2 \\
&\leq \|z_n - x^*\|^2 \quad (\text{since } \lambda_n < 2\beta,\ \forall n \geq 1), \\
\|z_n - x^*\|^2 &= \left\|T_{r_n}^{(F_1,\varphi_1)}(x_n - r_nAx_n) - x^*\right\|^2 \\
&= \left\|T_{r_n}^{(F_1,\varphi_1)}(x_n - r_nAx_n) - T_{r_n}^{(F_1,\varphi_1)}(x^* - r_nAx^*)\right\|^2 \\
&\leq \|(I - r_nA)x_n - (I - r_nA)x^*\|^2 \\
&\leq \|x_n - x^*\|^2 + r_n(r_n - 2\alpha)\|Ax_n - Ax^*\|^2 \\
&\leq \|x_n - x^*\|^2.
\end{aligned} \tag{3.4}$$

Therefore,

$$\|u_n - x^*\| \leq \|x_n - x^*\|. \tag{3.5}$$

Let $n = 1$, then $C_{1,i} = K$ is closed convex for each $i = 1, 2, \ldots$. Now assume that $C_{n,i}$ is closed convex for some $n > 1$. Then, from definition of $C_{n+1,i}$, we know that $C_{n+1,i}$ is closed convex for the same $n > 1$. Hence, $C_{n,i}$ is closed convex for $n \geq 1$ and for each $i = 1, 2, \ldots$. This implies that $C_n$ is closed convex for $n \geq 1$. Furthermore, we show that $\Omega \subset C_n$. For $n = 1$, $\Omega \subset K = C_{1,i}$. For $n \geq 2$, let $x^* \in \Omega$. Then,

$$\begin{aligned}
\left\|y_{n,i} - x^*\right\|^2 &= \alpha_{n,i}\|w_n - x^*\|^2 + (1 - \alpha_{n,i})\|T_iw_n - x^*\|^2 - \alpha_{n,i}(1 - \alpha_{n,i})\|T_iw_n - w_n\|^2 \\
&\leq \alpha_{n,i}\|w_n - x^*\|^2 + (1 - \alpha_{n,i})\left[\|w_n - x^*\|^2 + k_i\|T_iw_n - w_n\|^2\right] \\
&\quad - \alpha_{n,i}(1 - \alpha_{n,i})\|T_iw_n - w_n\|^2 \\
&= \|w_n - x^*\|^2 + (1 - \alpha_{n,i})(k_i - \alpha_{n,i})\|T_iw_n - w_n\|^2 \\
&\leq \|u_n - x^*\|^2 \leq \|x_n - x^*\|^2,
\end{aligned} \tag{3.6}$$

which shows that $x^* \in C_{n,i}$, for all $n \geq 2$, for all $i = 1, 2, \ldots$. Thus, $\Omega \subset C_{n,i}$, for all $n \geq 1$, for all $i = 1, 2, \ldots$. Hence, it follows that $\emptyset \neq \Omega \subset C_n$, for all $n \geq 1$. Therefore, $\{x_n\}_{n=0}^{\infty}$ is well defined. Since $x_n = P_{C_n} x_0$, for all $n \geq 1$ and $x_{n+1} \in C_{n+1} \subset C_n$, for all $n \geq 1$, we have

$$\|x_n - x_0\| \leq \|x_{n+1} - x_0\|, \quad \forall n \geq 1. \tag{3.7}$$

Also, as $\Omega \subset C_n$, by (2.1) it follows that

$$\|x_n - x_0\| \leq \|v - x_0\|, \quad v \in \Omega, \ \forall n \geq 1. \tag{3.8}$$

From (3.7) and (3.8), we have that $\lim_{n\to\infty}\|x_n - x_0\|$ exists. Hence, $\{x_n\}_{n=0}^{\infty}$ is bounded and so are $\{z_n\}_{n=1}^{\infty}$, $\{Ax_n\}_{n=1}^{\infty}$, $\{u_n\}_{n=1}^{\infty}$, $\{Du_n\}_{n=1}^{\infty}$, $\{Bz_n\}_{n=1}^{\infty}$, $\{w_n\}_{n=1}^{\infty}$, $\{T_i w_n\}_{n=1}^{\infty}$ and $\{y_{n,i}\}_{n=1}^{\infty}$, $i = 1, 2, \ldots$. For $m > n \geq 1$, we have that $x_m = P_{C_m} x_0 \in C_m \subset C_n$. By (2.4), we obtain

$$\|x_m - x_n\|^2 \leq \|x_m - x_0\|^2 - \|x_n - x_0\|^2. \tag{3.9}$$

Letting $m, n \to \infty$ and taking the limit in (3.9), we have $x_m - x_n \to 0$, $m, n \to \infty$, which shows that $\{x_n\}_{n=0}^{\infty}$ is Cauchy. In particular, $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Since, $\{x_n\}_{n=0}^{\infty}$ is Cauchy and $K$ is closed, there exists $z \in K$ such that $x_n \to z$, $n \to \infty$. Since $x_{n+1} = P_{C_{n+1}} x_0 \in C_{n+1}$, therefore

$$\left\|y_{n,i} - x_{n+1}\right\| \leq \|x_n - x_{n+1}\|, \tag{3.10}$$

and it follows that

$$\begin{aligned} \left\|y_{n,i} - x_n\right\| &\leq \left\|y_{n,i} - x_{n+1}\right\| + \|x_n - x_{n+1}\| \\ &\leq 2\|x_n - x_{n+1}\|. \end{aligned} \tag{3.11}$$

