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Research Article

Some Normality Criteria of Meromorphic Functions

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This paper studies some normality criteria for a family of meromorphic functions, which improve some results of Lahiri, Lu and Gu, as well as Charak and Rieppo.

1. Introduction and Main Results

Let f be a nonconstant meromorphic function in the complex plane $\mathcal{C}$. We shall use the standard notations in Nevanlinna's value distribution theory of meromorphic functions such as $T(r, f)$, $N(r, f)$, and $m(r, f)$ (see, e.g., [1, 2]). The notation $S(r, f)$ is defined to be any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$ possibly outside a set of E of finite linear measure.

Let $\mathfrak{F}$ be a family of meromorphic functions on a domain $\mathfrak{D} \subset \mathcal{C}$. We say that $\mathfrak{F}$ is normal in $\mathfrak{D}$ if every sequence of functions $\{f_n\} \subset \mathfrak{F}$ contains either a subsequence which converges to a meromorphic function f uniformly on each compact subset of $\mathfrak{D}$ or a subsequence which converges to ∞ uniformly on each compact subset of $\mathfrak{D}$. (See [1, 3].)

The Bloch principle [3] is the hypothesis that a family of analytic (meromorphic) functions which have a common property P in a domain $\mathfrak{D}$ will in general be a normal family if P reduces an analytic (meromorphic) function in the open complex plane $\mathcal{C}$ to a constant. Unfortunately the Bloch principle is not universally true. But it is also very difficult to find some counterexamples about the converse of the Bloch principle, and hence it is interesting to study the problem.

In 2005, Lahiri [4] proved the following criterion for the normality, and gave a counterexample to the converse of the Bloch principle by using the result.

Theorem A. *Let $\mathfrak{F}$ be a family of meromorphic functions in a domain $\mathfrak{D}$, and let $a(\neq 0)$, b be two finite constants. Define*

$$E_f = \left\{ z : z \in \mathfrak{D}, f'(z) + \frac{a}{f(z)} = b \right\}. \tag{1.1}$$

If there exists a positive number M such that for every $f \in \mathfrak{F}$, one has $|f(z)| \geq M$ whenever $z \in E_f$, then $\mathfrak{F}$ is normal.

In this direction, Lahiri and Dewan [5] as well as Xu and Zhang [6] proved the following result.

Theorem B. *Let $\mathfrak{F}$ be a family of meromorphic functions in a domain $\mathfrak{D}$, and let $a(\neq 0)$, b be two finite constants. Suppose that*

$$E_f = \left\{ z : z \in \mathfrak{D}, f^{(k)} - af^{-n} = b \right\}, \tag{1.2}$$

where k and n are positive integers.

If for every $f \in \mathfrak{F}$

(i) *all zeros of f have multiplicity at least k,*

(ii) *there exists a positive number M such that for every $f \in \mathfrak{F}$ one has $|f(z)| \geq M$ whenever $z \in E_f$,*

then $\mathfrak{F}$ is normal in $\mathfrak{D}$ so long as $(A) n \geq 2$; or $(B) n = 1$ and $k = 1$.

Here, we also give a counterexample to the converse of the Bloch principle by considering Theorem B, which is similar to an example in [7].

Example 1.1. Let $f(z) = \cot z$, then $f'(z) = 1 + \cot^2 z \neq 0$ for all $z \in \mathcal{C}$. Now we can see that

$$f'(z) + f^{-2}(z) + 1 = \frac{(1 + \cot^2 z)^2}{\cot^2 z} = \frac{4}{\sin^2 2z} \neq 0, \tag{1.3}$$

but Theorem B is true especially when E_f is an empty set for every f in the family.

In the following, we continue to study the normal family when $n = 1$ and $k \geq 2$ in Theorem B.

Theorem 1.2. *Let $\mathfrak{F}$ be a family of meromorphic functions in a domain $\mathfrak{D}$, and $a(\neq 0)$, b be two finite constants. Suppose that*

$$E_f = \left\{ z : z \in \mathfrak{D}, f^{(k)} - af^{-1} = b \right\}, \tag{1.4}$$

where $k \geq 2$ is a positive integer.

