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Research Article

Iterative Methods for Finding Common Solution of Generalized Equilibrium Problems and Variational Inequality Problems and Fixed Point Problems of a Finite Family of Nonexpansive Mappings

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We introduce a new method for a system of generalized equilibrium problems, system of variational inequality problems, and fixed point problems by using S-mapping generated by a finite family of nonexpansive mappings and real numbers. Then, we prove a strong convergence theorem of the proposed iteration under some control condition. By using our main result, we obtain strong convergence theorem for finding a common element of the set of solution of a system of generalized equilibrium problems, system of variational inequality problems, and the set of common fixed points of a finite family of strictly pseudocontractive mappings.

1. Introduction

Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H. Let $A : C \to H$ be a nonlinear mapping, and let $F : C \times C \to \mathbb{R}$ be a bifunction. A mapping T of H into itself is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$. We denote by $F(T)$ the set of fixed points of T (i.e., $F(T) = \{x \in H : Tx = x\}$). Goebel and Kirk [1] showed that $F(T)$ is always closed convex, and also nonempty provided T has a bounded trajectory.

A bounded linear operator A on H is called *strongly positive* with coefficient $\overline{\gamma}$ if there is a constant $\overline{\gamma} > 0$ with the property

$$\langle Ax, \ x\rangle \geq \overline{\gamma}\|x\|^2. \tag{1.1}$$

The equilibrium problem for F is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{1.2}$$

The set of solutions of (1.2) is denoted by $\mathrm{EP}(F)$. Many problems in physics, optimization, and economics are seeking some elements of $\mathrm{EP}(F)$, see [2, 3]. Several iterative methods have been proposed to solve the equilibrium problem, see, for instance, [2–4]. In 2005, Combettes and Hirstoaga [3] introduced an iterative scheme of finding the best approximation to the initial data when $\mathrm{EP}(F)$ is nonempty and proved a strong convergence theorem.

The variational inequality problem is to find a point $u \in C$ such that

$$\langle v - u,\ \mathrm{A}u \rangle \geq 0 \quad \forall\ v \in C. \tag{1.3}$$

The set of solutions of the variational inequality is denoted by $\mathrm{VI}(C, A)$, and we consider the following generalized equilibrium problem.

$$\text{Find } z \in C \quad \text{such that } F(z, y) + \langle Az, y - z \rangle \geq 0, \quad \forall y \in C. \tag{1.4}$$

The set of such $z \in C$ is denoted by $\mathrm{EP}(F, A)$, that is,

$$\mathrm{EP}(F, A) = \{z \in C : F(z, y) + \langle Az, y - z \rangle \geq 0,\ \forall y \in C\}. \tag{1.5}$$

In the case of $A \equiv 0$, $\mathrm{EP}(F, A) = \mathrm{EP}(F)$. Numerous problems in physics, optimization, variational inequalities, minimax problems, the Nash equilibrium problem in noncooperative games reduce to find element of (1.5)

A mapping A of C into H is called *inverse-strongly monotone*, see [5], if there exists a positive real number α such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2 \tag{1.6}$$

for all $x, y \in C$.

The problem of finding a common fixed point of a family of nonexpansive mappings has been studied by many authors. The well-known convex feasibility problem reduces to finding a point in the intersection of the fixed point sets of a family of nonexpansive mapping (see [6, 7]).

The ploblem of finding a common element of $\mathrm{EP}(F, A)$ and the set of all common fixed points of a family of nonexpansive mappings is of wide interdisciplinary interest and importance. Many iterative methods are purposed for finding a common element of the solutions of the equilibrium problem and fixed point problem of nonexpansive mappings, see [8–10].

In 2008, S.Takahashi and W.Takahashi [11] introduced a general iterative method for finding a common element of $\mathrm{EP}(F,\ A)$ and $F(T)$. They defined $\{x_n\}$ in the following way:

$$\begin{gathered}
u, x_1 \in C, \quad \text{arbitrarily;} \\
F(z_n,\ y) + \langle Ax_n, y - z_n\rangle + \frac{1}{\lambda_n}\langle y - z_n,\ z_n - x_n\rangle \geq 0, \quad \forall y \in C, \\
x_{n+1} = \beta_n x_n + (1-\beta_n) T(a_n u + (1-a_n) z_n), \quad \forall n \in \mathbb{N},
\end{gathered} \tag{1.7}$$

where A is an α-inverse strongly monotone mapping of C into H with positive real number α, and $\{a_n\} \in [0,1]$, $\{\beta_n\} \subset [0,1]$, $\{\lambda_n\} \subset [0, 2\alpha]$, and proved strong convergence of the scheme (1.7) to $z \in \bigcap_{i=1}^{N} F(T_i) \cap \mathrm{EP}(F, A)$, where $z = P_{\bigcap_{i=1}^{N} F(T_i) \cap \mathrm{EP}(F,\ A)} u$ in the framework of a Hilbert space, under some suitable conditions on $\{a_n\}$, $\{\beta_n\}$, $\{\lambda_n\}$ and bifunction F.

Very recently, in 2010, Qin, et al. [12] introduced a iterative scheme method for finding a common element of $\mathrm{EP}(F_1, A)$, $\mathrm{EP}(F_2, B)$ and common fixed point of infinite family of nonexpansive mappings. They defined $\{x_n\}$ in the following way:

$$\begin{gathered}
x_1 \in C, \quad \text{arbitrarily;} \\
F_1(u_n, u) + \langle Ax_n, u - u_n\rangle + \frac{1}{r}\langle u - u_n, u_n - x_n\rangle \geq 0, \quad \forall u \in C, \\
F_2(v_n, v) + \langle Bx_n, v - v_n\rangle + \frac{1}{s}\langle v - v_n, v_n - x_n\rangle \geq 0, \quad \forall v \in C, \\
y_n = \delta_n u_n + (1-\delta_n) v_n, \\
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n x_n, \quad \forall n \in \mathbb{N},
\end{gathered} \tag{1.8}$$

where $f : C \to C$ is a contraction mapping and W_n is W-mapping generated by infinite family of nonexpansive mappings and infinite real number. Under suitable conditions of these parameters they proved strong convergence of the scheme (1.8) to $z = P_{\mathfrak{F}} f(z)$, where $\mathfrak{F} = \bigcap_{i=1}^{\infty} F(T_i) \cap \mathrm{EP}(F_1, A) \cap \mathrm{EP}(F_2, B)$.

In this paper, motivated by [11, 12], we introduced a general iterative scheme $\{x_n\}$ defined by

$$\begin{gathered}
F(u_n, u) + \langle Ax_n, u - u_n\rangle + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, \\
G(v_n, v) + \langle Bx_n, v - v_n\rangle + \frac{1}{s_n}\langle v - v_n, v_n - x_n\rangle \geq 0, \\
y_n = \delta_n P_C(u_n - \lambda_n A u_n) + (1-\delta_n) P_C(v_n - \eta_n B v_n), \\
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n y_n, \quad \forall n \geq 0,
\end{gathered} \tag{1.9}$$

where $f : C \to C$ and S_n is S-mapping generated by $T_0, \ldots, T_n$ and $\alpha_n, \alpha_{n-1}, \ldots, \alpha_0$. Under suitable conditions, we proved strong convergence of $\{x_n\}$ to $z = P_{\mathfrak{F}} f(z)$, and z is solution of

$$\begin{aligned} \langle Ax^*, x - x^* \rangle &\geq 0, \\ \langle Bx^*, x - x^* \rangle &\geq 0. \end{aligned} \tag{1.10}$$

2. Preliminaries

In this section, we collect and give some useful lemmas that will be used for our main result in the next section.

Let C be closed convex subset of a real Hilbert space H, and let P_C be the metric projection of H onto C, that is, for $x \in H$, $P_C x$ satisfies the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|. \tag{2.1}$$

The following characterizes the projection P_C.

Lemma 2.1 (see [13]). *Given $x \in H$ and $y \in C$. Then $P_C x = y$ if and only if there holds the inequality*

$$\langle x - y, y - z \rangle \geq 0 \quad \forall z \in C. \tag{2.2}$$

Lemma 2.2 (see [14]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} = (1 - \alpha_n) s_n + \beta_n, \quad \forall n \geq 0 \tag{2.3}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ satisfy the conditions

(1) $\{\alpha_n\} \subset (0,1), \sum_{n=1}^{\infty} \sum \alpha_n = \infty$,

(2) $\limsup_{n \to \infty} \beta_n / \alpha_n \leq 0$.

Then $\lim_{n \to \infty} s_n = 0$.

Lemma 2.3 (see [15]). *Let C be a closed convex subset of a strictly convex Banach space E. Let $\{T_n : n \in \mathbb{N}\}$ be a sequence of nonexpansive mappings on C. Suppose that $\bigcap_{n=1}^{\infty} F(T_n)$is nonempty. Let $\{\lambda_n\}$ be a sequence of positive numbers with $\Sigma_{n=1}^{\infty} \lambda_n = 1$. Then a mapping S on C defined by*

$$S(x) = \Sigma_{n=1}^{\infty} \lambda_n T_n x \tag{2.4}$$

for $x \in C$ is well defined, nonexpansive, and $F(S) = \bigcap_{n=1}^{\infty} F(T_n)$ hold.

Lemma 2.4 (see [16]). *Let E be a uniformly convex Banach space, C a nonempty closed convex subset of E, and $S : C \to C$ a nonexpansive mapping. Then $I - S$ is demiclosed at zero.*

Lemma 2.5 (see [17]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X, and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$. Suppose that*

$$x_{n+1} = \beta_n x_n + (1-\beta_n) z_n \tag{2.5}$$

for all integer $n \ge 0$ and

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0. \tag{2.6}$$

Then $\lim_{n\to\infty}\|x_n - z_n\| = 0$.

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone, that is, $F(x,y) + F(y,x) \le 0, \quad \forall x,y \in C$,

(A3) for all $x,y,z \in C$,

$$\lim_{t\to 0^+} F(tz + (1-t)x, y) \le F(x,y), \tag{2.7}$$

(A4) for all $x \in C, y \mapsto F(x,y)$ is convex and lower semicontinuous.

The following lemma appears implicitly in [2].

Lemma 2.6 (see [2]). *Let C be a nonempty closed convex subset of H, and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)–(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z,\ y) + \frac{1}{r}\langle y - z, z - x\rangle \tag{2.8}$$

for all $x \in C$.

Lemma 2.7 (see [3]). *Assume that $F : C\times C \to \mathbb{R}$ satisfies (A1)–(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{ z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0,\ \forall y \in C \right\}. \tag{2.9}$$

for all $z \in H$. Then, the following hold:

(1) *T_r is single-valued;*

(2) *T_r is firmly nonexpansive, that is,*

$$\|T_r(x) - T_r(y)\|^2 \le \langle T_r(x) - T_r(y), x - y\rangle \quad \forall x,y \in H; \tag{2.10}$$

(3) $F(T_r) = \mathrm{EP}(F)$;

(4) $\mathrm{EP}(F)$ *is closed and convex.*

In 2009, Kangtunyakarn and Suantai [18] defined a new mapping and proved their lemma as follows.