Thus,

$$\lim_{n\to\infty}\left\|y_{n,i} - x_n\right\| = 0, \quad i = 1, 2, \ldots. \tag{3.12}$$

Furthermore,

$$\begin{aligned}
\|y_{n,i} - x^*\|^2 &= \alpha_{n,i}\|w_n - x^*\|^2 + (1-\alpha_{n,i})\|T_i w_n - x^*\|^2 \\
&\quad - \alpha_{n,i}(1-\alpha_{n,i})\|T_i w_n - w_n\|^2 \\
&\le \alpha_{n,i}\|w_n - x^*\|^2 + (1-\alpha_{n,i})\left[\|w_n - x^*\|^2 + k_i\|T_i w_n - w_n\|^2\right] \\
&\quad - \alpha_{n,i}(1-\alpha_{n,i})\|T_i w_n - w_n\|^2 \\
&= \alpha_{n,i}\|w_n - x^*\|^2 + (1-\alpha_{n,i})\|w_n - x^*\|^2 \\
&\quad - (1-\alpha_{n,i})(\alpha_{n,i} - k_i)\|T_i w_n - w_n\|^2 \\
&\le \alpha_{n,i}\|u_n - x^*\|^2 + (1-\alpha_{n,i})\|u_n - x^*\|^2 \\
&\le \alpha_{n,i}\|u_n - x^*\|^2 + (1-\alpha_{n,i})\left\|T_{\lambda_n}^{(F_2,\varphi_2)}(z_n - \lambda_n B z_n) - T_{\lambda_n}^{(F_2,\varphi_2)}(x^* - \lambda_n B x^*)\right\|^2 \\
&\le \alpha_{n,i}\|u_n - x^*\|^2 + (1-\alpha_{n,i})\|(z_n - \lambda_n B z_n) - (x^* - \lambda_n B x^*)\|^2 \\
&\le \alpha_{n,i}\|u_n - x^*\|^2 + (1-\alpha_{n,i})\left[\|z_n - x^*\|^2 + \lambda_n(\lambda_n - 2\beta)\|B z_n - B x^*\|^2\right] \\
&\le \|u_n - x^*\|^2 + (1-\alpha_{n,i})\lambda_n(\lambda_n - 2\beta)\|B z_n - B x^*\|^2 \\
&\le \|x_n - x^*\|^2 + (1-\alpha_{n,i})\lambda_n(\lambda_n - 2\beta)\|B z_n - B x^*\|^2.
\end{aligned} \tag{3.13}$$

Since $0 < c \le \lambda_n \le f < 2\beta$, $0 \le k_i \le \alpha_{n,i} \le d_i < 1$, we have

$$\begin{aligned}
(1-d_i)c(2\beta - f)\|B z_n - B x^*\|^2 &\le \|x_n - x^*\|^2 - \|y_{n,i} - x^*\|^2 \\
&\le \|y_{n,i} - x_n\|\left(\|x_n - x^*\| + \|y_{n,i} - x^*\|\right).
\end{aligned} \tag{3.14}$$

Hence, $\lim_{n\to\infty}\|B z_n - B x^*\| = 0$. From (3.1), we have

$$\begin{aligned}
\|y_{n,i} - x^*\|^2 &\le \alpha_{n,i}\|u_n - x^*\|^2 + (1-\alpha_{n,i})\left[\|u_n - x^*\|^2 + k_i\|T_i w_n - w_n\|^2\right] \\
&\quad - \alpha_{n,i}(1-\alpha_{n,i})\|T_i w_n - w_n\|^2 \\
&= \alpha_{n,i}\|u_n - x^*\|^2 + (1-\alpha_{n,i})\|u_n - x^*\|^2 - (1-\alpha_{n,i})(\alpha_{n,i} - k_i)\|T_i w_n - w_n\|^2 \\
&\le \|u_n - x^*\|^2.
\end{aligned} \tag{3.15}$$

On the other hand,

$$\begin{aligned}
\|u_n - x^*\|^2 &\leq \left\| T_{\lambda_n}^{(F_2,\varphi_2)}(z_n - \lambda_n B z_n) - T_{\lambda_n}^{(F_2,\varphi_2)}(x^* - \lambda_n B x^*) \right\|^2 \\
&\leq \langle (z_n - \lambda_n B z_n) - (x^* - \lambda_n B x^*), u_n - x^* \rangle \\
&= \frac{1}{2}\Big[ \|(z_n - \lambda_n B z_n) - (x^* - \lambda_n B x^*)\|^2 + \|u_n - x^*\|^2 \\
&\qquad - \|(z_n - \lambda_n B z_n) - (x^* - \lambda_n B x^*) - (u_n - x^*)\|^2 \Big] \\
&\leq \frac{1}{2}\Big[ \|z_n - x^*\|^2 + \|u_n - x^*\|^2 - \|(z_n - \lambda_n B z_n) - (x^* - \lambda_n B x^*) - (u_n - x^*)\|^2 \Big] \\
&= \frac{1}{2}\Big[ \|z_n - x^*\|^2 + \|u_n - x^*\|^2 - \|u_n - z_n\|^2 + 2\lambda_n \langle z_n - u_n, B z_n - B x^* \rangle \\
&\qquad - \lambda_n^2 \|B z_n - B x^*\|^2 \Big],
\end{aligned} \tag{3.16}$$