If for every $f \in \mathfrak{F}$

(i) *all zeros of f have multiplicity at least $k + 1$,*

(ii) *there exists a positive number M such that for every $f \in \mathfrak{F}$, one has $|f(z)| \geq M$ whenever $z \in E_f$, then $\mathfrak{F}$ is normal in $\mathfrak{D}$.*

Corollary 1.3. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain $\mathfrak{D}$, all of whose zeros have multiplicity at least $k+1$, and let $a(\neq 0)$, b be two finite constants. Suppose that $f^{(k)} - af^{-1} \neq b$, where $k \geq 2$ is a positive integer. Then $\mathcal{F}$ is normal in $\mathfrak{D}$.*

Recently, Lu and Gu [8] considered two related normal families.

Theorem C. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain $\mathfrak{D}$; all of whose zeros have multiplicity at least $k+2$. Suppose that, for each $f \in \mathcal{F}$, $ff^{(k)} \neq a$ for $z \in \mathfrak{D}$, then $\mathcal{F}$ is a normal family on $\mathfrak{D}$, where a is a nonzero finite complex number and $k \geq 1$ is an integer number.*

Theorem D. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain $\mathfrak{D}$; all of whose zeros have multiplicity at least $k+1$, and all of whose poles are multiple. Suppose that, for each $f \in \mathcal{F}$, $ff^{(k)} \neq a$ for $z \in \mathfrak{D}$, then $\mathcal{F}$ is a normal family on $\mathfrak{D}$, where a is a nonzero finite complex number and $k \geq 1$ is an integer number.*

In this paper, we give a simple proof and improve the above results.

Theorem 1.4. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain $\mathfrak{D}$; all of whose zeros have multiplicity at least $k+1$. Suppose that, for each $f \in \mathcal{F}$, $ff^{(k)} \neq a$ for $z \in \mathfrak{D}$, then $\mathcal{F}$ is a normal family on $\mathfrak{D}$, where a is a nonzero finite complex number and $k \geq 1$ is an integer number.*

In 2009, Charak and Rieppo [7] generalized Theorem A and obtained two normality criteria of Lahiri's type.

Theorem E. *Let $\mathcal{F}$ be a family of meromorphic functions in a complex domain $\mathfrak{D}$. Let $a, b \in \mathbb{C}$ such that $a \neq 0$. Let m_1, m_2, n_1, n_2 be nonnegative integers such that $m_1 n_2 - m_2 n_1 > 0$, $m_1 + m_2 \geq 1$, and $n_1 + n_2 \geq 2$, and put*

$$E_f = \left\{ z \in \mathfrak{D} : (f(z))^{n_1} (f'(z))^{m_1} + \frac{a}{(f(z))^{n_2} (f'(z))^{m_2}} = b \right\}. \tag{1.5}$$

If there exists a positive constant M such that $|f(z)| \geq M$ for all $f \in \mathcal{F}$ whenever $z \in E_f$, then $\mathcal{F}$ is a normal family.

Theorem F. *Let $\mathcal{F}$ be a family of meromorphic functions in a complex domain $\mathfrak{D}$. Let $a, b \in \mathbb{C}$ such that $a \neq 0$. Let m_1, m_2, n_1, n_2 be nonnegative integers such that $m_1 n_2 = m_2 n_1 > 0$, and put*

$$E_f = \left\{ z \in \mathfrak{D} : (f(z))^{n_1} (f'(z))^{m_1} + \frac{a}{(f(z))^{n_2} (f'(z))^{m_2}} = b \right\}. \tag{1.6}$$

If there exists a positive constant M such that $|f(z)| \geq M$ for all $f \in \mathcal{F}$ whenever $z \in E_f$, then $\mathcal{F}$ is a normal family.

Naturally, we ask whether the above results are still true when f' is replaced by $f^{(k)}$ in Theorems E and F. We obtain the following results.

