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Research Article

On Asymptotically (λ, σ)-Statistical Equivalent Sequences of Fuzzy Numbers

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The goal of this paper is to give the asymptotically (λ, σ)-statistical equivalent which is a natural combination of the definition for asymptotically equivalent, invariant mean and λ-statistical convergence of fuzzy numbers.

1. Introduction

The concepts of fuzzy sets and fuzzy set operation were first introduced by Zadeh [1] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces similarity relations and fuzzy orderings, fuzzy measures of fuzzy events and fuzzy mathematical programming. Matloka [2] introduced bounded and convergent sequences of fuzzy numbers and studied their some properties. For sequences of fuzzy numbers, Nuray and Savaş [3] introduced and discussed the concepts of statistically convergent and statistically Cauchy sequences.

Quite recently, Savaş [4] introduced the idea of asymptotically λ-statistically equivalent sequences of fuzzy numbers. In this paper we extend his result by using invariant means.

2. Preliminaries

Before we enter the motivation for this paper and presentation of the main results we give some preliminaries.

By l_∞ and c, we denote the Banach spaces of bounded and convergent sequences $x = (x_k)$ normed by $\|x\| = \sup_n |x_n|$, respectively. A linear functional L on l_∞ is said to be a Banach limit (see [5]) if it has the following properties:

(1) $L(x) \geq 0$ if $x_n \geq 0$ for all n;

(2) $L(e) = 1$ where $e = (1, 1, \ldots)$;

(3) $L(Dx) = L(x)$, where the shift operator D is defined by $D(x_n) = \{x_{n+1}\}$.

Let B be the set of all Banach limits on l_∞. A sequence $x \in l_\infty$ is said to be almost convergent if all Banach limits of x coincide. Let $\hat{c}$ denote the space of almost convergent sequences.

Let σ be a one-to-one mapping from the set of natural numbers into itself. A continuous linear functional ϕ on l_∞ is said to be an invariant mean or a σ-mean if and only if

(1) $\phi(x) \geq 0$ when the sequence $x = (x_k)$ is such that $x_k \geq 0$ for all k,

(2) $\phi(e) = 1$ where $e = (1, 1, 1, \ldots)$, and

(3) $\phi(x) = \phi(x_{\sigma(k)})$ for all $x \in l_\infty$.

Throughout this paper we shall consider the mapping σ has having on finite orbits, that is, $\sigma^m(k) \neq k$ for all nonnegative integers with $m \geq 1$, where $\sigma^m(k)$ is the m th iterate of σ at k. Thus σ-mean extends the limit functional on c in the sense that $\phi(x) = \lim x$ for all $x \in c$. Consequently, $c \subset V_\sigma$ where V_σ is the set of bounded sequences all of whose σ-mean are equal.

In the case when $\sigma(k) = k + 1$, the σ-mean is often called the Banach limit and V_σ is the set of almost convergent sequences.

A fuzzy real number X is a fuzzy set on R, that is, a mapping $X : R \to I = [0, 1]$, associating each real number t with its grade of membership $X(t)$.

The α-cut of fuzzy real number X is denoted by $[X]_\alpha$, $0 < \alpha \leq 1$, where $[X]_\alpha = \{t \in R : X(t) \geq \alpha\}$. If $\alpha = 0$, then it is the closure of the strong 0-*cut*. A fuzzy real number X is said to be upper semicontinuous if for each $\varepsilon > 0$, $X^{-1}([0, a + \varepsilon))$, for all $a \in I$ is open in the usual topology of R. If there exists $t \in R$ such that $X(t) = 1$, then the fuzzy real number X is called normal.