and hence

$$\begin{aligned}
\|u_n - x^*\|^2 &\leq \|z_n - x^*\|^2 - \|u_n - z_n\|^2 + 2\lambda_n \langle z_n - u_n, B z_n - B x^* \rangle \\
&\quad - \lambda_n^2 \|B z_n - B x^*\|^2 \\
&\leq \|z_n - x^*\|^2 - \|u_n - z_n\|^2 + 2\lambda_n \|z_n - u_n\| \|B z_n - B x^*\|.
\end{aligned} \tag{3.17}$$

Putting (3.17) into (3.15), we have

$$\left\| y_{n,i} - x^* \right\|^2 \leq \|z_n - x^*\|^2 - \|u_n - z_n\|^2 + 2\lambda_n \|z_n - u_n\| \|B z_n - B x^*\|. \tag{3.18}$$

It follows that

$$\begin{aligned}
\|z_n - u_n\|^2 &\leq \|z_n - x^*\|^2 - \left\| y_{n,i} - x^* \right\|^2 + 2\lambda_n \|z_n - u_n\| \|B z_n - B x^*\| \\
&\leq \|x_n - x^*\|^2 - \left\| y_{n,i} - x^* \right\|^2 + 2\lambda_n \|z_n - u_n\| \|B z_n - B x^*\| \\
&\leq \left\| y_{n,i} - x_n \right\| \left( \|x_n - x^*\| + \left\| y_{n,i} - x^* \right\| \right) \\
&\quad + 2\lambda_n \|z_n - u_n\| \|B z_n - B x^*\|.
\end{aligned} \tag{3.19}$$

Therefore, $\lim_{n\to\infty}\|z_n - u_n\| = 0$. Furthermore,

$$\begin{aligned}
\left\|y_{n,i} - x^*\right\|^2 &\le \alpha_{n,i}\|w_n - x^*\|^2 + (1-\alpha_{n,i})\|T_i w_n - x^*\|^2 \\
&\quad - \alpha_{n,i}(1-\alpha_{n,i})\|T_i w_n - w_n\|^2 \\
&\le \alpha_{n,i}\|x_n - x^*\|^2 + (1-\alpha_{n,i})\|u_n - x^*\|^2 \\
&\quad - (1-\alpha_{n,i})(\alpha_{n,i} - k_i)\|T_i w_n - w_n\|^2 \\
&\le \alpha_{n,i}\|x_n - x^*\|^2 + (1-\alpha_{n,i})\|u_n - x^*\|^2 \\
&\le \alpha_{n,i}\|x_n - x^*\|^2 + (1-\alpha_{n,i})\|z_n - x^*\|^2 \\
&\le \alpha_{n,i}\|x_n - x^*\|^2 + (1-\alpha_{n,i})\left\|T_{r_n}^{(F_1,\varphi_1)}(x_n - r_n A x_n) - T_{r_n}^{(F_1,\varphi_1)}(x^* - r_n A x^*)\right\|^2 \\
&\le \alpha_{n,i}\|x_n - x^*\|^2 + (1-\alpha_{n,i})\|(x_n - r_n A x_n) - (x^* - r_n A x^*)\|^2 \\
&\le \alpha_{n,i}\|x_n - x^*\|^2 + (1-\alpha_{n,i})\left[\|x_n - x^*\|^2 + r_n(r_n - 2\alpha)\|Ax_n - Ax^*\|^2\right] \\
&= \|x_n - x^*\|^2 + (1-\alpha_{n,i})r_n(r_n - 2\alpha)\|Ax_n - Ax^*\|^2.
\end{aligned} \tag{3.20}$$

Since $0 < a \le r_n \le b < 2\alpha$ and $0 \le k_i \le \alpha_{n,i} \le d_i < 1$, we have

$$\begin{aligned}
(1-d_i)a(2\alpha - b)\|Ax_n - Ax^*\|^2 &\le \|x_n - x^*\|^2 - \left\|y_{n,i} - x^*\right\|^2 \\
&\le \left\|y_{n,i} - x_n\right\|\left(\|x_n - x^*\| + \left\|y_{n,i} - x^*\right\|\right).
\end{aligned} \tag{3.21}$$

Hence, $\lim_{n\to\infty}\|Ax_n - Ax^*\| = 0$. From (3.1), we have

$$\begin{aligned}
\left\|y_{n,i} - x^*\right\|^2 &\le \alpha_{n,i}\|w_n - x^*\|^2 + (1-\alpha_{n,i})\|T_i w_n - x^*\|^2 \\
&\quad - \alpha_{n,i}(1-\alpha_{n,i})\|T_i w_n - w_n\|^2 \\
&\le \alpha_{n,i}\|x_n - x^*\|^2 + (1-\alpha_{n,i})\|u_n - x^*\|^2 \\
&\quad - (1-\alpha_{n,i})(\alpha_{n,i} - k_i)\|T_i w_n - w_n\|^2 \\
&\le \alpha_{n,i}\|x_n - x^*\|^2 + (1-\alpha_{n,i})\|u_n - x^*\|^2 \\
&\le \alpha_{n,i}\|x_n - x^*\|^2 + (1-\alpha_{n,i})\|z_n - x^*\|^2.
\end{aligned} \tag{3.22}$$