Theorem 1.5. *Let $\mathcal{F}$ be a family of meromorphic functions in a complex domain $\mathfrak{D}$; all of whose zeros have multiplicity at least k. Let $a, b \in C$ such that $a \neq 0$. Let m_1, m_2, n_1, n_2 be nonnegative integers such that $m_1 n_2 - m_2 n_1 > 0$, $m_1 + m_2 \geq 1$, and $n_1 + n_2 \geq 2$ (if $n_1 = n_2 = 1$, $k \geq 5$), and put*

$$E_f = \left\{ z \in \mathfrak{D} : (f(z))^{n_1} \left(f^{(k)}(z)\right)^{m_1} + \frac{a}{(f(z))^{n_2} (f^{(k)}(z))^{m_2}} = b \right\}. \tag{1.7}$$

If there exists a positive constant M such that $|f(z)| \geq M$ for all $f \in \mathcal{F}$ whenever $z \in E_f$, then $\mathcal{F}$ is a normal family.

Theorem 1.6. *Let $\mathcal{F}$ be a family of meromorphic functions in a complex domain $\mathfrak{D}$; all of whose zeros have multiplicity at least k. Let $a, b \in C$ such that $a \neq 0$. Let $m_1 \geq 2$, m_2, n_1, n_2 be nonnegative integers such that $m_1 n_2 = m_2 n_1$, and put*

$$E_f = \left\{ z \in \mathfrak{D} : (f(z))^{n_1} \left(f^{(k)}(z)\right)^{m_1} + \frac{a}{(f(z))^{n_2} (f^{(k)}(z))^{m_2}} = b \right\}. \tag{1.8}$$

If there exists a positive constant M such that $|f(z)| \geq M$ for all $f \in \mathcal{F}$ whenever $z \in E_f$, then $\mathcal{F}$ is a normal family.

2. Some Lemmas

Lemma 2.1 (see [9]). *Let $\mathcal{F}$ be a family of functions meromorphic on the unit disc, all of whose zeros have multiplicity at least k, then if $\mathcal{F}$ is not normal, there exist, for each $0 \leq \alpha < k$,*

(a) *a number $0 < r < 1$,*

(b) *points $z_n, z_n < 1$,*

(c) *functions $f_n \in \zeta$,*

(d) *positive number $\rho_n \to \infty$ such that $\rho_n^{-\alpha} f_n(z_n + \rho_n \xi) = g_n(\xi) \to g(\xi)$ locally uniformly, where g is a nonconstant meromorphic on C, all of whose zeros have multiplicity at least k, such that $g^{\#}(\xi) \leq g^{\#}(0)$.*

Here, as usual, $g^{\#}(\xi) = |g'(\xi)|/(1 + |g(\xi)|^2)$ is the spherical derivative.

Lemma 2.2. *Let f be rational in the complex plane and m, n positive integers. If f has only zero with multiplicity at least k, then $f^n (f^{(k)})^m$ takes on each nonzero value $a \in \mathbb{C}$.*

Proof. In Lemma 6 of [7], the case of $k = 1$ is proved. We just consider the case of $k \geq 2$ by a different way which comes from [10].

If f is a polynomial, obviously the conclusion holds. If f is a nonpolynomial rational function, then we can set

$$f^n \left(f^{(k)}\right)^m = A \frac{(z - \alpha_1)^{m_1} (z - \alpha_2)^{m_2} \cdots (z - \alpha_s)^{m_s}}{(z - \beta_1)^{n_1} (z - \beta_2)^{n_2} \cdots (z - \beta_t)^{n_t}}, \tag{2.1}$$

where A is a nonzero constant. Since f has only zero with multiplicity at least k, we find that

$$m_i \geq kn \quad (i = 1, 2, \ldots, s), \qquad n_j \geq k + 1 + n \quad (j = 1, 2, \ldots, t). \tag{2.2}$$

For convenience, we denote

$$\begin{aligned} M &= m_1 + m_2 + \cdots + m_s \geq kns, \\ N &= n_1 + n_2 + \cdots + n_t \geq (k + 1 + n)t. \end{aligned} \tag{2.3}$$

Differentiating (2.1), we obtain

$$\left[f^n\left(f^{(k)}\right)^m\right]' = \frac{(z-\alpha_1)^{m_1-1}(z-\alpha_2)^{m_2-1}\cdots(z-\alpha_s)^{m_s-1}}{(z-\beta_1)^{n_1+1}(z-\beta_2)^{n_2+1}\cdots(z-\beta_t)^{n_t+1}}g(z), \tag{2.4}$$

where $g(z)$ is a polynomial with $\deg(g) \leq s + t - 1$.