Definition 2.8. Let C be a nonempty convex subset of real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself. For each $j = 1, 2, \dots, N$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I \in [0,1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. We define the mapping $S : C \to C$ as follows:

$$\begin{aligned}
U_0 &= I, \\
U_1 &= \alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I, \\
U_2 &= \alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I, \\
U_3 &= \alpha_1^3 T_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I, \\
&\vdots \\
U_{N-1} &= \alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I, \\
S = U_N &= \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I.
\end{aligned} \tag{2.11}$$

This mapping is called *S-mapping* generated by $T_1, \dots, T_N$ and $\alpha_1, \alpha_2, \dots, \alpha_N$.

Lemma 2.9. *Let C be a nonempty closed convex subset of strictly convex. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpanxive mappings of C into itself with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$, and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, 3, \dots, N$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0,1)$ for all $j = 1, 2, \dots,\ N-1, \alpha_1^N \in (0,1]$ $\alpha_2^j, \alpha_3^j \in [0,1)$ for all $j = 1, 2, \dots, N$. Let S be the mapping generated by $T_1, \dots, T_N$ and $\alpha_1, \alpha_2, \dots, \alpha_N$. Then $F(S) = \bigcap_{i=1}^N F(T_i)$.*

Lemma 2.10. *Let C be a nonempty closed convex subset of Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself and $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j})$, $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I = [0,1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$ such that $\alpha_i^{n,j} \to \alpha_i^j \in [0,1]$ as $n \to \infty$ for $i = 1, 3$ and $j = 1, 2, 3, \dots, N$. Moreover, for every $n \in \mathbb{N}$, let S and S_n be the S-mappings generated by $T_1, T_2, \dots, T_N$ and $\alpha_1, \alpha_2, \dots, \alpha_N$ and $T_1, T_2, \dots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \dots, \alpha_N^{(n)}$, respectively. Then $\lim_{n\to\infty} \|S_n x - Sx\| = 0$ for every $x \in C$.*

Lemma 2.11 (see [19]). *Let C be a nonempty closed convex subset of a Hilbert space H, and let $G : C \to C$ be defined by*

$$G(x) = P_C(x - \lambda A x), \quad \forall x \in C, \tag{2.12}$$

with $\forall \lambda > 0$. Then $x^* \in VI(C, A)$ if and only if $x^* \in F(G)$.

3. Main Result

Theorem 3.1. *Let C be a nonempty closed convex subset of a Hilbert space H. Let F and G be two bifunctions from $C \times C$ into $\mathbb{R}$ satisfying (A1)–(A4), respectively. Let $A : C \to H$ a α-inverse strongly monotone mapping and $B : C \to H$ be a β-inverse strongly monotone mapping. Let $\{T_i\}_{i=1}^N$ be finite* *family of nonexpansive mappings with* $\mathfrak{F} = \bigcap_{i=1}^{N} F(T_i) \cap \mathrm{EP}(F, A) \cap \mathrm{EP}(G, B) \cap F(G_1) \cap F(G_2) \neq \emptyset$, *where* $G_1, G_2 : C \to C$ *are defined by* $G_1(x) = P_C(x - \lambda_n Ax)$, $G_2(x) = P_C(x - \eta_n Bx)$, $\forall x \in C$. *Let* $f : C \to C$ *be a contraction with the coefficient* $\theta \in (0,1)$. *Let* S_n *be the S-mappings generated by* $T_1, T_2, \ldots, T_N$ *and* $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$, *where* $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j}) \in I \times I \times I$, $I = [0,1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$ *and* $0 < \eta_1 \leq \alpha_1^{n,j} \leq \theta_1 < 1$ $\forall n \in \mathbb{N}, \forall j = 1, 2, \ldots, N-1$, $0 < \eta_N \leq \alpha_1^{n,N} \leq 1$ *and* $0 \leq \alpha_2^{n,j}, \alpha_3^{n,j} \leq \theta_3 < 1$ $\forall n \in \mathbb{N}$, $\forall j = 1, 2, \ldots, N$. *Let* $\{x_n\}, \{u_n\}, \{v_n\}, \{y_n\}$ *be sequences generated by* $x_1, u, v \in C$

$$
\begin{aligned}
&F(u_n, u) + \langle Ax_n, u - u_n \rangle + \frac{1}{r_n} \langle u - u_n, u_n - x_n \rangle \geq 0, \\
&G(v_n, v) + \langle Bx_n, v - v_n \rangle + \frac{1}{s_n} \langle v - v_n, v_n - x_n \rangle \geq 0, \\
&y_n = \delta_n P_C(u_n - \lambda_n Au_n) + (1 - \delta_n) P_C(v_n - \eta_n Bv_n), \\
&x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n y_n, \quad \forall n \geq 1,
\end{aligned}
\tag{3.1}
$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \in (0,1)$ *such that* $\alpha_n + \beta_n + \gamma_n = 1$, $r_n \in [a,b] \subset (0, 2\alpha)$, $s_n \in [c,d] \subset (0, 2\beta)$, $\lambda_n \in [e,f] \subset (0, 2\alpha)$, $\eta_n \in [g,h] \subset (0, 2\beta)$. *Assume that*

(i) $\lim_{n\to\infty} n = 0$ *and* $\Sigma_{n=0}^{\infty} \alpha_n = \infty$,

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

(iii) $\lim_{n\to\infty} \delta_n = \delta \in (0,1)$,

(iv) $\Sigma_{n=0}^{\infty} |s_{n+1} - s_n|$, $\Sigma_{n=0}^{\infty} |r_{n+1} - r_n|$, $\Sigma_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n|$, $\Sigma_{n=0}^{\infty} |\eta_{n+1} - \eta_n|$, $\Sigma_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n|$, $\Sigma_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,

(v) $|\alpha_1^{n+1,j} - \alpha_1^{n,j}| \to 0$, *and* $|\alpha_3^{n+1,j} - \alpha_3^{n,j}| \to 0$ *as* $n \to \infty$, *for all* $j \in \{1, 2, 3, \ldots, N\}$.

Then the sequence $\{x_n\}$, $\{y_n\}$, $\{u_n\}$, $\{v_n\}$ *converge strongly to* $z = P_{\mathfrak{F}} f(z)$, *and* z *is solution of*

$$
\begin{aligned}
&\langle Ax^*, x - x^* \rangle \geq 0, \\
&\langle Bx^*, x - x^* \rangle \geq 0.
\end{aligned}
\tag{3.2}
$$

Proof. First, we show that $(I - \lambda_n A)$, $(I - \eta_n B)(I - r_n A)$ and $(I - s_n B)$ are nonexpansive. Let $x, y \in C$. Since A is α-strongly monotone and $\lambda_n < 2\alpha$ for all $n \in \mathbb{N}$, we have

$$
\begin{aligned}
\|(I - \lambda_n A)x - (I - \lambda_n A)y\|^2 &= \|x - y - \lambda_n (Ax - Ay)\|^2 \\
&= \|x - y\|^2 - 2\lambda_n \langle x - y, Ax - Ay \rangle + \lambda_n^2 \|Ax - Ay\|^2 \\
&\leq \|x - y\|^2 - 2\alpha\lambda_n \|Ax - Ay\|^2 + \lambda_n^2 \|Ax - Ay\|^2 \\
&= \|x - y\|^2 + \lambda_n (\lambda_n - 2\alpha) \|Ax - Ay\|^2 \\
&\leq \|x - y\|^2.
\end{aligned}
\tag{3.3}
$$

Thus $(I-\lambda_n A)$ is nonexpansive. By using the same proof, we obtain that $(I-\eta_n B)$ $(I-r_n A)$ and $(I-s_n B)$ are nonexpansive.

We will divide our proof into 6 steps.

Step 1. We will show that the sequence $\{x_n\}$ is bounded. Since

$$F(u_n,u)+\langle Ax_n,u-u_n\rangle+\frac{1}{r_n}\langle u-u_n,u_n-x_n\rangle\geq 0,\quad \forall u\in C, \tag{3.4}$$

then we have

$$F(u_n,u)+\frac{1}{r_n}\langle u-u_n,u_n-(I-r_nA)x_n\rangle\geq 0. \tag{3.5}$$

By Lemma 2.7, we have $u_n=T_{r_n}(I-r_nA)x_n$. By the same argument as above, we obtaine that $v_n=T_{s_n}(I-s_nB)x_n$

Let $z\in\mathfrak{F}$. Then $F(z,y)+\langle y-z,Az\rangle\geq 0$ and $G(z,\ y)+\langle y-z,Bz\rangle\geq 0$. Hence

$$\begin{aligned} &F(z,y)+\frac{1}{r_n}\langle y-z,z-z+r_nAz\rangle\geq 0,\\ &G(z,y)+\frac{1}{s_n}\langle y-z,z-z+s_nBz\rangle\geq 0. \end{aligned} \tag{3.6}$$

Again by Lemma 2.7, we have $z=T_{r_n}(z-r_nAz)=T_{s_n}(z-s_nBz)$. Since $z\in\mathfrak{F}$, we have $z=P_C(I-\lambda_nA)z=P_C(I-\eta_nB)z$. By nonexpansiveness of T_{r_n}, T_{s_n}, $I-r_nA$, $I-s_nB$, we have

$$\begin{aligned} \|x_{n+1}-z\| &\leq \alpha_n\|f(x_n)-z\|+\beta_n\|x_n-z\|+\gamma_n\|S_ny_n-z\|\\ &\leq \alpha_n\|f(x_n)-f(z)\|+\alpha_n\|f(z)-z\|+\beta_n\|x_n-z\|+\gamma_n\|y_n-z\|\\ &\leq \alpha_n\theta\|x_n-z\|+\alpha_n\|f(z)-z\|+\beta_n\|x_n-z\|\\ &\quad+\gamma_n\|\delta_n(P_C(u_n-\lambda_nAu_n)-z)+(1-\delta_n)(P_C(v_n-\eta_nBv_n)-z)\|\\ &\leq \alpha_n\theta\|x_n-z\|+\alpha_n\|f(z)-z\|+\beta_n\|x_n-z\|+\gamma_n(\delta_n\|u_n-z\|+(1-\delta_n)\|v_n-z\|)\\ &= \alpha_n\theta\|x_n-z\|+\alpha_n\|f(z)-z\|+\beta_n\|x_n-z\|\\ &\quad+\gamma_n(\delta_n\|T_{r_n}(I-r_nA)x_n-T_{r_n}(I-r_nA)z\|\\ &\quad+(1-\delta_n)\|T_{s_n}(I-s_nB)x_n-T_{s_n}(I-s_nB)z\|)\\ &\leq \alpha_n\theta\|x_n-z\|+\alpha_n\|f(z)-z\|+\beta_n\|x_n-z\|+\gamma_n\|x_n-z\|\\ &= \alpha_n\theta\|x_n-z\|+\alpha_n\|f(z)-z\|+(1-\alpha_n)\|x_n-z\|\\ &= (1-\alpha_n(1-\theta))\|x_n-z\|+\alpha_n\|f(z)-z\|\\ &\leq \max\left\{\|x_n-z\|,\frac{\|f(z)-z\|}{1-\theta}\right\}. \end{aligned} \tag{3.7}$$

