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Research Article

A Note on Kantorovich Inequality for Hermite Matrices

Zhibing Liu,[1, 2] Kanmin Wang,[2] and Chengfeng Xu[2]

[1] *School of Mathematical Sciences, Xiamen University, Xiamen 361005, China*
[2] *Department of Mathematics, Jiujiang University, Jiujiang 332005, China*

Correspondence should be addressed to Zhibing Liu, liuzhibingjju@126.com

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A new Kantorovich type inequality for Hermite matrices is proposed in this paper. It holds for the invertible Hermite matrices and provides refinements of the classical results. Elementary methods suffice to prove the inequality.

1. Introduction

We first state the well-known Kantorovich inequality for a positive definite Hermite matrix (see [1, 2]), let $A \in M_n$ be a positive definite Hermite matrix with real eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Then

$$1 \leq x^* A x x^* A^{-1} x \leq \frac{(\lambda_1 + \lambda_n)^2}{4\lambda_1 \lambda_n}, \tag{1.1}$$

for any $x \in C^n$, $\|x\| = 1$, where A^* denotes the conjugate transpose of the matrix A. An equivalent form of this result is incorporated in

$$0 \leq x^* A x x^* A^{-1} x - 1 \leq \frac{(\lambda_n - \lambda_1)^2}{4\lambda_1 \lambda_n}, \tag{1.2}$$

for any $x \in C^n$, $\|x\| = 1$.

This famous inequality plays an important role in statistics and numerical analysis, for example, in discussions of converging rates and error bounds of solving systems of equations (see [2–4]). Motivated by interests in applied mathematics outlined above, we establish in this paper a new Kantorovich type inequality, the classical Kantorovich inequality is modified to apply not only to positive definite but also to all invertible Hermitian matrices. An elementary proof of this result is also presented.

In the next section, we will state the main theorem and its proof. Before starting, we quickly review some basic definitions and notations. Let $A \in M_n$ be an invertible Hermite matrix with real eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, and the corresponding orthonormal eigenvectors $\varphi_1, \varphi_2, \ldots, \varphi_n$ with $\|\varphi_i\| = 1$ $(i = 1, 2, \ldots, n)$, where $\|\varphi_i\|$ denotes 2-norm of the vector of C_n.

For A, we define the following transform

$$C(A, x) = x^*(\lambda_n I - A)(A - \lambda_1 I)x. \tag{1.3}$$

If $\lambda_1\lambda_n > 0$, then,

$$C\left(A^{-1}, x\right) = x^*\left(\frac{1}{\lambda_1}I - A^{-1}\right)\left(A^{-1} - \frac{1}{\lambda_n}I\right)x. \tag{1.4}$$

Otherwise, $\lambda_1\lambda_n < 0$, then,

$$C\left(A^{-1}, x\right) = x^*\left(\frac{1}{\lambda_{k+1}}I - A^{-1}\right)\left(A^{-1} - \frac{1}{\lambda_k}I\right)x, \tag{1.5}$$

where

$$\lambda_1 \leq \cdots \leq \lambda_k < 0 < \lambda_{k+1} \leq \cdots \leq \lambda_n. \tag{1.6}$$

2. New Kantorovich Inequality for Hermite Matrices

Theorem 2.1. *Let A be an $n \times n$ invertible Hermite matrix with real eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Then*

$$\left|x^*Axx^*A^{-1}x - 1\right| \leq \begin{cases} \dfrac{(\lambda_n - \lambda_1)^2}{4\lambda_1\lambda_n} - \sqrt{C(A,x)C\left(A^{-1},x\right)} & \text{if } \lambda_1\lambda_n > 0, \\ \dfrac{(\lambda_n - \lambda_1)(\lambda_{k+1} - \lambda_k)}{4|\lambda_k\lambda_{k+1}|} - \sqrt{C(A,x)C\left(A^{-1},x\right)} & \text{if } \lambda_1\lambda_n < 0 \end{cases} \tag{2.1}$$

for any $x \in C^n$, $\|x\| = 1$, where $C(A,x)$, $C(A^{-1},x)$, λ_k, λ_{k+1} defined by (1.3), (1.4), (1.5), *and* (1.6).

To simplify the proof, we first introduce some lemmas.

Lemma 2.2. *With the assumptions in Theorem 2.1, then*

$$\lambda_1 \le x^* A x \le \lambda_n, \quad 0 \le (\lambda_n - x^* A x)(x^* A x - \lambda_1) \le \frac{1}{4}(\lambda_n - \lambda_1)^2,$$
$$\frac{1}{\lambda_n} \le x^* A^{-1} x \le \frac{1}{\lambda_1}, \quad 0 \le \left(\frac{1}{\lambda_1} - x^* A^{-1} x\right)\left(x^* A^{-1} x - \frac{1}{\lambda_n}\right) \le \frac{(\lambda_n - \lambda_1)^2}{4(\lambda_n \lambda_1)^2} \quad \text{if } \lambda_1 \lambda_n > 0,$$
$$\frac{1}{\lambda_k} \le x^* A^{-1} x \le \frac{1}{\lambda_{k+1}}, \quad 0 \le \left(\frac{1}{\lambda_{k+1}} - x^* A^{-1} x\right)\left(x^* A^{-1} x - \frac{1}{\lambda_k}\right) \le \frac{(\lambda_{k+1} - \lambda_k)^2}{4(\lambda_{k+1} \lambda_k)^2} \quad \text{if } \lambda_1 \lambda_n < 0 \tag{2.2}$$

for any $x \in C^n$, $\|x\| = 1$.