A fuzzy number X is said to be convex, if $X(t) \geq X(s) \wedge X(r) = \min(X(s), X(r))$ where $s < t < r$. The class of all upper semi-continuous, normal, convex fuzzy real numbers is denoted by $R(I)$ and throughout the article, by a fuzzy real number we mean that the number belongs to $R(I)$. Let $X, Y \in R(I)$ and the α-level sets be

$$[X]_\alpha = \left[a_1^\alpha, a_2^\alpha\right], \quad [Y]_\alpha = \left[b_1^\alpha, b_2^\alpha\right], \quad \alpha \in [0, 1]. \tag{2.1}$$

Then the arithmetic operations on $R(I)$ are defined as follows:

$$\begin{aligned}
(X \oplus Y)(t) &= \sup\{X(s) \wedge Y(t - s)\}, \quad t \in R, \\
(X \ominus Y)(t) &= \sup\{X(s) \wedge Y(s - t)\}, \quad t \in R, \\
(X \otimes Y)(t) &= \sup\left\{X(s) \wedge Y\left(\frac{t}{s}\right)\right\}, \quad t \in R, \\
(X / Y)(t) &= \sup\{X(st) \wedge Y(s)\}, \quad t \in R.
\end{aligned} \tag{2.2}$$

The above operations can be defined in terms of α-level sets as follows:

$$
\begin{aligned}
[X \oplus Y]_\alpha &= \left[a_1^\alpha + b_1^\alpha, a_2^\alpha + b_2^\alpha\right],\\
[X \ominus Y]_\alpha &= \left[a_1^\alpha - b_2^\alpha, a_2^\alpha - b_1^\alpha\right],\\
[X \otimes Y]_\alpha &\left[\min_{i,j\in\{1,2\}} a_i^\alpha \cdot b_j^\alpha, \max_{i,j\in\{1,2\}} a_i^\alpha \cdot b_j^\alpha\right],\\
\left[X^{-1}\right]_\alpha &= \left[\left(a_2^\alpha\right)^{-1}, \left(a_1^\alpha\right)^{-1}\right], \quad 0 \notin \{X\}.
\end{aligned}
\tag{2.3}
$$

The additive identity and multiplicative identity in $R(I)$ are denoted by $\overline{0}$ and $\overline{1}$, respectively.

Let D the set of all closed and bounded intervals $X = [X^L, X^R]$. Then we write $X \leq Y$, if and only if $X^L \leq Y^L$ and $X^R \leq Y^R$, and

$$
\rho(X, Y) = \max\left\{\left|X^L - Y^L\right|, \left|X^R - Y^R\right|\right\}. \tag{2.4}
$$

It is obvious that (D, ρ) is a complete metric space. Now we define the metric $d : R(I) \times R(I) \to R$ by

$$
d(X, Y) = \sup_{0\leq\alpha\leq 1} \rho([X]_\alpha, [Y]_\alpha) \tag{2.5}
$$

for $X, Y \in R(I)$.

We now give the following definitions (see [6]) for fuzzy real-valued sequences.

Definition 2.1. A fuzzy real-valued sequence $X = (X_k)$ is a function X from the set N of natural numbers into R(I). The fuzzy real-valued sequence X_n denotes the value of the function at $n \in N$ and is called the n th term of the sequence. We denote by $w(F)$ the set of all fuzzy real-valued sequences $X = (X_k)$.

Definition 2.2. A fuzzy real-valued sequence $X = (X_k)$ is said to be convergent to a fuzzy number X_0, written as $\lim_k X_k = X_0$, if for every $\epsilon > 0$ there exists a positive integer N_0 such that

$$
d(X_k, X_0) < \epsilon \quad \text{for } k > N_0. \tag{2.6}
$$

Let $c(F)$ denote the set of all convergent sequences of fuzzy numbers.

Definition 2.3. A sequence $X = (X_k)$ of fuzzy numbers is said to be bounded if the set $\{X_k : k \in N\}$ of fuzzy numbers is bounded. We denote by $\ell_\infty(F)$ the set of all bounded sequences of fuzzy numbers.

It is easy to see that

$$
c(F) \subset \ell_\infty(F) \subset w(F). \tag{2.7}
$$

It was shown that $c(F)$ and $\ell_\infty(F)$ are complete metric spaces (see [7]).