By induction we can prove that $\{x_n\}$ is bounded and so are $\{u_n\}$, $\{v_n\}$, $\{y_n\}$, $\{S_n y_n\}$. Without of generality, assume that there exists a bounded set $K \subset C$ such that

$$\{u_n\}, \{v_n\}, \{y_n\}, \{S_n y_n\} \in K. \tag{3.8}$$

Step 2. We will show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

Putting $k_n = (x_{n+1} - \beta_n x_n)/(1-\beta_n)$, we have

$$x_{n+1} = (1-\beta_n)k_n + \beta_n x_n, \quad \forall n \geq 0. \tag{3.9}$$

From definition of k_n, we have

$$\begin{aligned}
\|k_{n+1} - k_n\| &= \left\| \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} \right\| \\
&= \left\| \frac{\alpha_{n+1} f(x_{n+1}) + \gamma_{n+1} S_{n+1} y_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n f(x_n) + \gamma_n S_n y_n}{1-\beta_n} \right\| \\
&= \left\| \frac{\alpha_{n+1} f(x_{n+1}) + (1-\beta_{n+1}-\alpha_{n+1}) S_{n+1} y_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n f(x_n) + (1-\beta_n-\alpha_n) S_n y_n}{1-\beta_n} \right\| \\
&= \left\| \frac{\alpha_{n+1}}{1-\beta_{n+1}} (f(x_{n+1}) - S_{n+1}y_{n+1}) - \frac{\alpha_n}{1-\beta_n}(f(x_n) - S_n y_n) + S_{n+1}y_{n+1} - S_n y_n \right\| \\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}} \|f(x_{n+1}) - S_{n+1}y_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|f(x_n) - S_n y_n\| + \|S_{n+1}y_{n+1} - S_n y_n\| \\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}} \|f(x_{n+1}) - S_{n+1}y_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|f(x_n) - S_n y_n\| \\
&\quad + \|S_{n+1}y_{n+1} - S_n y_n\| + \|x_{n+1} - x_n\|.
\end{aligned} \tag{3.10}$$

By definition of S_n, for $k \in \{2,3,\dots,N\}$, we have

$$
\begin{aligned}
\|U_{n+1,k}y_n - U_{n,k}y_n\| &= \Big\|\alpha_1^{n+1,k}T_kU_{n+1,k-1}y_n + \alpha_2^{n+1,k}U_{n+1,k-1}y_n + \alpha_3^{n+1,k}y_n \\
&\quad -\alpha_1^{n,k}T_kU_{n,k-1}y_n - \alpha_2^{n,k}U_{n,k-1}y_n - \alpha_3^{n,k}y_n\Big\| \\
&= \Big\|\alpha_1^{n+1,k}(T_kU_{n+1,k-1}y_n - T_kU_{n,k-1}y_n) + \left(\alpha_1^{n+1,k} - \alpha_1^{n,k}\right)T_kU_{n,k-1}y_n \\
&\quad + \left(\alpha_3^{n+1,k} - \alpha_3^{n,k}\right)y_n + \alpha_2^{n+1,k}(U_{n+1,k-1}y_n - U_{n,k-1}y_n) \\
&\quad +\left(\alpha_2^{n+1,k} - \alpha_2^{n,k}\right)U_{n,k-1}y_n\Big\| \\
&\le \alpha_1^{n+1,k}\|U_{n+1,k-1}y_n - U_{n,k-1}y_n\| + \left|\alpha_1^{n+1,k} - \alpha_1^{n,k}\right|\|T_kU_{n,k-1}y_n\| \\
&\quad + \left|\alpha_3^{n+1,k} - \alpha_3^{n,k}\right|\|y_n\| + \alpha_2^{n+1,k}\|U_{n+1,k-1}y_n - U_{n,k-1}y_n\| \\
&\quad + \left|\alpha_2^{n+1,k} - \alpha_2^{n,k}\right|\|U_{n,k-1}y_n\| \\
&= \left(\alpha_1^{n+1,k} + \alpha_2^{n+1,k}\right)\|U_{n+1,k-1}y_n - U_{n,k-1}y_n\| \\
&\quad + \left|\alpha_1^{n+1,k} - \alpha_1^{n,k}\right|\|T_kU_{n,k-1}y_n\| + \left|\alpha_3^{n+1,k} - \alpha_3^{n,k}\right|\|y_n\| \\
&\quad + \left|\alpha_2^{n+1,k} - \alpha_2^{n,k}\right|\|U_{n,k-1}y_n\| \\
&\le \|U_{n+1,k-1}y_n - U_{n,k-1}y_n\| + \left|\alpha_1^{n+1,k} - \alpha_1^{n,k}\right|\|T_kU_{n,k-1}y_n\| \\
&\quad + \left|\alpha_3^{n+1,k} - \alpha_3^{n,k}\right|\|y_n\| + \Big|1 - \left(\alpha_1^{n+1,k} + \alpha_3^{n+1,k}\right) \\
&\quad -\left(1 - \left(\alpha_1^{n,k} + \alpha_3^{n,k}\right)\right)\Big|\|U_{n,k-1}y_n\| \\
&= \|U_{n+1,k-1}y_n - U_{n,k-1}y_n\|\left|\alpha_1^{n+1,k} - \alpha_1^{n,k}\right|\|T_kU_{n,k-1}y_n\| \\
&\quad + \left|\alpha_3^{n+1,k} - \alpha_3^{n,k}\right|\|y_n\| + \left|\left(\alpha_1^{n,k} - \alpha_1^{n+1,k}\right) + \left(\alpha_3^{n,k} - \alpha_3^{n+1,k}\right)\right|\|U_{n,k-1}y_n\| \\
&\le \|U_{n+1,k-1}y_n - U_{n,k-1}y_n\| + \left|\alpha_1^{n+1,k} - \alpha_1^{n,k}\right|\|T_kU_{n,k-1}y_n\| \\
&\quad + \left|\alpha_3^{n+1,k} - \alpha_3^{n,k}\right|\|y_n\| + \left|\alpha_1^{n,k} - \alpha_1^{n+1,k}\right|\|U_{n,k-1}y_n\| \\
&\quad + \left|\alpha_3^{n,k} - \alpha_3^{n+1,k}\right|\|U_{n,k-1}y_n\| \\
&= \|U_{n+1,k-1}y_n - U_{n,k-1}y_n\| + \left|\alpha_1^{n+1,k} - \alpha_1^{n,k}\right|(\|T_kU_{n,k-1}y_n\| + \|U_{n,k-1}y_n\|) \\
&\quad + \left|\alpha_3^{n+1,k} - \alpha_3^{n,k}\right|(\|y_n\| + \|U_{n,k-1}y_n\|).
\end{aligned}
\tag{3.11}
$$

By (3.11), we obtain that for each $n \in \mathbb{N}$,

$$\begin{aligned}
\|S_{n+1}y_n - S_n y_n\| &= \|U_{n+1,N}y_n - U_{n,N}y_n\| \\
&\leq \|U_{n+1,N-1}y_n - U_{n,N-1}y_n\| + \left|\alpha_1^{n+1,N} - \alpha_1^{n,N}\right|(\|T_N U_{n,N-1}y_n\| \\
&\quad + \|U_{n,N-1}y_n\|) + \left|\alpha_3^{n+1,N} - \alpha_3^{n,N}\right|(\|y_n\| + \|U_{n,N-1}y_n\|) \\
&\leq \|U_{n+1,N-2}y_n - U_{n,N-2}y_n\| + \left|\alpha_1^{n+1,N-1} - \alpha_1^{n,N-1}\right| \\
&\quad \times (\|T_{N-1}U_{n,N-2}y_n\| + \|U_{n,N-2}y_n\|) \\
&\quad + \left|\alpha_3^{n+1,N-1} - \alpha_3^{n,N-1}\right|(\|y_n\| + \|U_{n,N-2}y_n\|) \\
&\quad + \left|\alpha_1^{n+1,N} - \alpha_1^{n,N}\right|(\|T_N U_{n,N-1}y_n\| + \|U_{n,N-1}y_n\|) \\
&\quad + \left|\alpha_3^{n+1,N} - \alpha_3^{n,N}\right|(\|y_n\| + \|U_{n,N-1}y_n\|) \\
&= \|U_{n+1,N-2}y_n - U_{n,N-2}y_n\| + \sum_{j=N-1}^{N}\left|\alpha_1^{n+1,j} - \alpha_1^{n,j}\right|(\|T_j U_{n,j-1}y_n\| \\
&\quad + \|U_{n,j-1}y_n\|) + \sum_{j=N-1}^{N}\left|\alpha_3^{n+1,j} - \alpha_3^{n,j}\right|(\|y_n\| + \|U_{n,j-1}y_n\|) \\
&\leq \\
&\vdots \\
&\leq \|U_{n+1,1}y_n - U_{n,1}y_n\| + \sum_{j=2}^{N}\left|\alpha_1^{n+1,j} - \alpha_1^{n,j}\right|(\|T_j U_{n,j-1}y_n\| + \|U_{n,j-1}y_n\|) \\
&\quad + \sum_{j=2}^{N}\left|\alpha_3^{n+1,j} - \alpha_3^{n,j}\right|(\|y_n\| + \|U_{n,j-1}y_n\|) \\
&= \left\|\left(1 - \alpha_1^{n+1,1}\right)y_n + \alpha_1^{n+1,1}T_1 y_n - \left(1 - \alpha_1^{n,1}\right)y_n - \alpha_1^{n,1}T_1 y_n\right\| \\
&\quad + \sum_{j=2}^{N}\left|\alpha_1^{n+1,j} - \alpha_1^{n,j}\right|(\|T_j U_{n,j-1}y_n\| + \|U_{n,j-1}y_n\|) \\
&\quad + \sum_{j=2}^{N}\left|\alpha_3^{n+1,j} - \alpha_3^{n,j}\right|(\|y_n\| + \|U_{n,j-1}y_n\|) \\
&= \left|\alpha_1^{n+1,1} - \alpha_1^{n,1}\right|\|T_1 y_n - y_n\| \\
&\quad + \sum_{j=2}^{N}\left|\alpha_1^{n+1,j} - \alpha_1^{n,j}\right|(\|T_j U_{n,j-1}y_n\| + \|U_{n,j-1}y_n\|) \\
&\quad + \sum_{j=2}^{N}\left|\alpha_3^{n+1,j} - \alpha_3^{n,j}\right|(\|y_n\| + \|U_{n,j-1}y_n\|).
\end{aligned} \tag{3.12}$$

This together with the condition (iv), we obtain

$$\lim_{n\to\infty}\left\|S_{n+1}y_n - S_n y_n\right\| = 0. \tag{3.13}$$

By (3.10), (3.13) and conditions (i), (ii), (iii), (iv), it implies that

$$\limsup_{n\to\infty}(\|k_{n+1} - k_n\| - \|x_{n+1} - x_n\|) \leq 0. \tag{3.14}$$