Lemma 2.3. *With the assumptions in Theorem 2.1, then*

$$0 \le C(A, x) \le \frac{1}{4}(\lambda_n - \lambda_1)^2,$$
$$0 \le C\left(A^{-1}, x\right) \le \begin{cases} \dfrac{(\lambda_n - \lambda_1)^2}{4(\lambda_1 \lambda_n)^2} & \text{if } \lambda_1 \lambda_n > 0, \\ \dfrac{(\lambda_{k+1} - \lambda_k)^2}{4(\lambda_k \lambda_{k+1})^2} & \text{if } \lambda_1 \lambda_n < 0 \end{cases} \tag{2.3}$$

for any $x \in C^n$, $\|x\| = 1$.

Proof. Let $x = \sum k_i \varphi_i$, then

$$C(A, x) = \left(\sum k_j^* \varphi_j^* (\lambda_n - \lambda_j)\right)\left(\sum k_i \varphi_i (\lambda_i - \lambda_1)\right) = \sum |k_i|^2 (\lambda_n - \lambda_i)(\lambda_i - \lambda_1) \ge 0 \tag{2.4}$$

while

$$\begin{aligned} C(A, x) &= x^* \left(\frac{\lambda_n - \lambda_1}{2} I - \left(A - \frac{\lambda_n + \lambda_1}{2} I\right)\right)\left(\frac{\lambda_n - \lambda_1}{2} I + \left(A - \frac{\lambda_n + \lambda_1}{2} I\right)\right) x \\ &= \frac{(\lambda_n - \lambda_1)^2}{4} - \left\| \left(A - \frac{\lambda_n + \lambda_1}{2} I\right) x \right\|^2 \le \frac{(\lambda_n - \lambda_1)^2}{4}. \end{aligned} \tag{2.5}$$

The proof about $C(A^{-1}, x)$ is similar. □

Lemma 2.4. *With the assumptions in Theorem 2.1, then*

$$\|Ax\|^2-(x^*Ax)^2\le\frac{(\lambda_n-\lambda_1)^2}{4}-C(A,x),$$
$$\left\|A^{-1}x\right\|^2-\left(x^*A^{-1}x\right)^2\le\begin{cases}\dfrac{(\lambda_n-\lambda_1)^2}{4(\lambda_n\lambda_1)^2}-C\left(A^{-1},x\right) & \text{if } \lambda_1\lambda_n>0,\\ \dfrac{(\lambda_{k+1}-\lambda_k)^2}{4(\lambda_{k+1}\lambda_k)^2}-C\left(A^{-1},x\right) & \text{if } \lambda_1\lambda_n<0\end{cases}\tag{2.6}$$

for any $x\in C^n$, $\|x\|=1$.

Proof. Thus,

$$\begin{aligned}\|Ax\|^2-(x^*Ax)^2&=(\lambda_n-x^*Ax)(x^*Ax-\lambda_1)+x^*\left(A^2-(\lambda_1+\lambda_n)A+\lambda_1\lambda_nI\right)x\\&=(\lambda_n-x^*Ax)(x^*Ax-\lambda_1)+x^*(A-\lambda_1I)(A-\lambda_nI)x\\&=(\lambda_n-x^*Ax)(x^*Ax-\lambda_1)-x^*(A-\lambda_1I)(\lambda_nI-A)x\\&\le\frac{(\lambda_n-\lambda_1)^2}{4}-C(A,x)\quad(\text{by Lemma 2.2}).\end{aligned}\tag{2.7}$$

The other inequality can be obtained similarly, the proof is completed. □

We are now ready to prove the theorem.

Proof of the Theorem 2.1. Thus,

$$\begin{aligned}\left|x^*Axx^*A^{-1}x-1\right|^2&=\left|x^*\left(\left(x^*A^{-1}x\right)I-A^{-1}\right)((x^*Ax)I-A)x\right|^2\\&\le\left\|\left(\left(x^*A^{-1}x\right)I-A^{-1}\right)x\right\|^2\|((x^*Ax)I-A)x\|^2,\end{aligned}\tag{2.8}$$

while

$$\begin{aligned}\|((x^*Ax)I-A)x\|^2&=x^*\left((x^*Ax)^2I-2(x^*Ax)A+A^2\right)x\\&=x^*A^2x-(x^*Ax)^2\\&=\|Ax\|^2-(x^*Ax)^2.\end{aligned}\tag{2.9}$$