3. Definitions and Notations

Definition 3.1. Two fuzzy real-valued sequences $X = (X_k)$ and $Y = (Y_k)$ are said to be *asymptotically equivalent* if

$$\lim_k d\left(\frac{X_k}{Y_k}, 1\right) = 0 \tag{3.1}$$

(denoted by $X \overset{F}{\sim} Y$).

Let $\Lambda = (\lambda_n)$ be a nondecreasing sequence of positive reals tending to infinity and $\lambda_1 = 1$ and $\lambda_{n+1} \leq \lambda_n + 1$.

In [4], Savaş introduced the concept of λ-statistical convergence of fuzzy numbers as follows.

Definition 3.2. A fuzzy real-valued sequences $X = (X_k)$ is said to be λ-statistically convergent or S_λ-convergent to L if for every $\epsilon > 0$,

$$\lim_n \frac{1}{\lambda_n} |\{k \in I_n : d(X_k, L) \geq \epsilon\}| = 0. \tag{3.2}$$

In this case we write S_λ-limit $X = L$ or $X_k \to L(S_\lambda)$, and

$$S_\lambda = \{X : \exists L \in R(I), S_\lambda\text{-limit}\, X = L\}. \tag{3.3}$$

The next definition is natural combination of Definitions 3.1 and 3.2., which was defined in [4].

Definition 3.3. Two fuzzy real-valued sequences $X = (X_k)$ and $Y = (Y_k)$ are said to be asymptotically S_λ-statistical equivalent of multiple L provided that for every $\epsilon > 0$

$$\lim_n \frac{1}{\lambda_n} \left| \left\{ k \in I_n : d\left(\frac{X_k}{Y_k}, L\right) \geq \epsilon \right\} \right| = 0 \tag{3.4}$$

(denoted by $X \overset{S_\lambda^L(F)}{\sim} Y$) and simply asymptotically λ-statistical equivalent if $L = 1$.

If we take $\lambda_n = n$, the above definition reduces to the following definition.

Definition 3.4. Two fuzzy real-valued sequences $X = (X_k)$ and $Y = (Y_k)$ are said to be asymptotically statistical equivalent of multiple L provided that for every $\epsilon > 0$,

$$\lim_n \frac{1}{n}\left\{\text{the number of } k < n : d\left(\frac{X_k}{Y_k}, L\right) \geq \epsilon\right\} = 0 \tag{3.5}$$

(denoted by $X \overset{S_L(F)}{\sim} Y$) and simply asymptotically statistical equivalent if $L = 1$.

It is quite naturel to expect the following definition.

Definition 3.5. Fuzzy real-valued sequences $X = (X_k)$ is said to be (λ, σ)-statistically convergent to L provided that for every $\epsilon > 0$

$$\lim_n \frac{1}{\lambda_n}\left|\left\{k \in I_n : d\left(X_{\sigma^k(m)}, L\right) \geq \epsilon\right\}\right| = 0 \tag{3.6}$$

uniformly in m.

In this case we write $S_{(\lambda,\sigma)}$-limit $X = L$ or $X_k \to L(S_{(\lambda,\sigma)})$, and

$$S_{(\lambda,\sigma)}(F) = \left\{X : \exists L \in R(I) : S_{(\lambda,\sigma)}\text{-limit } \ X = L\right\}. \tag{3.7}$$

Following this result we introduce two new notions asymptotically (λ, σ)-statistical equivalent of multiple L and strong (λ, σ)-asymptotically equivalent of multiple L.

Definition 3.6. Two fuzzy real-valued sequences $X = (X_k)$ and $Y = (Y_k)$ are said to be asymptotically (λ, σ)-statistical equivalent of multiple L provided that for every $\epsilon > 0$

$$\lim_n \frac{1}{\lambda_n}\left|\left\{k \in I_n : d\left(\frac{X_{\sigma^k(m)}}{Y_{\sigma^k(m)}}, L\right) \geq \epsilon\right\}\right| = 0, \tag{3.8}$$

uniformly in m, (denoted by $X \overset{S^L_{(\lambda,\sigma)}(F)}{\sim} Y$) and simply asymptotically (λ, σ)-statistical equivalent if $L = 1$.