On the other hand,

$$\begin{aligned}
\|z_n - x^*\|^2 &\le \left\| T_{r_n}^{(F_1,\varphi_1)}(x_n - r_n A x_n) - T_{r_n}^{(F_1,\varphi_1)}(x^* - r_n A x^*) \right\|^2 \\
&\le \langle (x_n - r_n A x_n) - (x^* - r_n A x^*), z_n - x^* \rangle \\
&= \frac{1}{2}\Big[ \|(x_n - r_n A x_n) - (x^* - r_n A x^*)\|^2 + \|z_n - x^*\|^2 \\
&\qquad - \|(x_n - r_n A x_n) - (x^* - r_n A x^*) - (z_n - x^*)\|^2 \Big] \\
&\le \frac{1}{2}\Big[ \|x_n - x^*\|^2 + \|z_n - x^*\|^2 - \|(x_n - r_n A x_n) - (x^* - r_n A x^*) - (z_n - x^*)\|^2 \Big] \\
&= \frac{1}{2}\Big[ \|x_n - x^*\|^2 + \|z_n - x^*\|^2 - \|z_n - x_n\|^2 + 2r_n \langle x_n - z_n, A x_n - A x^* \rangle \\
&\qquad - r_n^2 \|A x_n - A x^*\|^2 \Big],
\end{aligned} \tag{3.23}$$

and hence

$$\begin{aligned}
\|z_n - x^*\|^2 &\le \|x_n - x^*\|^2 - \|z_n - x_n\|^2 + 2r_n \langle x_n - z_n, A x_n - A x^* \rangle - r_n^2 \|A x_n - A x^*\|^2 \\
&\le \|x_n - x^*\|^2 - \|z_n - x_n\|^2 + 2r_n \|x_n - z_n\| \|A x_n - A x^*\|.
\end{aligned} \tag{3.24}$$

Putting (3.24) into (3.22), we have

$$\left\| y_{n,i} - x^* \right\|^2 \le \|x_n - x^*\|^2 - (1 - \alpha_{n,i}) \|z_n - x_n\|^2 + 2r_n \|x_n - z_n\| \|A x_n - A x^*\|. \tag{3.25}$$

It follows that

$$\begin{aligned}
(1 - d_i)\|x_n - z_n\|^2 &\le \|x_n - x^*\|^2 - \left\| y_{n,i} - x^* \right\|^2 + 2r_n \|x_n - z_n\| \|A x_n - A x^*\| \\
&\le \left\| y_{n,i} - x_n \right\| \left( \|x_n - x^*\| + \left\| y_{n,i} - x^* \right\| \right) + 2r_n \|x_n - z_n\| \|A x_n - A x^*\|.
\end{aligned} \tag{3.26}$$

Therefore, $\lim_{n\to\infty}\|x_n - z_n\| = 0$. Then, we obtain that

$$\|x_{n+1} - u_n\| \le \|x_{n+1} - x_n\| + \|x_n - z_n\| + \|z_n - u_n\| \longrightarrow 0, \quad n \longrightarrow \infty. \tag{3.27}$$

Since $x_{n+1} \in C_{n+1}$, then

$$\left\| y_{n,i} - x_{n+1} \right\| \le \|x_n - x_{n+1}\|. \tag{3.28}$$

But $y_{n,i} = \alpha_{n,i} w_n + (1 - \alpha_{n,i}) T_i w_n$ implies that

$$\left\| y_{n,i} - x_{n+1} \right\|^2 = \alpha_{n,i} \|w_n - x_{n+1}\|^2 + (1 - \alpha_{n,i}) \|T_i w_n - x_{n+1}\|^2 - \alpha_{n,i}(1 - \alpha_{n,i}) \|w_n - T_i w_n\|^2. \tag{3.29}$$

Putting (3.29) into (3.28), we have

$$(1-\alpha_{n,i})\|T_iw_n - x_{n+1}\|^2 \le \alpha_{n,i}(1-\alpha_{n,i})\|w_n - T_iw_n\|^2 + \|x_n - x_{n+1}\|^2 - \alpha_{n,i}\|w_n - x_{n+1}\|^2. \quad (3.30)$$

Thus, we get

$$\begin{aligned}\|T_iw_n - x_{n+1}\|^2 &\le \alpha_{n,i}\|w_n - T_iw_n\|^2 + \frac{1}{(1-\alpha_{n,i})}\|x_n - x_{n+1}\|^2 \\ &\le \alpha_{n,i}\|w_n - T_iw_n\|^2 + \frac{1}{1-d_i}\|x_n - x_{n+1}\|^2.\end{aligned} \quad (3.31)$$

But

$$\|T_iw_n - x_{n+1}\|^2 = \|x_{n+1} - w_n\|^2 + 2\langle x_{n+1} - w_n, w_n - T_iw_n\rangle + \|w_n - T_iw_n\|^2. \quad (3.32)$$