From Lemma 2.5, (3.9), (3.14) and condition (ii), we have

$$\lim_{n\to\infty}\|x_n - k_n\| = 0. \tag{3.15}$$

From (3.9), we can rewrite

$$x_{n+1} - x_n = (1-\beta_n)(k_n - x_n). \tag{3.16}$$

By (3.15), we have

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.17}$$

On the other hand, we have

$$\begin{aligned}
\left\|x_n - S_n y_n\right\| &\leq \|x_n - x_{n+1}\| + \left\|x_{n+1} - S_n y_n\right\| \\
&= \|x_n - x_{n+1}\| + \left\|\alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n y_n - S_n y_n\right\| \\
&= \|x_n - x_{n+1}\| + \left\|\alpha_n(f(x_n) - S_n y_n) + \beta_n(x_n - S_n y_n)\right\| \\
&= \|x_n - x_{n+1}\| + \alpha_n\left\|f(x_n) - S_n y_n\right\| + \beta_n\left\|x_n - S_n y_n\right\|.
\end{aligned} \tag{3.18}$$

This implies that

$$(1-\beta_n)\left\|x_n - S_n y_n\right\| \leq \|x_n - x_{n+1}\| + \alpha_n\left\|f(x_n) - S_n y_n\right\|. \tag{3.19}$$

By (3.17) and condition (ii), we have

$$\lim_{n\to\infty}\left\|x_n - S_n y_n\right\| = 0. \tag{3.20}$$

Step 3. Let $z \in \mathfrak{F}$; we show that

$$\lim_{n\to\infty}\|Au_n - Az\| = \lim_{n\to\infty}\|Bv_n - Bz\| = \lim_{n\to\infty}\|Ax_n - Az\| = \lim_{n\to\infty}\|Bx_n - Bz\| = 0. \tag{3.21}$$

From definition of y_n, we have

$$\begin{aligned}
\|y_n - z\|^2 &= \|\delta_n(P_C(u_n - \lambda_n Au_n) - P_C(I - \lambda_n A)z) + (1-\delta_n)(P_C(v_n - \eta_n Bv_n) \\
&\quad - P_C(I - \eta_n B)z)\|^2 \\
&\le \delta_n\|(P_C(u_n - \lambda_n Au_n) - P_C(I - \lambda_n A)z)\|^2 \\
&\quad + (1-\delta_n)\|(P_C(v_n - \eta_n Bv_n) - P_C(I - \eta_n B)z)\|^2 \\
&\le \delta_n\|u_n - \lambda_n Au_n - z + \lambda_n Az\|^2 + (1-\delta_n)\|v_n - \eta_n Bv_n - z + \eta_n Bz\|^2 \\
&= \delta_n\|(u_n - z) - \lambda_n(Au_n - Az)\|^2 + (1-\delta_n)\|(v_n - z) - \eta_n(Bv_n - Bz)\|^2 \\
&= \delta_n\Big(\|u_n - z\|^2 + \lambda_n^2\|(Au_n - Az)\|^2 - 2\lambda_n\langle u_n - z, Au_n - Az\rangle\Big) \\
&\quad + (1-\delta_n)\Big(\|v_n - z\|^2 + \eta_n^2\|Bv_n - Bz\|^2 - 2\eta_n\langle v_n - z, Bv_n - Bz\rangle\Big) \\
&\le \delta_n\Big(\|u_n - z\|^2 + \lambda_n^2\|(Au_n - Az)\|^2 - 2\lambda_n\alpha\|Au_n - Az\|^2\Big) \\
&\quad + (1-\delta_n)\Big(\|v_n - z\|^2 + \eta_n^2\|Bv_n - Bz\|^2 - 2\eta_n\beta\|Bv_n - Bz\|^2\Big) \\
&= \delta_n\Big(\|u_n - z\|^2 - \lambda_n(2\alpha - \lambda_n)\|(Au_n - Az)\|^2\Big) \\
&\quad + (1-\delta_n)\Big(\|v_n - z\|^2 - \eta_n(2\beta - \eta_n)\|Bv_n - Bz\|^2\Big)
\end{aligned} \tag{3.22}$$

$$\begin{aligned}
&= \delta_n\Big(\|T_{r_n}(I - r_n A)x_n - T_{r_n}(z - r_n Az)\|^2 - \lambda_n(2\alpha - \lambda_n)\|(Au_n - Az)\|^2\Big) \\
&\quad + (1-\delta_n)\Big(\|T_{s_n}(I - s_n B)x_n - T_{s_n}(z - s_n Bz)\|^2 - \eta_n(2\beta - \eta_n)\|Bv_n - Bz\|^2\Big) \\
&\le \delta_n\Big(\|x_n - z\|^2 - \lambda_n(2\alpha - \lambda_n)\|(Au_n - Az)\|^2\Big) \\
&\quad + (1-\delta_n)\Big(\|x_n - z\|^2 - \eta_n(2\beta - \eta_n)\|Bv_n - Bz\|^2\Big) \\
&= \|x_n - z\|^2 - \lambda_n\delta_n(2\alpha - \lambda_n)\|(Au_n - Az)\|^2 \\
&\quad - \eta_n(1-\delta_n)(2\beta - \eta_n)\|Bv_n - Bz\|^2.
\end{aligned} \tag{3.23}$$

By (3.23), we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \left\|\alpha_n\left(f(x_n) - z\right) + \beta_n(x_n - z) + \gamma_n\left(S_n y_n - z\right)\right\|^2 \\
&\le \alpha_n\left\|f(x_n) - z\right\|^2 + \beta_n\|x_n - z\|^2 + \gamma_n\left\|S_n y_n - z\right\|^2 \\
&\le \alpha_n\left\|f(x_n) - z\right\|^2 + \beta_n\|x_n - z\|^2 + \gamma_n\left\|y_n - z\right\|^2 \\
&\le \alpha_n\left\|f(x_n) - z\right\|^2 + \beta_n\|x_n - z\|^2 + \gamma_n\Big(\|x_n - z\|^2 - \lambda_n\delta_n(2\alpha - \lambda_n)\|(Au_n - Az)\|^2 \\
&\quad -\eta_n(1-\delta_n)(2\beta - \eta_n)\|Bv_n - Bz\|^2\Big) \\
&= \alpha_n\left\|f(x_n) - z\right\|^2 + \beta_n\|x_n - z\|^2 + \gamma_n\|x_n - z\|^2 - \lambda_n\gamma_n\delta_n(2\alpha - \lambda_n)\|(Au_n - Az)\|^2 \\
&\quad - \eta_n\gamma_n(1-\delta_n)(2\beta - \eta_n)\|Bv_n - Bz\|^2 \\
&\le \alpha_n\left\|f(x_n) - z\right\|^2 + \|x_n - z\|^2 - \lambda_n\gamma_n\delta_n(2\alpha - \lambda_n)\|(Au_n - Az)\|^2 \\
&\quad - \eta_n\gamma_n(1-\delta_n)(2\beta - \eta_n)\|Bv_n - Bz\|^2.
\end{aligned} \tag{3.24}$$

By (3.24), we have

$$\begin{aligned}
\lambda_n\gamma_n\delta_n(2\alpha - \lambda_n)\|(Au_n - Az)\|^2 &\le \alpha_n\left\|f(x_n) - z\right\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\
&\quad - \eta_n\gamma_n(1-\delta_n)(2\beta - \eta_n)\|Bv_n - Bz\|^2 \\
&\le \alpha_n\left\|f(x_n) - z\right\|^2 + (\|x_n - z\| + \|x_{n+1} - z\|)\|x_{n+1} - x_n\|.
\end{aligned} \tag{3.25}$$

From conditions (i)–(iii) and (3.17), we have

$$\lim_{n\to\infty}\|Au_n - Az\|^2 = 0. \tag{3.26}$$

By using the same method as (3.26), we have

$$\lim_{n\to\infty}\|Bv_n - Bz\|^2 = 0. \tag{3.27}$$

By nonexpansiveness of $T_{r_n}, T_{s_n}, I-\lambda_n A, I-\eta_n B$ and (3.23), we have

$$\begin{aligned}
\|y_n - z\|^2 &\le \delta_n \|(P_C(u_n-\lambda_n A u_n) - P_C(I-\lambda_n A)z)\|^2 \\
&\quad + (1-\delta_n)\|P_C(v_n-\eta_n B v_n) - P_C(I-\eta_n B)z\|^2 \\
&\le \delta_n\|(I-\lambda_n A)u_n - (I-\lambda_n A)z\|^2 + (1-\delta_n)\|(I-\eta_n B)v_n - (I-\eta_n B)z\|^2 \\
&\le \delta_n\|u_n - z\|^2 + (1-\delta_n)\|v_n - z\|^2 \\
&= \delta_n\|T_{r_n}(I-r_nA)x_n - T_{r_n}(I-r_nA)z\|^2 + (1-\delta_n)\|T_{s_n}(I-s_nB)x_n \\
&\quad - T_{s_n}(I-s_nB)z\|^2 \\
&\le \delta_n\|(I-r_nA)x_n - (I-r_nA)z\|^2 + (1-\delta_n)\|(I-s_nB)x_n - (I-s_nB)z\|^2 \\
&= \delta_n\|x_n - r_nAx_n - z + r_nAz\|^2 + (1-\delta_n)\|x_n - s_nBx_n - z + s_nBz\|^2 \\
&= \delta_n\|(x_n-z) - r_n(Ax_n - Az)\|^2 + (1-\delta_n)\|(x_n-z) - s_n(Bx_n - Bz)\|^2 \\
&= \delta_n\left(\|x_n-z\|^2 + r_n^2\|Ax_n - Az\|^2 - 2r_n\langle x_n - z, Ax_n - Az\rangle\right) \\
&\quad + (1-\delta_n)\left(\|x_n-z\|^2 + s_n^2\|Bx_n - Bz\|^2 - 2s_n\langle x_n - zBx_n - Bz\rangle\right) \\
&= \delta_n\|x_n-z\|^2 + r_n^2\delta_n\|Ax_n - Az\|^2 - 2\delta_n r_n\langle x_n - z, Ax_n - Az\rangle \\
&\quad + (1-\delta_n)\|x_n-z\|^2 + s_n^2(1-\delta_n)\|Bx_n - Bz\|^2 - 2s_n(1-\delta_n)\langle x_n - zBx_n - Bz\rangle \\
&\le \|x_n-z\|^2 + r_n^2\delta_n\|Ax_n - Az\|^2 - 2\delta_n r_n\alpha\|Ax_n - Az\|^2 \\
&\quad + s_n^2(1-\delta_n)\|Bx_n - Bz\|^2 - 2s_n(1-\delta_n)\beta\|Bx_n - Bz\|^2 \\
&= \|x_n-z\|^2 - \delta_n r_n(2\alpha - r_n)\|Ax_n - Az\|^2 - s_n(1-\delta_n)(2\beta - s_n)\|Bx_n - Bz\|^2.
\end{aligned} \tag{3.28}$$