In case $\lambda_n = n$, the above definition reduces to the following definition.

Definition 3.7. Two fuzzy real-valued sequences $X = (X_k)$ and $Y = (Y_k)$ are said to be asymptotically (σ)-statistical equivalent of multiple L provided that for every $\epsilon > 0$

$$\lim_n \frac{1}{n}\left\{\text{the number of } k < n : d\left(\frac{X_{\sigma^k(m)}}{Y_{\sigma^k(m)}}, L\right) \geq \epsilon\right\} = 0 \tag{3.9}$$

uniformly in m, (denoted by $X \overset{S^L_{(\sigma)}(F)}{\sim} Y$) and simply asymptotically (σ)-statistical equivalent if $L = 1$.

We now define the following.

Definition 3.8. Let $p = (p_k)$ be a sequence of positive real numbers; two fuzzy real-valued sequences $X = (X_k)$ and $Y = (Y_k)$ are strongly asymptotically (λ, σ)-equivalent of multiple L, provided that

$$\lim_n \frac{1}{\lambda_n}\sum_{k\in I_n} d\left(\frac{X_{\sigma^k(m)}}{Y_{\sigma^k(m)}}, L\right)^{p_k} = 0 \tag{3.10}$$

(denoted by $X \overset{[V^{L(p)}_{(\lambda,\sigma)}](F)}{\sim} Y$) and simply strongly asymptotically (λ, σ)-equivalent if $L = 1$.

If we take $p_k = p$ for all $k \in \mathbb{N}$ we write ($X \overset{[V^{Lp}_{(\lambda,\sigma)}](F)}{\sim} Y$) instead of ($X \overset{[V^{L(p)}_{(\lambda,\sigma)}](F)}{\sim} Y$).
In case $\lambda_n = n$ in above definition we get following.

Definition 3.9. Let $p = (p_k)$ be a sequence of positive numbers and let us consider two fuzzy real-valued sequences $X = (X_k)$ and $Y = (Y_k)$. Two fuzzy real-valued sequences $X = (X_k)$ and $Y = (Y_k)$ are said to be strongly asymptotically Cesáro equivalent of multiple L provided that

$$\lim_n \frac{1}{n}\sum_{k=1}^{n} d\left(\frac{X_{\sigma^k(m)}}{Y_{\sigma^k(m)}}, L\right)^{p_k} = 0 \tag{3.11}$$

(denoted by $X \overset{[V^{L(p)}_{(\sigma)}](F)}{\sim} Y$), and simply strong Cesáro asymptotically equivalent if $L = 1$.

4. Main Results

Theorem 4.1. *Let $\lambda_n \in \Lambda$. Then*

(1) *If $X \overset{[V^{Lp}_{(\lambda,\sigma)}](F)}{\sim} Y$ then $X \overset{S^L_{(\lambda,\sigma)}(F)}{\sim} Y$;*

(2) *If X and $Y \in l_\infty(F)$ and $X \overset{S^L_{(\lambda,\sigma)}(F)}{\sim} Y$ then $X \overset{[V^{Lp}_{(\lambda,\sigma)}](F)}{\sim} Y$*

(3) $X \overset{S^L_{(\sigma)}(F)}{\sim} Y \cap l_\infty(F) = X \overset{[V^{Lp}_{(\lambda,\sigma)}](F)}{\sim} Y \cap l_\infty(F)$.