By the Lemma 2.4, we have

$$\|((x^*Ax)I-A)x\|^2\le\frac{(\lambda_n-\lambda_1)^2}{4}-C(A,x).\tag{2.10}$$

Similarly, we can get that,

$$\left\|\left(\left(x^*A^{-1}x\right)I-A^{-1}\right)x\right\|^2=\left\|A^{-1}x\right\|^2-\left(x^*A^{-1}x\right)^2\le\begin{cases}\dfrac{(\lambda_n-\lambda_1)^2}{4(\lambda_n\lambda_1)^2}-C\left(A^{-1},x\right) & \text{if } \lambda_1\lambda_n>0,\\ \dfrac{(\lambda_{k+1}-\lambda_k)^2}{4(\lambda_{k+1}\lambda_k)^2}-C\left(A^{-1},x\right) & \text{if } \lambda_1\lambda_n<0.\end{cases}\tag{2.11}$$

Therefore,

$$\left|x^*Axx^*A^{-1}x-1\right|^2\le\begin{cases}\left(\dfrac{(\lambda_n-\lambda_1)^2}{4}-C(A,x)\right)\left(\dfrac{(\lambda_n-\lambda_1)^2}{4(\lambda_n\lambda_1)^2}-C\left(A^{-1},x\right)\right) & \text{if } \lambda_1\lambda_n>0,\\ \left(\dfrac{(\lambda_n-\lambda_1)^2}{4}-C(A,x)\right)\left(\dfrac{(\lambda_{k+1}-\lambda_k)^2}{4(\lambda_{k+1}\lambda_k)^2}-C\left(A^{-1},x\right)\right) & \text{if } \lambda_1\lambda_n<0.\end{cases}\tag{2.12}$$

From $(a^2-b^2)(c^2-d^2)\le(ac-bd)^2$ for real numbers a,b,c,d, we have

$$\left|x^*Axx^*A^{-1}x-1\right|^2\le\begin{cases}\left(\dfrac{(\lambda_n-\lambda_1)^2}{4\lambda_1\lambda_n}-\sqrt{C(A,x)C\left(A^{-1},x\right)}\right)^2, & \text{if } \lambda_1\lambda_n>0\\ \left(\dfrac{(\lambda_n-\lambda_1)(\lambda_{k+1}-\lambda_k)}{4|\lambda_k\lambda_{k+1}|}-\sqrt{C(A,x)C\left(A^{-1},x\right)}\right)^2, & \text{if } \lambda_1\lambda_n<0.\end{cases}\tag{2.13}$$

The proof of Theorem 2.1 is completed. □

Remark 2.5. Let $A\in M_n$ be a positive definite Hermite matrix. From Theorem 2.1, we have

$$0\le x^*Axx^*A^{-1}x-1\le\frac{(\lambda_n-\lambda_1)^2}{4\lambda_1\lambda_n}-\sqrt{C(A,x)C\left(A^{-1},x\right)}\le\frac{(\lambda_n-\lambda_1)^2}{4\lambda_1\lambda_n},\tag{2.14}$$

our result improves the Kantorovich inequality (1.2), so we conclude that Theorem 2.1 gives an improvement of the Kantorovich inequality that applies all invertible Hermite matrices.

Example 2.6. Let

$$A=\begin{pmatrix}2&0&0\\0&3&2\\0&2&4\end{pmatrix},\qquad A^{-1}=\begin{pmatrix}\frac{1}{2}&0&0\\0&\frac{1}{2}&-\frac{1}{4}\\0&-\frac{1}{4}&\frac{3}{8}\end{pmatrix},\qquad x=\frac{1}{\sqrt{2}}(1,1,0)^T.\tag{2.15}$$

The eigenvalues of A are: $\lambda_1 = (7-\sqrt{17})/2$, $\lambda_2 = 2$, $\lambda_3 = (7+\sqrt{17})/2$ by easily calculating, we have

$$x^* A x x^* A^{-1} x - 1 = \frac{1}{4}, \qquad \frac{(\lambda_3-\lambda_1)^2}{4\lambda_1\lambda_3} = \frac{17}{32}, \qquad C(A,x)C\left(A^{-1},x\right) = \frac{1}{32}. \tag{2.16}$$

Therefore,

$$\frac{(\lambda_3-\lambda_1)^2}{4\lambda_1\lambda_3} - \sqrt{C(A,x)C(A^{-1},x)} < \frac{(\lambda_3-\lambda_1)^2}{4\lambda_1\lambda_3}, \tag{2.17}$$

we get a sharpen upper bound.

3. Conclusion

In this paper, we introduce a new Kantorovich type inequality for the invertible Hermite matrices. In Theorem 2.1, if $\lambda_1 > 0$, $\lambda_n > 0$, the result is well-known Kantorovich inequality. Moreover, it holds for negative definite Hermite matrices, even for any invertible Hermite matrix, there exists a similar inequality.

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