Suppose that $f^n(f^{(k)})^m - a$ has no zero, then we can write

$$f^n\left(f^{(k)}\right)^m - a = A\frac{B}{(z-\beta_1)^{n_1}(z-\beta_2)^{n_2}\cdots(z-\beta_t)^{n_t}}, \tag{2.5}$$

where B is a nonzero constant.

Differentiating (2.5), we obtain

$$\left[f^n\left(f^{(k)}\right)^m\right]' = \frac{Bg_1(z)}{(z-\beta_1)^{n_1+1}(z-\beta_2)^{n_2+1}\cdots(z-\beta_t)^{n_t+1}}, \tag{2.6}$$

where $g_1(z)$ is a polynomial of the form $-BNz^{t-1} + B_{t-2}z^{t-2} + \cdots + B_0$, in which $B_0, \ldots, B_{t-2}$ are constants.

Comparing (2.1) and (2.5), we can obtain $M = N$. From (2.4) and (2.6), we have

$$\begin{aligned} \sum_{i=1}^{s}(m_i - 1) = M - s \leq \deg\left(g_1(z)\right) = t - 1, \\ M &\leq s + t - 1 \\ &\leq \frac{M}{kn} + \frac{N}{k+1+n} - 1 \\ &= \frac{M}{kn} + \frac{M}{k+1+n} - 1 \\ &= \left(\frac{1}{kn} + \frac{1}{k+1+n}\right)M - 1. \end{aligned} \tag{2.7}$$

It is a contradiction with $n \geq 1$ and $k \geq 2$. This proves the lemma. □

Lemma 2.3 (see [11]). *Let f be a transcendental meromorphic function all of whose zeros have multiplicity at least t, then $ff^{(k)}$ assumes every finite nonzero value infinitely often, where $t = k+1$ if $k \le 4$, and $t = 5$ if $k \ge 5$.*

Remark 2.4. The lemma was first proved by Wang as $t = 5$ if $k = 5$ and $t = 6$ if $k \ge 6$ in [12]. Recently, the result is improved by [11].

Lemma 2.5. *Let f be a meromorphic function all of whose zeros have multiplicity with at least $k+1$ in the complex plane, then $ff^{(k)} - a$ must have zeros for any constant $a \neq 0, \infty$.*

Proof. If f is rational, then by Lemma 2.2 the conclusion holds.

If f is transcendental, supposing that $ff^{(k)} - a$ has no zeros, then by Lemma 2.3, we can get a contradiction. This completes the proof of the lemma. □

Lemma 2.6. *Let f be meromorphic in the complex plane, and let $a \neq 0$ be a constant, for any positive integer k; if $ff^{(k)} \equiv a$, then f is a constant.*

Proof. If f is not a constant, and from $a \neq 0$, we know that $f \neq 0$, then with the identity $ff^{(k)} \equiv a$, we can get that, if $r \to \infty$,

$$T\left(r, \frac{1}{f}\right) = m\left(r, \frac{1}{f}\right) \le \log^{+} \frac{1}{|a|} + m\left(r, \frac{f^{(k)}}{f}\right) = o(T(r, f)), \tag{2.8}$$

and $r \notin E$ with E being a set of r values of finite linear measure. It is a contradiction. □

Lemma 2.7 (see [13]). *Let f be a transcendental meromorphic function, and let $n \ge 2$, $n_k \ge 1$ be two integers. Then for any nonzero value c, the function $f^{n}(f^{(k)})^{n_k} - c$ has infinitely many zeros.*