By (3.28), we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\alpha_n(f(x_n) - z) + \beta_n(x_n - z) + \gamma_n(S_n y_n - z)\|^2 \\
&\le \alpha_n\|f(x_n) - z\|^2 + \beta_n\|x_n - z\|^2 + \gamma_n\|S_n y_n - z\|^2 \\
&\le \alpha_n\|f(x_n) - z\|^2 + \beta_n\|x_n - z\|^2 + \gamma_n\|y_n - z\|^2 \\
&\le \alpha_n\|f(x_n) - z\|^2 + \beta_n\|x_n - z\|^2 \\
&\quad + \gamma_n\Big(\|x_n - z\|^2 - \delta_n r_n(2\alpha - r_n)\|Ax_n - Az\|^2 \\
&\qquad - s_n(1-\delta_n)(2\beta - s_n)\|Bx_n - Bz\|^2\Big) \\
&= \alpha_n\|f(x_n) - z\|^2 + \beta_n\|x_n - z\|^2 + \gamma_n\|x_n - z\|^2 - \delta_n\gamma_n r_n(2\alpha - r_n)\|Ax_n - Az\|^2
\end{aligned}$$

$$
\begin{aligned}
&\quad - s_n\gamma_n(1-\delta_n)(2\beta - s_n)\|Bx_n - Bz\|^2 \\
&\leq \alpha_n\left\|f(x_n) - z\right\|^2 + \|x_n - z\|^2 - \delta_n\gamma_n r_n(2\alpha - r_n)\|Ax_n - Az\|^2 \\
&\quad - s_n\gamma_n(1-\delta_n)(2\beta - s_n)\|Bx_n - Bz\|^2.
\end{aligned}
\tag{3.29}
$$

By (3.29), we have

$$
\begin{aligned}
\delta_n\gamma_n r_n(2\alpha - r_n)\|Ax_n - Az\|^2 &\leq \alpha_n\left\|f(x_n) - z\right\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\
&\quad - s_n\gamma_n(1-\delta_n)(2\beta - s_n)\|Bx_n - Bz\|^2 \\
&\leq \alpha_n\left\|f(x_n) - z\right\|^2 + (\|x_n - z\| + \|x_{n+1} - z\|)\|x_{n+1} - x_n\|.
\end{aligned}
\tag{3.30}
$$

From (3.17) and conditions (i)–(iii), we have

$$
\lim_{n\to\infty}\|Ax_n - Az\| = 0. \tag{3.31}
$$

By using the same method as (3.31), we have

$$
\lim_{n\to\infty}\|Bx_n - Bz\| = 0. \tag{3.32}
$$

Step 4. We will show that

$$
\lim_{n\to\infty}\left\|y_n - x_n\right\| = 0. \tag{3.33}
$$

Putting $M_n = P_C(u_n - \lambda_n Au_n)$ and $N_n = P_C(v_n - \eta_n Bv_n)$, we will show that

$$
\lim_{n\to\infty}\|u_n - x_n\| = \lim_{n\to\infty}\|v_n - x_n\| = \lim_{n\to\infty}\|M_n - u_n\| = \lim_{n\to\infty}\|N_n - v_n\| = 0. \tag{3.34}
$$

Let $z \in \mathfrak{F}$; by (3.28), we have

$$
\begin{aligned}
\left\|y_n - z\right\|^2 &\leq \delta_n\|M_n - z\|^2 + (1-\delta_n)\|N_n - z\|^2 \\
&\leq \delta_n\|u_n - z\|^2 + (1-\delta_n)\|v_n - z\|^2.
\end{aligned}
\tag{3.35}
$$

By nonexpansiveness of $I - r_n A$, we have

$$\begin{aligned}
\|u_n - z\|^2 &= \|T_{r_n}(x_n - r_n A x_n) - T_{r_n}(z - r_n A z)\|^2 \\
&\le \langle (x_n - r_n A x_n) - (z - r_n A z), u_n - z \rangle \\
&= \frac{1}{2}\Big(\|(x_n - r_n A x_n) - (z - r_n A z)\|^2 + \|u_n - z\|^2 \\
&\quad - \|(x_n - r_n A x_n) - (z - r_n A z) - (u_n - z)\|^2 \Big) \\
&\le \frac{1}{2}\Big(\|x_n - z\|^2 + \|u_n - z\|^2 - \|(x_n - u_n) - r_n(A x_n - A z)\|^2 \Big) \\
&= \frac{1}{2}\Big(\|x_n - z\|^2 + \|u_n - z\|^2 - \|x_n - u_n\|^2 \\
&\quad + 2 r_n \langle x_n - u_n, A x_n - A z \rangle - r_n^2 \|A x_n - A z\|^2 \Big).
\end{aligned} \tag{3.36}$$

This implies

$$\|u_n - z\|^2 \le \|x_n - z\|^2 - \|x_n - u_n\|^2 + 2 r_n \langle x_n - u_n, A x_n - A z \rangle - r_n^2 \|A x_n - A z\|^2. \tag{3.37}$$

By using the same method as (3.37), we have

$$\|v_n - z\|^2 \le \|x_n - z\|^2 - \|x_n - v_n\|^2 + 2 s_n \langle x_n - v_n, B x_n - B z \rangle - s_n^2 \|B x_n - B z\|^2. \tag{3.38}$$

Substituting (3.37) and (3.38) into (3.35), we have

$$\begin{aligned}
\|y_n - z\|^2 &\le \delta_n \|u_n - z\|^2 + (1 - \delta_n)\|v_n - z\|^2 \\
&\le \delta_n \Big(\|x_n - z\|^2 - \|x_n - u_n\|^2 + 2 r_n \langle x_n - u_n, A x_n - A z \rangle - r_n^2 \|A x_n - A z\|^2 \Big) \\
&\quad + (1 - \delta_n)\Big(\|x_n - z\|^2 - \|x_n - v_n\|^2 + 2 s_n \langle x_n - v_n, B x_n - B z \rangle - s_n^2 \|B x_n - B z\|^2 \Big) \\
&\le \delta_n \|x_n - z\|^2 - \delta_n \|x_n - u_n\|^2 + 2 \delta_n r_n \|x_n - u_n\| \|A x_n - A z\| + (1 - \delta_n)\|x_n - z\|^2 \\
&\quad - (1 - \delta_n)\|x_n - v_n\|^2 + 2 s_n (1 - \delta_n)\|x_n - v_n\| \|B x_n - B z\| \\
&= \|x_n - z\|^2 - \delta_n \|x_n - u_n\|^2 + 2 \delta_n r_n \|x_n - u_n\| \|A x_n - A z\| - (1 - \delta_n)\|x_n - v_n\|^2 \\
&\quad + 2 s_n (1 - \delta_n)\|x_n - v_n\| \|B x_n - B z\|.
\end{aligned} \tag{3.39}$$

By (3.39), we have

$$\begin{aligned}
\|x_{n+1}-z\|^2 &\le \alpha_n\left\|f(x_n)-z\right\|^2+\beta_n\|x_n-z\|^2+\gamma_n\left\|y_n-z\right\|^2\\
&\le \alpha_n\left\|f(x_n)-z\right\|^2+\beta_n\|x_n-z\|^2\\
&\quad+\gamma_n\Big(\|x_n-z\|^2-\delta_n\|x_n-u_n\|^2\\
&\qquad+2\delta_n r_n\|x_n-u_n\|\|Ax_n-Az\|-(1-\delta_n)\|x_n-v_n\|^2\\
&\qquad+2s_n(1-\delta_n)\|x_n-v_n\|\|Bx_n-Bz\|\Big)\\
&=\alpha_n\left\|f(x_n)-z\right\|^2+\beta_n\|x_n-z\|^2+\gamma_n\|x_n-z\|^2-\gamma_n\delta_n\|x_n-u_n\|^2\\
&\quad+2\gamma_n\delta_n r_n\|x_n-u_n\|\|Ax_n-Az\|-(1-\delta_n)\gamma_n\|x_n-v_n\|^2\\
&\quad+2s_n\gamma_n(1-\delta_n)\|x_n-v_n\|\|Bx_n-Bz\|\\
&\le \alpha_n\left\|f(x_n)-z\right\|^2+\|x_n-z\|^2-\gamma_n\delta_n\|x_n-u_n\|^2\\
&\quad+2\gamma_n\delta_n r_n\|x_n-u_n\|\|Ax_n-Az\|-(1-\delta_n)\gamma_n\|x_n-v_n\|^2\\
&\quad+2s_n\gamma_n(1-\delta_n)\|x_n-v_n\|\|Bx_n-Bz\|.
\end{aligned}\tag{3.40}$$

It follows that

$$\begin{aligned}
\gamma_n\delta_n\|x_n-u_n\|^2 &\le \alpha_n\left\|f(x_n)-z\right\|^2+\|x_n-z\|^2-\|x_{n+1}-z\|^2\\
&\quad+2\gamma_n\delta_n r_n\|x_n-u_n\|\|Ax_n-Az\|-(1-\delta_n)\gamma_n\|x_n-v_n\|^2\\
&\quad+2s_n\gamma_n(1-\delta_n)\|x_n-v_n\|\|Bx_n-Bz\|\\
&\le \alpha_n\left\|f(x_n)-z\right\|^2+(\|x_n-z\|+\|x_{n+1}-z\|)\|x_{n+1}-x_n\|\\
&\quad+2\gamma_n\delta_n r_n\|x_n-u_n\|\|Ax_n-Az\|+2s_n\gamma_n(1-\delta_n)\|x_n-v_n\|\|Bx_n-Bz\|.
\end{aligned}\tag{3.41}$$

By conditions (i)–(iii), (3.41), (3.31), (3.32), and (3.17), we have

$$\lim_{n\to\infty}\|x_n-u_n\|=0.\tag{3.42}$$

By using the same method as (3.42), we have

$$\lim_{n\to\infty}\|x_n-v_n\|=0.\tag{3.43}$$

By nonexpansiveness of $T_{r_n}(I - r_n A)$, we have

$$\begin{aligned}
\|M_n - z\|^2 &= \|P_C(u_n - \lambda_n Au_n) - P_C(z - \lambda_n Az)\|^2 \\
&\leq \langle (u_n - \alpha_n Au_n) - (z - \alpha_n Az), M_n - z\rangle \\
&= \frac{1}{2}\Big(\|(u_n - \alpha_n Au_n) - (z - \alpha_n Az)\|^2 + \|M_n - z\|^2 - \|(u_n - \alpha_n Au_n) \\
&\qquad -(z - \alpha_n Az) - (M_n - z)\|^2\Big) \\
&\leq \frac{1}{2}\Big(\|u_n - z\|^2 + \|M_n - z\|^2 - \|(u_n - M_n) - \alpha_n(Au_n - Az)\|^2\Big) \\
&= \frac{1}{2}\Big(\|T_{r_n}(I - r_n A)x_n - T_{r_n}(I - r_n A)z\|^2 + \|M_n - z\|^2 - \|u_n - M_n\|^2 \\
&\qquad +2\alpha_n\langle u_n - M_n, Au_n - Az\rangle - \alpha_n^2\|Au_n - Az\|^2\Big) \\
&\leq \frac{1}{2}\Big(\|x_n - z\|^2 + \|M_n - z\|^2 - \|u_n - M_n\|^2 + 2\alpha_n\langle u_n - M_n, Au_n - Az\rangle \\
&\qquad -\alpha_n^2\|Au_n - Az\|^2\Big).
\end{aligned} \tag{3.44}$$

Hence, we have

$$\begin{aligned}
\|M_n - z\|^2 \leq \|x_n - z\|^2 - \|u_n - M_n\|^2 + 2\alpha_n\langle u_n - M_n, Au_n - Az\rangle \\
- \alpha_n^2\|Au_n - Az\|^2.
\end{aligned} \tag{3.45}$$

By using the same method as (3.45), we have

$$\|N_n - z\|^2 \leq \|x_n - z\|^2 - \|v_n - N_n\|^2 + 2\eta_n\langle v_n - N_n, Bv_n - Bz\rangle - \eta_n^2\|Bv_n - Bz\|^2. \tag{3.46}$$