Putting (3.32) into (3.31) and rearranging, we have

$$\begin{aligned}(1-d_i)\|w_n - T_iw_n\|^2 &\le \frac{1}{1-d_i}\|x_n - x_{n+1}\|^2 - 2\langle x_{n+1} - w_n, w_n - T_iw_n\rangle \\ &\le \frac{1}{1-d_i}\|x_n - x_{n+1}\|^2 + 2\|x_{n+1} - w_n\|\|w_n - T_iw_n\|.\end{aligned} \quad (3.33)$$

Hence, $\lim_{n\to\infty}\|w_n - T_iw_n\| = 0$, $i = 1, 2, \ldots$. Now,

$$\begin{aligned}\|w_n - x^*\|^2 &\le \|(I - s_nD)u_n - (I - s_nD)x^*\|^2 \\ &= \|u_n - x^*\|^2 - 2s_n\langle u_n - x^*, Du_n - Dx^*\rangle + s_n^2\|Du_n - Dx^*\|^2 \\ &\le \|x_n - x^*\|^2 - 2s_n\left[-\lambda\|Du_n - Dx^*\|^2 + \gamma\|u_n - x^*\|^2\right] + s_n^2\|Du_n - Dx^*\|^2 \\ &\le \|x_n - x^*\|^2 + \left(2s_n\lambda + s_n^2 - \frac{2s_n\gamma}{\mu^2}\right)\|Du_n - Dx^*\|^2.\end{aligned} \quad (3.34)$$

Furthermore,

$$\begin{aligned}\left\|y_{n,i} - x^*\right\|^2 &\le \alpha_{n,i}\|w_n - x^*\|^2 + (1-\alpha_{n,i})\|T_iw_n - x^*\|^2 \\ &\le \alpha_{n,i}\|x_n - x^*\|^2 + (1-\alpha_{n,i})\|T_iw_n - x^*\|^2 \\ &\le \alpha_{n,i}\|x_n - x^*\|^2 + (1-\alpha_{n,i})\|w_n - x^*\|^2 \\ &\le \alpha_{n,i}\|x_n - x^*\|^2 + (1-\alpha_{n,i})\left[\|x_n - x^*\|^2 + \left(2s_n\lambda + s_n^2 - \frac{2s_n\gamma}{\mu^2}\right)\|Du_n - Dx^*\|^2\right].\end{aligned} \quad (3.35)$$

Thus,

$$-\left(2s_n\lambda + s_n^2 - \frac{2s_n\gamma}{\mu^2}\right)\|Du_n - Dx^*\|^2 \le \|x_n - x^*\|^2 - \|y_{n,i} - x^*\|^2 \\ \le \|y_{n,i} - x_n\|\left(\|x_n - x^*\| + \|y_{n,i} - x^*\|\right). \tag{3.36}$$

By conditions (iii) and (iv), we have that $\lim_{n\to\infty}\|Du_n - Dx^*\| = 0$. Now, (2.2), we obtain

$$\begin{aligned}\|w_n - x^*\|^2 &\le \|P_K(u_n - s_nDu_n) - P_K(x^* - s_nDx^*)\|^2 \\ &\le \langle (u_n - s_nDu_n) - (x^* - s_nDx^*), w_n - x^*\rangle \\ &= \frac{1}{2}\Big[\|(u_n - s_nDu_n) - (x^* - s_nDx^*)\|^2 + \|w_n - x^*\|^2 \\ &\quad -\|(u_n - s_nDu_n) - (x^* - s_nDx^*) - (w_n - x^*)\|^2\Big] \\ &\le \frac{1}{2}\Big[\|w_n - x^*\|^2 + \|u_n - x^*\|^2 - \|(u_n - s_nDu_n) - (x^* - s_nDx^*) - (w_n - x^*)\|^2\Big] \\ &= \frac{1}{2}\Big[\|x_n - x^*\|^2 + \|w_n - x^*\|^2 - \|w_n - u_n\|^2 + 2s_n\langle u_n - w_n, Du_n - Dx^*\rangle \\ &\quad -s_n^2\|Du_n - Dx^*\|^2\Big].\end{aligned} \tag{3.37}$$

Thus,

$$\|w_n - x^*\|^2 \le \|x_n - x^*\|^2 - \|w_n - u_n\|^2 + 2s_n\|w_n - u_n\|\|Du_n - Dx^*\|. \tag{3.38}$$

Using this last inequality, we obtain from (3.1) that

$$\begin{aligned}\|y_{n,i} - x^*\|^2 &\le \alpha_{n,i}\|x_n - x^*\|^2 + (1 - \alpha_{n,i})\|T_iw_n - x^*\|^2 \\ &\le \alpha_{n,i}\|x_n - x^*\|^2 + (1 - \alpha_{n,i})\|w_n - x^*\|^2 \\ &\le \|x_n - x^*\|^2 - (1 - \alpha_{n,i})\|w_n - u_n\|^2 \\ &\quad + 2s_n(1 - \alpha_{n,i})\|w_n - u_n\|\|Du_n - Dx^*\|.\end{aligned} \tag{3.39}$$