Lemma 2.8 (see [14]). *Let f be a transcendental meromorphic function, and let $n \ge 2$ be an integer. Then for any nonzero value c, the function $f(f^{(k)})^{n} - c$ has infinitely many zeros.*

Lemma 2.9. *Let $\mathfrak{F}$ be a family of meromorphic functions in a complex domain $\mathfrak{D}$. Let $a, b \in \mathcal{C}$ such that $a \neq 0$. Let $m_1 \ge 2$, m_2, n_1, n_2 be nonnegative integers such that $m_1 n_2 = m_2 n_1$, and put*

$$E_f = \left\{ z \in \mathfrak{D} : (f(z))^{n_1} \left(f^{(k)}(z)\right)^{m_1} + \frac{a}{\left[(f(z))^{n_2} (f^{(k)}(z))^{m_2}\right]^{n_2/n_1}} = b \right\}. \tag{2.9}$$

has a finite zero.

Proof. The algebraic complex equation

$$x + \frac{a}{x^{n_2/n_1}} - b = 0 \tag{2.10}$$

has always a nonzero solution; say $x_0 \in \mathcal{C}$. By [14, Corollary 3] or [15], Lemmas 2.2, 2.7, and 2.8, the meromorphic function $f^{n_1}(f^{(k)})^{m_1}$ cannot avoid it and thus there exists $z_0 \in \mathcal{C}$ such that $(f(z_0))^{n_1}(f^{(k)}(z_0))^{m_1} = x_0$.

By assumption, we may write $m_2 = (n_2/n_1)m_1$ and $n_2 = (n_2/n_1)n_1$. Consequently

$$\Psi(z_0) = \left[(f(z_0))^{n_1}\left(f^{(k)}(z_0)\right)^{m_1}\right] + \frac{a}{\left[(f(z_0))^{n_1}(f^{(k)}(z_0))^{m_1}\right]^{n_2/n_1}} - b \tag{2.11}$$

and we complete the proof of the lemma. □

Remark 2.10. If $m_1 = 1$, we need $k \geq 5$ when $n_1 = 1$ by Lemma 2.3. We can get a similar result.

3. Proof of Theorems

Proof of Theorem 1.2. Let $\alpha = k/2 < k$. Suppose that $\mathcal{F}$ is not normal at $z_0 \in \mathfrak{D}$. Then by Lemma 2.1, there exist a sequence of functions $f_j \in \mathcal{F}$ $(j = 1, 2, \ldots)$, a sequence of complex numbers $z_j \to z_0$, and $\rho_j(>0) \to 0$ such that

$$g_j(\zeta) = \rho_j^{-\alpha} f_j(z_j + \rho_j \zeta) \tag{3.1}$$

converges spherically and locally uniformly to a nonconstant meromorphic function $g(\zeta)$ in $\mathcal{C}$. Also the zeros of $g(z)$ are of multiplicity at least $\geq k+1$. So $g^{(k)} \not\equiv 0$. Applying Lemma 2.5 to the function $g(z)$, we know that

$$\begin{aligned} g(\zeta_0)g^{(k)}(\zeta_0) - a &= 0, \\ g^{(k)}(\zeta_0) - \frac{a}{g(\zeta_0)} &= 0 \end{aligned} \tag{3.2}$$

for some $\zeta_0 \in \mathcal{C}$. Clearly ζ_0 is neither a zero nor a pole of g. So in some neighborhood of ζ_0, $g_j(\zeta)$ converges uniformly to $g(\zeta)$. Now in some neighborhood of ζ_0 we see that $g^{(k)}(\zeta) - ag(\zeta)^{-1}$ is the uniform limit of

$$g_j^{(k)}(\zeta_0) - ag_j(\zeta_0)^{-1} - \rho_j^{\alpha} b = \rho_j^{k/2}\left\{f_j^{(k)}(z_j + \rho_j\zeta_0) - af_j^{-1}(z_j + \rho_j\zeta_0) - b\right\}. \tag{3.3}$$

By (3.2) and Hurwitz's theorem, there exists a sequence $\zeta_j \to \zeta_0$ such that for all large values of j

$$f_j^{(k)}(z_j + \rho_j\zeta_j) - af_j^{-1}(z_j + \rho_j\zeta_j) = b. \tag{3.4}$$

Therefore for all large values of j, it follows from the given condition that $|g_j(\zeta_j)| \geq |f_j(z_j + \rho_j\zeta_j)|/\rho_j^{\alpha} \geq M/\rho_j^{\alpha}$.