Substituting (3.45) and (3.46) into (3.35), we have

$$\begin{aligned}
\|y_n - z\|^2 &\leq \delta_n\|M_n - z\|^2 + (1 - \delta_n)\|N_n - z\|^2 \\
&\leq \delta_n\Big(\|x_n - z\|^2 - \|u_n - M_n\|^2 + 2\alpha_n\langle u_n - M_n, Au_n - Az\rangle - \alpha_n^2\|Au_n - Az\|^2\Big) \\
&\quad + (1 - \delta_n)\Big(\|x_n - z\|^2 - \|v_n - N_n\|^2 + 2\eta_n\langle v_n - N_n, Bv_n - Bz\rangle - \eta_n^2\|Bv_n - Bz\|^2\Big) \\
&\leq \delta_n\|x_n - z\|^2 - \delta_n\|u_n - M_n\|^2 + 2\delta_n\alpha_n\|u_n - M_n\|\|Au_n - Az\| \\
&\quad + (1 - \delta_n)\|x_n - z\|^2 - (1 - \delta_n)\|v_n - N_n\|^2 + 2(1 - \delta_n)\eta_n\|v_n - N_n\|\|Bv_n - Bz\| \\
&= \|x_n - z\|^2 - \delta_n\|u_n - M_n\|^2 + 2\delta_n\alpha_n\|u_n - M_n\|\|Au_n - Az\| - (1 - \delta_n)\|v_n - N_n\|^2 \\
&\quad + 2(1 - \delta_n)\eta_n\|v_n - N_n\|\|Bv_n - Bz\|.
\end{aligned} \tag{3.47}$$

By (3.47), we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\le \alpha_n \left\| f(x_n) - z \right\|^2 + \beta_n \|x_n - z\|^2 + \gamma_n \left\| y_n - z \right\|^2 \\
&\le \alpha_n \left\| f(x_n) - z \right\|^2 + \beta_n \|x_n - z\|^2 \\
&\quad + \gamma_n \Big(\|x_n - z\|^2 - \delta_n \|u_n - M_n\|^2 \\
&\qquad + 2\delta_n \alpha_n \|u_n - M_n\| \|Au_n - Az\| - (1-\delta_n) \|v_n - N_n\|^2 \\
&\qquad + 2(1-\delta_n) \eta_n \|v_n - N_n\| \|Bv_n - Bz\| \Big) \\
&= \alpha_n \left\| f(x_n) - z \right\|^2 + \beta_n \|x_n - z\|^2 + \gamma_n \|x_n - z\|^2 - \delta_n \gamma_n \|u_n - M_n\|^2 \\
&\quad + 2\delta_n \gamma_n \alpha_n \|u_n - M_n\| \|Au_n - Az\| - (1-\delta_n) \gamma_n \|v_n - N_n\|^2 \\
&\quad + 2(1-\delta_n) \gamma_n \eta_n \|v_n - N_n\| \|Bv_n - Bz\| \\
&\le \alpha_n \left\| f(x_n) - z \right\|^2 + \|x_n - z\|^2 - \delta_n \gamma_n \|u_n - M_n\|^2 \\
&\quad + 2\delta_n \gamma_n \alpha_n \|u_n - M_n\| \|Au_n - Az\| - (1-\delta_n) \gamma_n \|v_n - N_n\|^2 \\
&\quad + 2(1-\delta_n) \gamma_n \eta_n \|v_n - N_n\| \|Bv_n - Bz\|.
\end{aligned} \tag{3.48}$$

It follows that

$$\begin{aligned}
\delta_n \gamma_n \|u_n - M_n\|^2 &\le \alpha_n \left\| f(x_n) - z \right\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\
&\quad + 2\delta_n \gamma_n \alpha_n \|u_n - M_n\| \|Au_n - Az\| - (1-\delta_n) \gamma_n \|v_n - N_n\|^2 \\
&\quad + 2(1-\delta_n) \gamma_n \eta_n \|v_n - N_n\| \|Bv_n - Bz\| \\
&\le \alpha_n \left\| f(x_n) - z \right\|^2 + (\|x_n - z\| + \|x_{n+1} - z\|) \|x_{n+1} - x_n\| \\
&\quad + 2\delta_n \gamma_n \alpha_n \|u_n - M_n\| \|Au_n - Az\| + 2(1-\delta_n) \gamma_n \eta_n \|v_n - N_n\| \|Bv_n - Bz\|.
\end{aligned} \tag{3.49}$$

From (3.17), (3.26), (3.27), and conditions (i)–(iii), we have

$$\lim_{n \to \infty} \|u_n - M_n\| = 0. \tag{3.50}$$

By using the same method as (3.50), we have

$$\lim_{n \to \infty} \|v_n - N_n\| = 0. \tag{3.51}$$

By (3.42) and (3.50), we have

$$\lim_{n\to\infty} \|M_n - x_n\| = 0. \tag{3.52}$$

By (3.43) and (3.51), we have

$$\lim_{n\to\infty} \|N_n - x_n\| = 0. \tag{3.53}$$

Since $M_n = P_C(u_n - \lambda_n A u_n)$ and $N_n = P_C(v_n - \eta_n B v_n)$, we have

$$y_n - x_n = \delta_n(M_n - x_n) + (1-\delta_n)(N_n - x_n). \tag{3.54}$$

By (3.52) and (3.53), we obtain

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.55}$$

Note that

$$\begin{aligned} \|x_n - S_n x_n\| &\le \|x_n - S_n y_n\| + \|S_n y_n - S_n x_n\| \\ &\le \|x_n - S_n y_n\| + \|y_n - x_n\|. \end{aligned} \tag{3.56}$$

From (3.20) and (3.55), we have

$$\lim_{n\to\infty} \|x_n - S_n x_n\| = 0. \tag{3.57}$$

Step 5. We will show that

$$\limsup_{n\to\infty} \langle f(z) - z, x_n - z\rangle \le 0, \tag{3.58}$$

where $z = P_{\mathfrak{F}} f(z)$. To show this inequality, take subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle f(z) - z, x_n - z\rangle = \limsup_{i\to\infty} \langle f(z) - z, x_{n_i} - z\rangle. \tag{3.59}$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to q. Without loss of generality, we can assume that $x_{n_i} \rightharpoonup q$. Since C is closed convex, C is weakly closed. So, we have $q \in C$. Let us show that $q \in \mathfrak{F} = \bigcap_{i=1}^{N} F(T_i) \cap \mathrm{EP}(F, A) \cap \mathrm{EP}(G, B) \cap F(G_1) \cap F(G_2)$. We first show that $q \in \mathrm{EP}(F, A) \cap \mathrm{EP}(G, B) \cap F(G_1) \cap F(G_2)$. From (3.42), we have $u_{n_i} \rightharpoonup q$. Since $u_n = T_{r_n}(I - r_n A)x_n$, for any $y \in C$, we have

$$F(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0. \tag{3.60}$$

From (A2), we have

$$\langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_n). \tag{3.61}$$

This implies that

$$\langle Ax_{n_i}, y - u_{n_i}\rangle + \frac{1}{r_{n_i}}\langle y - u_{n_i}, u_{n_i} - x_{n_i}\rangle \geq F(y, u_{n_i}). \tag{3.62}$$

Put $z_t = ty + (1-t)q$ for all $t \in (0,1]$ and $y \in C$. Then, we have $z_t \in C$. So, from (3.62), we have

$$\begin{aligned}\langle z_t - u_{n_i}, Az_t\rangle &\geq \langle z_t - u_{n_i}, Az_t\rangle - \langle z_t - u_{n_i}, Ax_{n_i}\rangle - \left\langle z_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle + F(z_t, u_{n_i}) \\ &= \langle z_t - u_{n_i}, Az_t - Au_{n_i}\rangle + \langle z_t - u_{n_i}, Au_{n_i} - Ax_{n_i}\rangle \\ &\quad - \left\langle z_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle + F(z_t, u_{n_i}).\end{aligned} \tag{3.63}$$

Since $\|u_{n_i} - x_{n_i}\| \to 0$, we have $\|Au_{n_i} - Ax_{n_i}\| \to 0$. Further, from monotonicity of A, we have $\langle z_t - u_{n_i}, Az_t - Au_{n_i}\rangle \geq 0$. So, from (A4), we have

$$\langle z_t - q, Az_t\rangle \geq F(z_t, q) \quad \text{as } i \longrightarrow \infty. \tag{3.64}$$

From (A1), (A4), and (3.64), we also have

$$\begin{aligned}0 = F(z_t, z_t) &\leq tF(z_t, y) + (1-t)F(z_t, q) \\ &\leq tF(z_t, y) + (1-t)\langle z_t - q, Az_t\rangle \\ &= tF(z_t, y) + (1-t)t\langle y - q, Az_t\rangle.\end{aligned} \tag{3.65}$$

Thus

$$0 \leq F(z_t, y) + (1-t)\langle y - q, Az_t\rangle. \tag{3.66}$$

Letting $t \to 0$, we have, for each $y \in C$,

$$0 \leq F(q, y) + \langle y - q, Aq\rangle. \tag{3.67}$$

This implies that

$$q \in \mathrm{EP}(F, A). \tag{3.68}$$

From (3.43), we have $v_{ni} \rightharpoonup q$. Since $v_n = T_{s_n}(I - s_n B)x_n$, for any $y \in C$, we have

$$G(v_n, y) + \langle Bx_n, y - v_n \rangle + \frac{1}{s_n}\langle y - v_n, v_n - x_n \rangle \geq 0. \tag{3.69}$$

From (A2), we have

$$\langle Bx_n, y - v_n \rangle + \frac{1}{s_n}\langle y - v_n, v_n - x_n \rangle \geq G(y, v_n). \tag{3.70}$$

This implies that

$$\langle Bx_{n_i}, y - v_{n_i} \rangle + \frac{1}{s_{n_i}}\langle y - v_{n_i}, v_{n_i} - x_{n_i} \rangle \geq G(y, v_{n_i}). \tag{3.71}$$

Put $z_t = ty + (1-t)q$ for all $t \in (0,1]$ and $y \in C$. Then, we have $z_t \in C$. So, from (3.71) we have

$$\begin{aligned}\langle z_t - v_{n_i}, Bz_t \rangle &\geq \langle z_t - v_{n_i}, Bz_t \rangle - \langle z_t - v_{n_i}, Bx_{n_i} \rangle - \left\langle z_t - v_{n_i}, \frac{v_{n_i} - x_{n_i}}{s_{n_i}} \right\rangle + G(z_t, v_{n_i}) \\ &= \langle z_t - v_{n_i}, Bz_t - Bv_{n_i} \rangle + \langle z_t - v_{n_i}, Bv_{n_i} - Bx_{n_i} \rangle - \left\langle z_t - v_{n_i}, \frac{v_{n_i} - x_{n_i}}{s_{n_i}} \right\rangle \\ &\quad + G(z_t, v_{n_i}).\end{aligned} \tag{3.72}$$