Proof. Part (1): if $\epsilon > 0$ and $X \overset{[V^{Lp}_{(\lambda,\sigma)}](F)}{\sim} Y$ then

$$\sum_{k\in I_n} d\left(\frac{X_{\sigma^k(m)}}{Y_{\sigma^k(m)}}, L\right)^p \geq \sum_{k\in I_n \& d\left(X_{(\sigma^k\ (m))}/Y_{(\sigma^k\ (m))}\ ,L\right)\geq\epsilon} d\left(\frac{X_{\sigma^k(m)}}{Y_{\sigma^k(m)}}, L\right)^p$$
$$\geq \epsilon^p \left|\left\{k \in I_n : d\left(\frac{X_{\sigma^k(m)}}{Y_{\sigma^k(m)}}, L\right) \geq \epsilon\right\}\right|. \tag{4.1}$$

Therefore $X \overset{S^L_{(\lambda,\sigma)}(F)}{\sim} Y$. Part (2): suppose that fuzzy real-valued sequences $X = (X_k)$ and $Y = (Y_k)$ are in $l_\infty(F)$ and $X \overset{S^L_{\lambda}(F)}{\sim} Y$. Then we can assume that $d(X_{(\sigma^k(m))}/Y_{(\sigma^k(m))}\ , L) \leq K$ for all k and m. Let $\epsilon > 0$ be given and N_ϵ be such that

$$\frac{1}{\lambda_n}\left|\left\{k \in I_n : d\left(\frac{X_{\sigma^k(m)}}{Y_{\sigma^k(m)}}, L\right) \geq \left(\frac{\epsilon}{2}\right)^{1/p}\right\}\right| \leq \frac{\epsilon}{2K^p} \tag{4.2}$$

for all $n > N_\epsilon$ and let

$$L_k := \left\{k \in I_n : d\left(\frac{X_{\sigma^k(m)}}{Y_{\sigma^k(m)}}, L\right) \geq \left(\frac{\epsilon}{2}\right)^{1/p}\right\}. \tag{4.3}$$

Now for all $n > N$ we have

$$\frac{1}{\lambda_n}\sum_{k\in I_n} d\left(\frac{X_{\sigma^k(m)}}{Y_{\sigma^k(m)}}, L\right)^p = \frac{1}{\lambda_n}\sum_{k\in L_k} d\left(\frac{X_{\sigma^k(m)}}{Y_{\sigma^k(m)}}, L\right)^p + \frac{1}{\lambda_n}\sum_{k\notin L_k} d\left(\frac{X_{\sigma^k(m)}}{Y_{\sigma^k(m)}}, L\right)^p$$
$$\leq \frac{1}{\lambda_n}\frac{\lambda_n\epsilon}{2K^p}K^p + \frac{1}{\lambda_n}\lambda_n\frac{\epsilon}{2}. \tag{4.4}$$

Hence $X \overset{[V^{Lp}_{(\lambda,\sigma)}](F)}{\sim} Y$. This completes the proof.Part (3): this immediately follows from (1) and (2). □

In the next theorem we prove the following relation.

Theorem 4.2. *Let* $0 < h = \inf_k p_k \leq \sup_k p_k = H < \infty$. *Then* $X \overset{[V^{L(p)}_{(\lambda,\sigma)}](F)}{\sim} Y$ *implies* $X \overset{S^L_{(\lambda,\sigma)}(F)}{\sim} Y$.

Proof. Let $X \overset{[V_{(\lambda,\sigma)}^{L(p)}](F)}{\sim} Y$ and $\epsilon > 0$ be given. Then