Since ζ_0 is not a pole of g, there exists a positive number K such that in some neighborhood of ζ_0 we get $|g(\zeta)| \leq K$.

Since $g_j(\zeta)$ converges uniformly to $g(\zeta)$ in some neighborhood of ζ_0, we get for all large values of j and for all ζ in that neighborhood of ζ_0

$$\left|g_j(\zeta) - g(\zeta)\right| < 1. \tag{3.5}$$

Since $\zeta_j \to \zeta$, we get for all large values of j

$$K \geq \left|g(\zeta_j)\right| \geq \left|g_j(\zeta_j)\right| - \left|g(\zeta_j) - g_j(\zeta_j)\right| > \frac{M}{\rho_j^{\alpha}} - 1, \tag{3.6}$$

which is a contradiction. This proves the theorem. □

Proof of Theorem 1.4. If $\mathcal{F}$ is not normal at $z_0 \in \mathfrak{D}$. We assume without loss of generality that $z_0 = 0$, then by Lemma 2.1, for $\alpha = k/2$, there exist a sequence of points $z_n \to z_0$, a sequence of positive numbers $\rho_n \to 0^+$, and a sequence of functions $\{f_n\}$ of $\mathcal{F}$ such that

$$g_j(\zeta) = \rho_j^{-\alpha} f_j(z_j + \rho_j\zeta) \longrightarrow g(z) \tag{3.7}$$

spherically uniformly on compact subsets of $\mathcal{C}$, where $g(z)$ is a nonconstant meromorphic function on $\mathcal{C}$; all of whose zeros have multiplicity $k+1$ at least. By (3.7),

$$g_j(\zeta)g_j^{(k)}(\zeta) - a = f_j(\zeta)f_j^{(k)}(\zeta) - a \neq 0. \tag{3.8}$$

It follows that $g(\zeta)g^{(k)}(\zeta) \neq a$ or $g(\zeta)g^{(k)}(\zeta) \equiv a$ by Hurwitz's theorem. From Lemma 2.6, we obtain that $gg^{(k)} \neq a$. By Lemma 2.5, we get a contradiction. This completes the proof of the theorem. □

Proof of Theorem 1.5. Suppose that $\mathcal{F}$ is not normal at $z_0 \in \mathfrak{D}$. Then by Lemma 2.1, for $0 \leq \alpha < k$, there exist a sequence of functions $f_j \in \mathcal{F}$ $(j = 1, 2, \ldots)$, a sequence of complex number $z_j \to z_0$, and $\rho_j(>0) \to 0$ such that

$$g_j(\zeta) = \rho_j^{-\alpha} f_j(z_j + \rho_j\zeta) \tag{3.9}$$

converges spherically and locally uniformly to a nonconstant meromorphic function $g(\zeta)$ in $\mathcal{C}$. Also the zeros of $g(z)$ are of multiplicity at least $\geq k$. So $g^{(k)} \not\equiv 0$. By Lemmas 2.2, 2.3, 2.7, and 2.8, we get

$$\left(g(\zeta_0)\right)^{n_1}\left(g^{(k)}(\zeta_0)\right)^{m_1} + \frac{a}{\left(g(\zeta_0)\right)^{n_2}\left(g^{(k)}(\zeta_0)\right)^{m_2}} = 0, \tag{3.10}$$

for some $\zeta_0 \in \mathcal{C}$. Clearly ζ_0 is neither a zero nor a pole of g. So in some neighborhood of ζ_0, $g_j(\zeta)$ converges uniformly to $g(\zeta)$. Now in some neighborhood of ζ_0 we have