Since $\|v_{n_i} - x_{n_i}\| \to 0$, we have $\|Bv_{n_i} - Bx_{n_i}\| \to 0$. Further, from monotonicity of B, we have $\langle z_t - v_{n_i}, Bz_t - Bv_{n_i} \rangle \geq 0$. So, from (A4), we have

$$\langle z_t - q, Bz_t \rangle \geq G(z_t, q). \tag{3.73}$$

From (A1), (A4), and (3.64), we also have

$$\begin{aligned}0 = G(z_t, z_t) &\leq tG(z_t, y) + (1-t)G(z_t, q) \\ &\leq tG(z_t, y) + (1-t)\langle z_t - q, Bz_t \rangle \\ &= tG(z_t, y) + (1-t)t\langle y - q, Bz_t \rangle,\end{aligned} \tag{3.74}$$

hence

$$0 \leq G(z_t, y) + (1-t)\langle y - q, Bz_t \rangle. \tag{3.75}$$

Letting $t \to 0$, we have, for each $y \in C$,

$$0 \leq G(q, y) + \langle y - q, Bq \rangle. \tag{3.76}$$

This implies that

$$q \in \mathrm{EP}(G, B). \tag{3.77}$$

Define a mapping $Q : C \to C$ by

$$Qx = \delta P_C(I - \lambda_n A)x + (1 - \delta)P_C\left(I - \eta_n B\right)x, \quad \forall x \in C, \tag{3.78}$$

where $\lim_{n\to\infty}\delta_n = \delta \in (0,1)$. From Lemma 2.3, we have that Q is nonexpansive with

$$F(Q) = F(P_C(I - \lambda_n A)) \bigcap F\left(P_C\left(I - \eta_n B\right)\right). \tag{3.79}$$

Next, we show that

$$\lim_{n\to n}\|x_n - Qx_n\| = 0. \tag{3.80}$$

By nonexpansiveness of $I - \eta_n B$ and $I - \lambda_n A$, we have

$$\begin{aligned}
\|x_n - Qx_n\| &\le \left\|x_n - y_n\right\| + \left\|y_n - Qx_n\right\| \\
&= \left\|x_n - y_n\right\| + \left\|\delta_n P_C(u_n - \lambda_n A u_n) + (1 - \delta_n)P_C\left(v_n - \eta_n B v_n\right) - \delta P_C(I - \lambda_n A)x_n \right.\\
&\qquad\qquad \left. -(1 - \delta)P_C\left(I - \eta_n B\right)x_n\right\| \\
&= \left\|x_n - y_n\right\| + \|\delta_n P_C(I - \lambda_n A)u_n - \delta_n P_C(I - \lambda_n A)x_n + \delta_n P_C(I - \lambda_n A)x_n \\
&\qquad + (1 - \delta_n)P_C\left(I - \eta_n B\right)v_n - (1 - \delta_n)P_C\left(I - \eta_n B\right)x_n \\
&\qquad +(1 - \delta_n)P_C\left(I - \eta_n B\right)x_n - \delta P_C(I - \lambda_n A)x_n - (1 - \delta)P_C\left(I - \eta_n B\right)x_n\right\| \\
&= \left\|x_n - y_n\right\| + \|\delta_n(P_C(I - \lambda_n A)u_n - P_C(I - \lambda_n A)x_n) + (\delta_n - \delta)P_C(I - \lambda_n A)x_n \\
&\qquad + (1 - \delta_n)\left(P_C\left(I - \eta_n B\right)v_n - P_C\left(I - \eta_n B\right)x_n\right) \\
&\qquad +(\delta - \delta_n)P_C\left(I - \eta_n B\right)x_n\right\| \\
&\le \left\|x_n - y_n\right\| + \delta_n\|P_C(I - \lambda_n A)u_n - P_C(I - \lambda_n A)x_n\| + |\delta_n - \delta|\|P_C(I - \lambda_n A)x_n\| \\
&\quad + (1 - \delta_n)\left\|P_C\left(I - \eta_n B\right)v_n - P_C\left(I - \eta_n B\right)x_n\right\| + |\delta_n - \delta|\left\|P_C\left(I - \eta_n B\right)x_n\right\| \\
&\le \left\|x_n - y_n\right\| + \delta_n\|u_n - x_n\| + |\delta_n - \delta|\|P_C(I - \lambda_n A)x_n\| + (1 - \delta_n)\|v_n - x_n\| \\
&\quad + |\delta_n - \delta|\left\|P_C\left(I - \eta_n B\right)x_n\right\| \\
&\le \left\|x_n - y_n\right\| + \delta_n\|u_n - x_n\| + 2|\delta_n - \delta|M_1 + (1 - \delta_n)\|v_n - x_n\|,
\end{aligned} \tag{3.81}$$

where $M_1 = \sup_{n\geq 0}\{\|P_C(I-\lambda_n A)x_n\| + \|P_C(I-\eta_n B)x_n\|\}$. From (3.17), (3.42), (3.43), (3.55), and condition (iii), we have $\lim_{n\to n}\|x_n - Qx_n\| = 0$. Since $x_{n_i} \rightharpoonup q$, it follows from (3.80) that, $\lim_{i\to\infty}\|x_{n_i} - Qx_{n_i}\| = 0$. By Lemma 2.4, we obtain that

$$q \in F(Q) = F(P_C(I-\lambda_n A)) \cap F\big(P_C\big(I-\eta_n B\big)\big) = F(G_1)\cap F(G_2). \tag{3.82}$$

Assume that $q \neq Sq$. Using Opial s' property, (3.57) and Lemma 2.10 we have

$$\begin{aligned}\liminf_{i\to\infty}\|x_{n_i} - q\| &< \liminf_{i\to\infty}\|x_{n_i} - Sq\| \\ &\leq \liminf_{i\to\infty}\big(\|x_{n_i} - S_{n_i}x_{n_i}\| + \big\|S_{n_i}x_{n_i} - S_{n_i}q\big\| + \big\|S_{n_i}q - Sq\big\|\big) \\ &\leq \liminf_{i\to\infty}\|x_{n_i} - q\|.\end{aligned} \tag{3.83}$$

This is a contradiction, so we have

$$q \in \bigcap_{i=1}^{N} F(T_i) = F(S). \tag{3.84}$$

From (3.68), (3.77) (3.82), and (3.84), we have $q \in \mathfrak{F}$. Since $P_{\mathfrak{F}}f$ is contraction with the coefficient $\theta \in (0,1)$, $P_{\mathfrak{F}}$ has a unique fixed point. Let z be a fixed point of $P_{\mathfrak{F}}f$, that is $z = P_{\mathfrak{F}}f(z)$. Since $x_{n_i} \rightharpoonup q$ and $q \in \mathfrak{F}$, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle f(z) - z, x_n - z\rangle &= \limsup_{i\to\infty}\langle f(z) - z, x_{n_i} - z\rangle \\ &= \langle f(z) - z, q - z\rangle \leq 0.\end{aligned} \tag{3.85}$$

Step 6. Finally, we will show that $x_n \to z$ as $n \to \infty$. By nonexpansiveness of $T_{r_n}, T_{s_n}, I - \lambda_n A, I - \eta_n B, I - r_n A, I - s_n B$, we can show that $\|y_n - z\| \leq \|x_n - z\|$. Then

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \langle \alpha_n\big(f(x_n) - z\big) + \beta_n(x_n - z) + \gamma_n\big(S_n y_n - z\big), x_{n+1} - z\rangle \\ &= \alpha_n\langle f(x_n) - z, x_{n+1} - z\rangle + \beta_n\langle x_n - z, x_{n+1} - z\rangle + \gamma_n\langle S_n y_n - z, x_{n+1} - z\rangle \\ &\leq \alpha_n\langle f(x_n) - f(z), x_{n+1} - z\rangle + \alpha_n\langle f(z) - z, x_{n+1} - z\rangle + \beta_n\|x_n - z\|\|x_{n+1} - z\| \\ &\quad + \gamma_n\big\|S_n y_n - z\ \|x_{n+1} - z\| \\ &\leq \alpha_n\big\|f(x_n) - f(z)\big\|\|x_{n+1} - z\| + \alpha_n\langle f(z) - z, x_{n+1} - z\rangle + \beta_n\|x_n - z\|\|x_{n+1} - z\| \\ &\quad + \gamma_n\big\|y_n - z\big\|\|x_{n+1} - z\|\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha_n\theta\|x_n - z\|\|x_{n+1} - z\| + \alpha_n\langle f(z) - z, x_{n+1} - z\rangle + \beta_n\|x_n - z\|\|x_{n+1} - z\| \\
&\quad + \gamma_n\|x_n - z\|\|x_{n+1} - z\| \\
&= (1 - \alpha_n(1 - \theta))\|x_n - z\|\|x_{n+1} - z\| + \alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\
&\leq (1 - \alpha_n(1 - \theta))\left(\frac{\|x_n - z\|^2 + \|x_{n+1} - z\|^2}{2}\right) + \alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\
&\leq \frac{(1 - \alpha_n(1 - \theta))}{2}\|x_n - z\|^2 + \frac{\|x_{n+1} - z\|^2}{2} + \alpha_n\langle f(z) - z, x_{n+1} - z\rangle;
\end{aligned}
\tag{3.86}
$$

we have

$$\|x_{n+1} - z\|^2 \leq (1 - \alpha_n(1 - \theta))\|x_n - z\|^2 + 2\alpha_n\langle f(z) - z, x_{n+1} - z\rangle. \tag{3.87}$$

By Step 5, (3.87), and Lemma 2.2, we have $\lim_{n\to\infty} x_n = z$, where $z = P_{\mathbb{F}} f(z)$. It easy to see that sequences $\{y_n\}$, $\{u_n\}$, and $\{v_n\}$ converge strongly to $z = P_{\mathfrak{F}} f(z)$. □

4. Application

Using our main theorem (Theorem 3.1), we obtain the following strong convergence theorems involving finite family of κ-strict pseudocontractions.

To prove strong convergence theorem in this section, we need definition and lemma as follows.