$$\begin{aligned}
\frac{1}{\lambda_n}\sum_{k\in I_n} d\left(\frac{X_{\sigma^k(m)}}{Y_{\sigma^k(m)}},L\right)^{p_k} &= \frac{1}{\lambda_n}\sum_{k\in I_n \& d\left(X_{(\sigma^k(m))}/Y_{(\sigma^k(m))},L\right)\geq\epsilon} d\left(\frac{X_{\sigma^k(m)}}{Y_{\sigma^k(m)}},L\right)^{p_k} \\
&\quad + \frac{1}{\lambda_n}\sum_{k\in I_n \& d\left(X_{(\sigma^k(m))}/Y_{(\sigma^k(m))},L\right)<\epsilon} d\left(\frac{X_{\sigma^k(m)}}{Y_{\sigma^k(m)}},L\right)^{p_k} \\
&\geq \frac{1}{\lambda_n}\sum_{k\in I_n \& d\left(X_{(\sigma^k(m))}/Y_{(\sigma^k(m))},L\right)\geq\epsilon} d\left(\frac{X_{\sigma^k(m)}}{Y_{\sigma^k(m)}},L\right)^{p_k} \\
&\geq \frac{1}{\lambda_n}\sum_{k\in I_n \& d\left(X_{(\sigma^k(m))}/Y_{(\sigma^k(m))},L\right)\geq\epsilon} \min\left\{(\epsilon)^{\inf p_k},(\epsilon)^H\right\} \\
&\geq \frac{1}{\lambda_n}\left\{k\in I_n : d\left(\frac{X_{\sigma^k(m)}}{Y_{\sigma^k(m)}},L\right)\geq\epsilon\right\}\min\left\{(\epsilon)^{\inf p_k},(\epsilon)^H\right\}.
\end{aligned} \tag{4.5}$$

Hence $X \overset{S_{(\lambda,\sigma)}^{L}(F)}{\sim} Y$. □

Theorem 4.3. *Let fuzzy real-valued sequences* $X = (X_k)$ *and* $Y = (Y_k)$ *be bounded and* $0 < h = \inf_k p_k \leq \sup_k p_k = H < \infty$. *Then* $X \overset{S_{(\lambda,\sigma)}^{L}(F)}{\sim} Y$ *implies* $X \overset{[V_{(\lambda,\sigma)}^{L(p)}](F)}{\sim} Y$.

Proof. Suppose that fuzzy real-valued sequences $X = (X_k)$ and $Y = (Y_k)$ be bounded and $\epsilon > 0$ is given. Since $X = (X_k)$ and $Y = (Y_k)$ are bounded there exists an integer K such that $d(X_{(\sigma^k(m))}/Y_{(\sigma^k(m))},L) \leq K$ for all k and m; then

$$\begin{aligned}
\frac{1}{\lambda_n}\sum_{k\in I_n} d\left(\frac{X_{\sigma^k(m)}}{Y_{\sigma^k(m)}},L\right)^{p_k} &= \frac{1}{\lambda_n}\sum_{k\in I_n \& d\left(X_{(\sigma^k(m))}/Y_{(\sigma^k(m))},L\right)\geq\epsilon} d\left(\frac{X_{\sigma^k(m)}}{Y_{\sigma^k(m)}},L\right)^{p_k} \\
&\quad + \frac{1}{\lambda_n}\sum_{k\in I_n \& d\left(X_{(\sigma^k(m))}/Y_{(\sigma^k(m))},L\right)<\epsilon} d\left(\frac{X_{\sigma^k(m)}}{Y_{\sigma^k(m)}},L\right)^{p_k} \\
&\leq \frac{1}{\lambda_n}\sum_{k\in I_n \& d\left(X_{(\sigma^k(m))}/Y_{(\sigma^k(m))},L\right)\geq\epsilon} \max\left\{K^h,K^H\right\} \\
&\quad + \frac{1}{\lambda_n}\sum_{k\in I_n \& d\left(X_{(\sigma^k(m))}/Y_{(\sigma^k(m))},L\right)<\epsilon} \max\{\epsilon\}^{p_k} \\
&\leq \max\left\{K^h,K^H\right\}\frac{1}{\lambda_n}\left\{k\in I_n : d\left(\frac{X_{\sigma^k(m)}}{Y_{\sigma^k(m)}},L\right)\geq\epsilon\right\} + \max\left\{\epsilon^h,\epsilon^H\right\}.
\end{aligned} \tag{4.6}$$

Hence $X \overset{[V_{(\lambda,\sigma)}^{L(p)}](F)}{\sim} Y$. □

Remark 4.4. If we take $\sigma(k) = k + 1$ in our results, all results reduce to the results of almost convergence which have not proved so far.

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