This implies that

$$\begin{aligned}(1 - \alpha_{n,i})\|w_n - u_n\|^2 &\le \|x_n - x^*\|^2 - \|y_{n,i} - x^*\|^2 \\ &\quad + 2s_n(1 - \alpha_{n,i})\|w_n - u_n\|\|Du_n - Dx^*\| \\ &\le \|y_{n,i} - x_n\|\left(\|x_n - x^*\| + \|y_{n,i} - x^*\|\right) \\ &\quad + 2s_n(1 - \alpha_{n,i})\|w_n - u_n\|\|Du_n - Dx^*\|.\end{aligned} \tag{3.40}$$

Since $0 \leq k_i \leq \alpha_{n,i} \leq d_i < 1$, we have $\lim_{n\to\infty}\|w_n - u_n\| = 0$. Also since $\lim_{n\to\infty}\|w_n - x_n\| = 0$ and $\lim_{n\to\infty}\|w_n - z\| = 0$, we have that $\lim_{n\to\infty}\|w_n - z\| = 0$. By $\lim_{n\to\infty}\|w_n - z\| = 0$ and $\lim_{n\to\infty}\|w_n - T_i w_n\| = 0$, $i = 1, 2, \ldots$, we have that $z \in \cap_{i=1}^{\infty} F(T_i)$.Since $z_n := T_{r_n}^{(F_1,\varphi_1)}(x_n - r_n A x_n)$, $n \geq 1$, we have for any $y \in K$ that

$$F_1(z_n, y) + \varphi_1(y) - \varphi_1(z_n) + \langle Ax_n, y - z_n\rangle + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geq 0. \tag{3.41}$$

Furthermore, from the last inequality and using (A2), we obtain

$$\varphi_1(y) - \varphi_1(z_n) + \langle Ax_n, y - z_n\rangle + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geq F_1(y, z_n). \tag{3.42}$$

Let $z_t := ty + (1-t)z$ for all $t \in (0,1]$ and $y \in K$. This implies that $z_t \in K$. Then, we have

$$\begin{aligned}\langle z_t - z_n, Az_t\rangle &\geq \varphi_1(z_n) - \varphi_1(z_t) + \langle z_t - z_n, Az_t\rangle - \langle z_t - z_n, Ax_n\rangle \\ &\quad - \left\langle z_t - z_n, \frac{z_n - x_n}{r_n}\right\rangle F_1(z_t, z_n) \\ &= \varphi_1(z_n) - \varphi_1(z_t) + \langle z_t - z_n, Az_t - Az_n\rangle \\ &\quad + \langle z_t - z_n, Az_n - Ax_n\rangle - \left\langle z_t - z_n, \frac{z_n - x_n}{r_n}\right\rangle + F_1(z_t, z_n).\end{aligned} \tag{3.43}$$

Since $\|x_n - z_n\| \to 0$, $n \to \infty$, we obtain $\|Ax_n - Az_n\| \to 0$, $n \to \infty$. Furthermore, by the monotonicity of $A$, we obtain $\langle z_t - z_n, Az_t - Az_n\rangle \geq 0$. Then, by (A4) we obtain (noting that $z_n \to z$, $n \to \infty$ since $\|z_n - z\| \leq \|x_n - z\| + \|x_n - z_n\|$),

$$\langle z_t - z, Az_t\rangle \geq \varphi_1(z) - \varphi_1(z_t) + F_1(z_t, z) \tag{3.44}$$

Using (A1), (A4) and (3.44), we also obtain

$$\begin{aligned}0 = F_1(z_t, z_t) + \varphi_1(z_t) - \varphi_1(z_t) &\leq tF_1(z_t, y) + (1-t)F_1(z_t, z) \\ &\quad + t\varphi_1(y) + (1-t)\varphi_1(z) - \varphi_1(z_t) \\ &\leq t[F_1(z_t, y) + \varphi_1(y) - \varphi_1(z_t)] + (1-t)\langle z_t - z, Az_t\rangle \\ &= t[F_1(z_t, y) + \varphi_1(y) - \varphi_1(z_t)] + (1-t)t\langle y - z, Az_t\rangle,\end{aligned} \tag{3.45}$$

and hence

$$0 \leq F_1(z_t, y) + \varphi_1(y) - \varphi_1(z_t) + (1-t)\langle y - z, Az_t\rangle. \tag{3.46}$$

Letting $t \to 0$, we have, for each $y \in K$,

$$0 \leq F_1(z, y) + \varphi_1(y) - \varphi_1(z) + \langle y - z, Az\rangle. \tag{3.47}$$

This implies that $z \in \mathrm{GMEP}(F_1, A, \varphi_1)$. By following the same arguments, we can show that $z \in \mathrm{GMEP}(F_2, B, \varphi_2)$.