Definition 4.1. A mapping $T : C \to C$ is said to be a κ-strongly pseudo contraction mapping, if there exist $\kappa \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa\|(I - T)x - (I - T)y\|^2, \quad \forall\, x, y \in C. \tag{4.1}$$

Lemma 4.2 (see [20]). *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ a κ-strict pseudo contraction. Define $S : C \to C$ by $Sx = \alpha x + (1 - \alpha)Tx$ for each $x \in C$. Then, as $\alpha \in [\kappa, 1)$ S is nonexpansive such that $F(S) = F(T)$.*

Theorem 4.3. *Let C be a nonempty closed convex subset of a Hilbert space H. Let F and G be two bifunctions from $C \times C$ into $\mathbb{R}$ satisfying (A1)–(A4), respectively. Let $A : C \to H$ is a α-inverse strongly monotone mapping and $B : C \to H$ be a β-inverse strongly monotone mapping. Let $\{T_i\}_{i=1}^N$ be a finite family of κ_i-psuedo contractions with $\mathfrak{F} = \bigcap_{i=1}^N F(T_i) \cap \mathrm{EP}(F, A) \cap EP(G, B) \cap F(G_1) \cap F(G_2) \neq \emptyset$, where G_1, $G_2 : C \to C$ are defined by $G_1(x) = P_C(x - \lambda_n Ax)$, $G_2(x) = P_C(x - \eta_n Bx)$, for all $x \in C$. Define a mapping T_{κ_i} by $T_{\kappa_i} = \kappa_i x + (1 - \kappa_i)T_i x$, for all $x \in C, i \in \{1, 2, \dots, N\}$. Let $f : C \to C$ be a contraction with the coefficient $\theta \in (0, 1)$. Let S_n be the S-mappings generated by $T_{\kappa_1}, T_{\kappa_2}, \dots, T_{\kappa_N}$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \dots, \alpha_N^{(n)}$, where $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j}) \in I \times I \times I$, $I = [0, 1]$,*

$\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$ *and* $0 < \eta_1 \le \alpha_1^{n,j} \le \theta_1 < 1$ *for all* $n \in \mathbb{N}$, *for all* $j = 1, 2, \ldots, N-1$, $0 < \eta_N \le \alpha_1^{n,N} \le 1$ *and* $0 \le \alpha_2^{n,j}$, $\alpha_3^{n,j} \le \theta_3 < 1$ *for all* $n \in \mathbb{N}$, *for all* $j = 1, 2, \ldots, N$. *Let* $\{x_n\}, \{u_n\}, \{v_n\}, \{y_n\}$ *be sequences generated by* $x_1, u, v \in C$

$$\begin{aligned} &F(u_n, u) + \langle Ax_n, u - u_n\rangle + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \ge 0, \\ &G(v_n, v) + \langle Bx_n, v - v_n\rangle + \frac{1}{s_n}\langle v - v_n, v_n - x_n\rangle \ge 0, \\ &y_n = \delta_n P_C(u_n - \lambda_n A u_n) + (1-\delta_n) P_C(v_n - \eta_n B v_n), \\ &x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n y_n, \quad \forall n \ge 1, \end{aligned} \tag{4.2}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \in (0,1)$ *such that* $\alpha_n + \beta_n + \gamma_n = 1$, $r_n \in [a,b] \subset (0, 2\alpha)$, $s_n \in [c,d] \subset (0, 2\beta)$, $\lambda_n \in [e,f] \subset (0,2\alpha)$, $\eta_n \in [g,h] \subset (0,2\beta)$. *Assume that*

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\Sigma_{n=0}^{\infty}\alpha_n = \infty$,

(ii) $\liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$,

(iii) $\lim_{n\to\infty}\delta_n = \delta \in (0,1)$,

(iv) $\Sigma_{n=0}^{\infty}|s_{n+1} - s_n|$, $\Sigma_{n=0}^{\infty}|r_{n+1} - r_n|$, $\Sigma_{n=0}^{\infty}|\lambda_{n+1} - \lambda_n|$, $\Sigma_{n=0}^{\infty}|\eta_{n+1} - \eta_n|$, $\Sigma_{n=0}^{\infty}|\alpha_{n+1} - \alpha_n|$, $\Sigma_{n=0}^{\infty}|\beta_{n+1} - \beta_n| < \infty$,

(v) $|\alpha_1^{n+1,j} - \alpha_1^{n,j}| \to 0$ *and* $|\alpha_3^{n+1,j} - \alpha_3^{n,j}| \to 0$ *as* $n \to \infty$, *for all* $j \in \{1, 2, 3, \ldots, N\}$.

Then the sequence $\{x_n\}, \{y_n\}, \{u_n\}, \{v_n\}$ *converges strongly to* $z = P_{\mathfrak{F}} f(z)$, *and* z *is solution of*

$$\begin{aligned} &\langle Ax^*, \ x - x^*\rangle \ge 0, \\ &\langle Bx^*, \ x - x^*\rangle \ge 0. \end{aligned} \tag{4.3}$$

Proof. For every $i \in \{1, 2, \ldots, N\}$, by Lemma 4.2, we have T_{κ_i} is nonexpansive mappings. From Theorem 3.1, we can concluded the desired conclusion. □

Theorem 4.4. *Let* C *be a nonempty closed convex subset of a Hilbert space* H. *Let* F *and* G *be two bifunctions from* $C \times C$ *into* $\mathbb{R}$ *satisfying (A1)–(A4), respectively. Let* $A : C \to H$ *be a* α*-inverse strongly monotone mapping. Let* $\{T_i\}_{i=1}^N$ *be a finite family of* κ_i*-strict pseudo contractions with* $\mathfrak{F} = \bigcap_{i=1}^N F(T_i) \cap \mathrm{EP}(F, A) \cap F(G_1) \ne \emptyset$, *where* $G_1 : C \to C$ *defined by* $G_1(x) = P_C(x - \lambda_n Ax)$, *for all* $x \in C$. *Define a mapping* T_{κ_i} *by* $T_{\kappa_i} = \kappa_i x + (1-\kappa_i)T_i x$, *for all* $x \in C, i \in \mathbb{N}$. *Let* $f : C \to C$ *a contraction with the coefficient* $\theta \in (0,1)$. *Let* S_n *be the S-mappings generated by* $T_{\kappa_1}, T_{\kappa_2}, \ldots, T_{\kappa_N}$ *and* $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$, *where* $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j}) \in I \times I \times I$, $I = [0,1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$ *and* $0 < \eta_1 \le \alpha_1^{n,j} \le \theta_1 < 1$ *for all* $n \in \mathbb{N}$, *for all* $j = 1, 2, \ldots, N-1$, $0 < \eta_N \le \alpha_1^{n,N} \le 1$ *and*

$0 \leq \alpha_2^{n,j}, \alpha_3^{n,j} \leq \theta_3 < 1$ *for all* $n \in \mathbb{N}$, *for all* $j = 1, 2, \ldots, N$. *Let* $\{x_n\}, \{u_n\}, \{y_n\}$ *be sequences generated by* $x_1, u, \in C$

$$F(u_n, u) + \langle Ax_n, u - u_n \rangle + \frac{1}{r_n}\langle u - u_n, u_n - x_n \rangle \geq 0,$$
$$y_n = P_C(u_n - \lambda_n A u_n), \tag{4.4}$$
$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n y_n, \quad \forall n \geq 1,$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \in (0,1)$ *such that* $\alpha_n + \beta_n + \gamma_n = 1$, $r_n \in [a, b] \subset (0, 2\alpha)$, $\lambda_n \in [e, f] \subset (0, 2\alpha)$. *Assume that*

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\Sigma_{n=0}^{\infty}\alpha_n = \infty$,

(ii) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,

(iii) $\Sigma_{n=0}^{\infty}|r_{n+1} - r_n|$, $\Sigma_{n=0}^{\infty}|\lambda_{n+1} - \lambda_n|$, $\Sigma_{n=0}^{\infty}|\alpha_{n+1} - \alpha_n|$, $\Sigma_{n=0}^{\infty}|\beta_{n+1} - \beta_n| < \infty$,

(iv) $|\alpha_1^{n+1,j} - \alpha_1^{n,j}| \to 0$ *and* $|\alpha_3^{n+1,j} - \alpha_3^{n,j}| \to 0$ *as* $n \to \infty$, *for all* $j \in \{1, 2, 3, \ldots, N\}$.

Then the sequence $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ *converges strongly to* $z = P_{\mathfrak{F}} f(z)$, *and* z *is solution of*

$$\langle Ax^*, x - x^* \rangle \geq 0. \tag{4.5}$$

Proof. For every $i \in \{1, 2, \ldots, N\}$, by Lemma 4.2, we have that T_{κ_i} is nonexpansive mappings, putting $F \equiv G$, $A \equiv B$, $s_n = r_n$, $\lambda_n = \eta_n$, and $u_n = v_n$. From Theorem 3.1, we can conclude the desired conclusion. □

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References

[1] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, vol. 28 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, UK, 1990.

[2] E. Blum and W. Oettli, "From optimization and variational inequalities to equilibrium problems," *The Mathematics Student*, vol. 63, no. 1–4, pp. 123–145, 1994.

[3] P. L. Combettes and S. A. Hirstoaga, "Equilibrium programming in Hilbert spaces," *Journal of Nonlinear and Convex Analysis*, vol. 6, no. 1, pp. 117–136, 2005.

[4] A. Moudafi and M. Théra, "Proximal and dynamical approaches to equilibrium problems," in *Ill-Posed Variational Problems and Regularization Techniques*, vol. 477 of *Lecture Notes in Econom. and Math. Systems*, pp. 187–201, Springer, Berlin, Germany, 1999.

[5] H. Iiduka and W. Takahashi, "Weak convergence theorems by Cesáro means for nonexpansive mappings and inverse-strongly-monotone mappings," *Journal of Nonlinear and Convex Analysis*, vol. 7, no. 1, pp. 105–113, 2006.

[6] H. H. Bauschke and J. M. Borwein, "On projection algorithms for solving convex feasibility problems," *SIAM Review*, vol. 38, no. 3, pp. 367–426, 1996.

[7] P. L. Combettes, "The foundations of set theoretic estimation," *Proceedings of the IEEE*, vol. 81, pp. 182–208, 1993.

[8] A. Kangtunyakarn and S. Suantai, "A new mapping for finding common solutions of equilibrium problems and fixed point problems of finite family of nonexpansive mappings," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 10, pp. 4448–4460, 2009.

[9] V. Colao, G. Marino, and H.-K. Xu, "An iterative method for finding common solutions of equilibrium and fixed point problems," *Journal of Mathematical Analysis and Applications*, vol. 344, no. 1, pp. 340–352, 2008.

[10] S. Takahashi and W. Takahashi, "Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 331, no. 1, pp. 506–515, 2007.

[11] S. Takahashi and W. Takahashi, "Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 3, pp. 1025–1033, 2008.

[12] X. Qin, Y. J. Cho, and S. M. Kang, "Viscosity approximation methods for generalized equilibrium problems and fixed point problems with applications," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 1, pp. 99–112, 2010.

[13] W. Takahashi, *Nonlinear Functional Analysis: Fixed Point Theory and Its Applications*, Yokohama Publishers, Yokohama, Japan, 2000.

[14] H.-K. Xu, "Iterative algorithms for nonlinear operators," *Journal of the London Mathematical Society. Second Series*, vol. 66, no. 1, pp. 240–256, 2002.

[15] R. E. Bruck Jr., "Properties of fixed-point sets of nonexpansive mappings in Banach spaces," *Transactions of the American Mathematical Society*, vol. 179, pp. 251–262, 1973.

[16] F. E. Browder, "Nonlinear operators and nonlinear equations of evolution in Banach spaces," in *Proceedings of Symposia in Pure Mathematics*, pp. 1–308, Amer. Math. Soc., Providence, RI, USA, 1976.

[17] T. Suzuki, "Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals," *Journal of Mathematical Analysis and Applications*, vol. 305, no. 1, pp. 227–239, 2005.

[18] A. Kangtunyakarn and S. Suantai, "Hybrid iterative scheme for generalized equilibrium problems and fixed point problems of finite family of nonexpansive mappings," *Nonlinear Analysis: Hybrid Systems*, vol. 3, no. 3, pp. 296–309, 2009.

[19] W. Takahashi, *Introduction to Nonlinear and Convex Analysis*, Yokohama Publishers, Yokohama, Japan, 2009.

[20] H. Zhou, "Convergence theorems of fixed points for κ-strict pseudo-contractions in Hilbert spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 2, pp. 456–462, 2008.