$$\begin{aligned}
&(g_j(\zeta))^{n_1}\left(g_j^{(k)}(\zeta)\right)^{m_1} + \frac{a}{(g_j(\zeta))^{n_2}\left(g_j^{(k)}(\zeta)\right)^{m_2}} - \rho_j^{\alpha k_2 - k m_2} b \\
&\quad = \rho_j^{-\alpha k_1 + k m_1}(f_j)^{n_1}\left(f_j^{(k)}\right)^{m_1} + \frac{a}{\rho_j^{-\alpha k_2 + k m_2}(f_j)^{n_2}\left(f_j^{(k)}\right)^{m_2}} - \rho_j^{\alpha k_2 - k m_2} b \\
&\quad = \rho_j^{\alpha k_2 - k m_2}\left(\rho_j^{-\alpha(k_1+k_2)+k(m_1+m_2)}(f_j)^{n_1}\left(f_j^{(k)}\right)^{m_1} + \frac{a}{(f_j)^{n_2}\left(f_j^{(k)}\right)^{m_2}} - b\right),
\end{aligned} \tag{3.11}$$

where $f_j(z_j + \rho_j\zeta)$ is replaced by f_j and $k_j = n_j + m_j$, $j = 1, 2$.

Taking $\alpha = (m_1 + m_2)k/(k_1 + k_2)$ and using the assumption $m_1 n_2 - n_1 m_2 > 0$, we see that

$$g^{n_1}\left(g^{(k)}\right)^{m_1} + \frac{a}{g^{n_2}\left(g^{(k)}\right)^{m_2}} \tag{3.12}$$

is the uniform limit of

$$\rho_j^{((m_1 n_2 - n_1 m_2)/k_1 + k_2)k}\left(f_j^{n_1}\left(f_j^{(k)}\right)^{m_1} + \frac{a}{f_j^{n_2}\left(f_j^{(k)}\right)^{m_2}} - b\right) \tag{3.13}$$

in some neighborhood of ζ_0. By (3.10) and Hurwitz's theorem, there exists a sequence $\zeta_j \to \zeta_0$ such that for all large values of j

$$(f_j(z_j + \rho_j\zeta_j))^{n_1}\left(f_j^{(k)}(z_j + \rho_j\zeta_j)\right)^{m_1} + \frac{a}{(f_j(z_j + \rho_j\zeta_j))^{n_2}\left(f_j^{(k)}(z_j + \rho_j\zeta_j)\right)^{m_2}} - b. \tag{3.14}$$

Hence, for all large j, it follows from the given condition that

$$|g_j(\zeta_j)| \geq \frac{|f_j(z_j + \rho_j\zeta_j)|}{\rho_j^{\alpha}} = \frac{M}{\rho_j^{\alpha}}. \tag{3.15}$$

In the following, we can get a contradiction in a similar way with the proof of the last part of Theorem 1.2. This completes the proof of the theorem. □

Proof of Theorem 1.6. Suppose that $\mathfrak{F}$ is not normal at $z_0 \in \mathfrak{D}$. Then by Lemma 2.1, for $0 \leq \alpha < k$, there exist a sequence of functions $f_j \in \mathfrak{F}$ $(j = 1, 2, \ldots)$, a sequence of complex numbers $z_j \to z_0$, and $\rho_j(>0) \to 0$ such that

$$g_j(\zeta) = \rho_j^{-\alpha} f_j(z_j + \rho_j\zeta) \tag{3.16}$$

converges spherically and locally uniformly to a nonconstant meromorphic function $g(\zeta)$ in $\mathbb{C}$. Also the zeros of $g(z)$ are of multiplicity at least $\geq k$. So $g^{(k)} \not\equiv 0$. By Lemma 2.9, we get

$$\left(g(\zeta_0)\right)^{n_1}\left(g^{(k)}(\zeta_0)\right)^{m_1} + \frac{a}{\left(g(\zeta_0)\right)^{n_2}\left(g^{(k)}(\zeta_0)\right)^{m_2}} - b = 0, \tag{3.17}$$

for some $\zeta_0 \in \mathbb{C}$.

In the following, we can get a contradiction in a similar way with the proof of the last part of Theorem 1.5. This completes the proof of the theorem. □

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