Next, we show $z \in V(K, D)$. Put

$$Mx = \begin{cases} Dx + N_K x, & x \in K, \\ \emptyset, & x \notin K. \end{cases} \tag{3.48}$$

Since $D$ is relaxed $(\lambda, \gamma)$-cocoercive and by condition (iv), we have

$$\langle Dx - Dy, x - y \rangle \geq (-\lambda)\|Dx - Dy\|^2 + \gamma\|x - y\|^2 \geq \left(\gamma - \lambda\mu^2\right)\|x - y\|^2 \geq 0, \tag{3.49}$$

which shows that $D$ is monotone. Thus, $M$ is maximal monotone. Let $(x, y) \in G(M)$. Since $y - Dx \in N_K x$ and $w_n \in K$, we have

$$\langle x - w_n, y - Dx \rangle \geq 0. \tag{3.50}$$

On the other hand, from $w_n = P_{\mathrm{K}}(I - s_n A)u_n$, we have

$$\langle x - w_n, w_n - (I - s_n D)u_n \rangle \geq 0, \tag{3.51}$$

and hence

$$\left\langle x - w_n, \frac{w_n - u_n}{s_n} + Du_n \right\rangle \geq 0. \tag{3.52}$$

It follows that

$$\begin{aligned} \langle x - w_n, y \rangle \geq \langle x - w_n, Dx \rangle &\geq \langle x - w_n, y \rangle - \left\langle x - w_n, \frac{w_n - u_n}{s_n} + Du_n \right\rangle \\ &= \left\langle x - w_n, Dx - \frac{w_n - u_n}{s_n} - Du_n \right\rangle \\ &\geq \langle x - w_n, Dw_n - Du_n \rangle - \left\langle x - w_n, \frac{\rho_n - u_n}{s_n} \right\rangle, \end{aligned} \tag{3.53}$$

which implies that $\langle x - z, y \rangle \geq 0$. We have $z \in M^{-1}0$ and hence $z \in \mathrm{VI}(K, D)$. Therefore, $z \in \cap_{i=1}^{\infty} F(T_i) \cap \mathrm{GMEP}(F_1, A, \varphi_1) \cap \mathrm{GMEP}(F_2, B, \varphi_2) \cap \mathrm{VI}(K, D)$.

Noting that $x_n = P_{C_n} x_0$, we have by (2.3) that

$$\langle x_0 - x_n, y - x_n \rangle \leq 0, \tag{3.54}$$

for all $y \in C_n$. Since $\Omega \subset C_n$ and by the continuity of inner product, we obtain from the above inequality that

$$\langle x_0 - z, y - z \rangle \leq 0, \tag{3.55}$$

for all $y \in \Omega$. By (2.3) again, we conclude that $z = P_\Omega x_0$. This completes the proof. □

**Corollary 3.2.** *Let $K$ be a nonempty closed and convex subset of a real Hilbert space $H$. For each $m = 1, 2$, let $F_m$ be a bifunction from $K \times K$ satisfying (A1)–(A4), $\varphi_m : K \to \mathbb{R} \cup \{+\infty\}$ a proper lower semicontinuous and convex function with assumption (B1) or (B2), $A$ an $\alpha$-inverse-strongly monotone mapping of $K$ into $H$, $B$ a $\beta$-inverse-strongly monotone mapping of $K$ into $H$ and for each $i = 1, 2, \ldots$, let $T_i : K \to K$ be a nonexpansive mapping such that $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $D$ be a $\mu$-Lipschitzian, relaxed $(\lambda, \gamma)$-cocoercive mapping of $K$ into $H$. Suppose $\Omega := \cap_{i=1}^{\infty} F(T_i) \cap \mathrm{GMEP}(F_1, A, \varphi_1) \cap \mathrm{GMEP}(F_2, B, \varphi_2) \cap \mathrm{VI}(K, D) \neq \emptyset$. Let $\{z_n\}_{n=1}^{\infty}$, $\{u_n\}_{n=1}^{\infty}$, $\{w_n\}_{n=1}^{\infty}$, $\{y_{n,i}\}_{n=1}^{\infty}$ $(i = 1, 2, \ldots)$ and $\{x_n\}_{n=0}^{\infty}$ be generated by $x_0 \in K$, $C_{1,i} = K$, $C_1 = \cap_{i=1}^{\infty} C_{1,i}$, $x_1 = P_{C_1} x_0$,*

$$\begin{aligned}
z_n &= T_{r_n}^{(F_1, \varphi_1)} (x_n - r_n A x_n), \\
u_n &= T_{\lambda_n}^{(F_2, \varphi_2)} (z_n - \lambda_n B z_n), \\
w_n &= P_K (u_n - s_n D u_n), \\
y_{n,i} &= \alpha_{n,i} w_n + (1 - \alpha_{n,i}) T_i w_n, \\
C_{n+1,i} &= \left\{ z \in C_{n,i} : \left\| y_{n,i} - z \right\| \leq \| x_n - z \| \right\}, \\
C_{n+1} &= \bigcap_{i=1}^{\infty} C_{n+1,i}, \\
x_{n+1} &= P_{C_{n+1}} x_0, \quad n \geq 1.
\end{aligned} \tag{3.56}$$

*Assume that $\{\alpha_{n,i}\}_{n=1}^{\infty} \subset [0, 1)$ $(i = 1, 2, \ldots)$, $\{r_n\}_{n=1}^{\infty} \subset [0, 2\alpha]$, and $\{\lambda_n\}_{n=1}^{\infty} \subset [0, 2\beta]$ satisfy*

(i) $0 < a \leq r_n \leq b < 2\alpha$,

(ii) $0 < c \leq \lambda_n \leq f < 2\beta$,

(iii) $0 \leq \alpha_{n,i} \leq d_i < 1$,

(iv) $0 < h \leq s_n \leq j < 2(\gamma - \lambda\mu^2)/\mu^2$.

*Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to $P_\Omega x_0$.*

Let $C$ be a nonempty closed and convex cone in $H$ and $D$ an operator of $C$ into $H$. We define the *polar* of $C$ in $H$ to be the set

$$K^* := \left\{ y^* \in H : \langle x, y^* \rangle \geq 0, \ \forall x \in C \right\}. \tag{3.57}$$

Then, the element $u \in C$ is called a solution of the *complementarity problem* if

$$Du \in K^*, \quad \langle u, Du \rangle = 0. \tag{3.58}$$

The set of solutions of the complementarity problem is denoted by $C(C,D)$. We shall assume that $D$ satisfies the following conditions:

(E1) $D$ is $\gamma$-inverse strongly monotone,

(E2) $C(C,D) \neq \emptyset$.

Also, we replace conditions (B1) and (B2) with

(D1) for each $x \in H$ and $r > 0$ there exist a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0, \tag{3.59}$$

(D2) $C$ is a bounded set.

**Theorem 3.3.** *Let $C$ be a nonempty closed and convex cone of a real Hilbert space $H$. For each $m = 1,2$, let $F_m$ be a bifunction from $C \times C$ satisfying (A1)–(A4), $\varphi_m : C \to \mathbb{R} \cup \{+\infty\}$ a proper lower semicontinuous and convex function with assumption (D1) or (D2), $A$ an $\alpha$-inverse-strongly monotone mapping of $C$ into $H$, $B$ a $\beta$-inverse-strongly monotone mapping of $C$ into $H$ and for each $i = 1,2,\ldots$, let $T_i : C \to C$ be a $k_i$-strictly pseudocontractive mapping for some $0 \le k_i < 1$ such that $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $D$ be a $\mu$-Lipschitzian, relaxed $(\lambda,\gamma)$-cocoercive mapping of $C$ into $H$. Suppose $\Omega := \cap_{i=1}^{\infty} F(T_i) \cap \mathrm{GMEP}(F_1, A, \varphi_1) \cap \mathrm{GMEP}(F_2, B, \varphi_2) \cap C(C,D) \neq \emptyset$. Let $\{z_n\}_{n=1}^{\infty}, \{u_n\}_{n=1}^{\infty}, \{w_n\}_{n=1}^{\infty}, \{y_{n,i}\}_{n=1}^{\infty}$ $(i = 1,2,\ldots)$, and $\{x_n\}_{n=0}^{\infty}$ be generated by $x_0 \in C, C_{1,i} = C, C_1 = \cap_{i=1}^{\infty} C_{1,i}, x_1 = P_{C_1} x_0$,*

$$\begin{aligned}
z_n &= T_{r_n}^{(F_1,\varphi_1)}(x_n - r_n A x_n),\\
u_n &= T_{\lambda_n}^{(F_2,\varphi_2)}(z_n - \lambda_n B z_n),\\
w_n &= P_C(u_n - s_n D u_n),\\
y_{n,i} &= \alpha_{n,i} w_n + (1 - \alpha_{n,i}) T_i w_n,\\
C_{n+1,i} &= \{z \in C_{n,i} : \|y_{n,i} - z\| \le \|x_n - z\|\},\\
C_{n+1} &= \bigcap_{i=1}^{\infty} C_{n+1,i},\\
x_{n+1} &= P_{C_{n+1}} x_0, \quad n \ge 1.
\end{aligned} \tag{3.60}$$

*Assume that $\{\alpha_{n,i}\}_{n=1}^{\infty} \subset [0,1)$ $(i = 1,2,\ldots)$, $\{r_n\}_{n=1}^{\infty} \subset [0,2\alpha]$, and $\{\lambda_n\}_{n=1}^{\infty} \subset [0,2\beta]$ satisfy*

(i) $0 < a \le r_n \le b < 2\alpha$,

(ii) $0 < c \le \lambda_n \le f < 2\beta$,

(iii) $0 \le k_i \le \alpha_{n,i} \le d_i < 1$,

(iv) $0 < h \le s_n \le j < 2(\gamma - \lambda\mu^2)/\mu^2$.

*Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to $P_{\Omega} x_0$.*

*Proof.* Using Lemma 7.1.1 of [34], we have that $\mathrm{VI}(C, D) = C(C, D)$. Hence, by Theorem 3.1, we obtain the desired conclusion. □

*Remark 3.4.* Our Corollary 3.2 extends Theorems 1.1 and 1.2.

*Remark 3.5.* Our iterative scheme (3.1) is simpler than the iterative schemes (5.1) and (5.11) of Acedo and Xu [6]. Furthermore, in our results, we use iterative scheme (3.1) to approximate a common fixed point of an *infinite family of $k$-strictly pseudocontractive mappings* while the iterative schemes (5.1) and (5.11) of Acedo and Xu [6] are used to approximate a common fixed point of a *finite family of $k$-strictly pseudocontractive mappings.*

*Remark 3.6.* Our results also hold for infinite family of uniformly continuous quasistrict pseudocontractions. Hence, we can adapt our results for an infinite family of uniformly continuous quasi-nonexpansive mappings in a real Hilbert space.

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