*Research Article*

# On the Symmetric Properties of the Multivariate $p$-Adic Invariant Integral on $\mathbb{Z}_p$ Associated with the Twisted Generalized Euler Polynomials of Higher Order

**Taekyun Kim,[1] Byungje Lee,[2] and Young-Hee Kim[1]**

[1] *Division of General Education-Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea*
[2] *Department of Wireless Communications Engineering, Kwangwoon University, Seoul 139-701, Republic of Korea*

Correspondence should be addressed to Young-Hee Kim, yhkim@kw.ac.kr

We study the symmetric properties for the multivariate $p$-adic invariant integral on $\mathbb{Z}_p$ related to the twisted generalized Euler polynomials of higher order.

## 1. Introduction

Let $p$ be a fixed prime number. Throughout this paper, the symbols $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ denote the ring of rational integers, the ring of $p$-adic integers, the field of $p$-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers. The normalized valuation in $\mathbb{C}_p$ is denoted by $|\cdot|_p$ with $|p|_p = 1/p$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the fermionic $p$-adic invariant integral on $\mathbb{Z}_p$ is defined as

$$I(f) = \int_{\mathbb{Z}_p} f(x)d\mu(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x)\mu\left(x + p^N\mathbb{Z}_p\right) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x \quad (1.1)$$

(see [1–25]). For $n \in \mathbb{N}$, we note that

$$\int_{\mathbb{Z}_p} f(x+n)d\mu(x) = (-1)^n \int_{\mathbb{Z}_p} f(x)d\mu(x) + \sum_{l=0}^{n-1} (-1)^{n-1-l} f(l) \quad \text{(see [5])}. \quad (1.2)$$

Let $d$ be a fixed odd positive integer. For $N \in \mathbb{N}$, we set

$$X = X_d = \varprojlim_N \mathbb{Z}/dp^N\mathbb{Z}, \quad X_1 = \mathbb{Z}_p,$$

$$X^* = \bigcup_{\substack{0 < a < dp \\ (a,p)=1}} (a + dp\mathbb{Z}_p),$$

$$a + dp^N\mathbb{Z}_p = \left\{ x \in X \mid x \equiv a \ (\mathrm{mod}\ dp^N) \right\},$$

(1.3)

where $a \in \mathbb{Z}$ lies in $0 \le a < dp^N$ (see [1–13]). It is well known that for $f \in UD(\mathbb{Z}_p)$,

$$\int_X f(x)d\mu(x) = \int_{\mathbb{Z}_p} f(x)d\mu(x).$$

(1.4)

For $n \in \mathbb{N}$, let $C_{p^n}$ be the cyclic group of order $p^n$. That is, $C_{p^n} = \{\xi \mid \xi^{p^n} = 1\}$. The $p$-adic locally constant space, $T_p$, is defined by $T_p = \lim_{n \to \infty} C_{p^n} = \bigcup_{n \ge 1} C_{p^n}$.

Let $\chi$ be Dirichlet's character with conductor $d \in \mathbb{N}$ and let $\xi \in T_p$. Then the generalized twisted Bernoulli polynomials $B_{n,\chi,\xi}(x)$ attached to $\chi$ are defined as

$$\frac{t\sum_{a=0}^{d-1}\chi(a)\xi^a e^{at}}{\xi^d e^{dt} - 1} e^{xt} = \sum_{n=0}^{\infty} B_{n,\chi,\xi}(x)\frac{t^n}{n!} \quad \text{(see [10])}.$$

(1.5)

In [4, 7, 10–12], the generalized twisted Bernoulli polynomials of order $k$ attached to $\chi$ are also defined as follows:

$$\underbrace{\left(\frac{t\sum_{a=0}^{d-1}\chi(a)\xi^a e^{at}}{\xi^d e^{dt} - 1}\right) \times \cdots \times \left(\frac{t\sum_{a=0}^{d-1}\chi(a)\xi^a e^{at}}{\xi^d e^{dt} - 1}\right)}_{k\text{-times}} e^{xt} = \sum_{n=0}^{\infty} B_{n,\chi,\xi}^{(k)}(x)\frac{t^n}{n!}.$$

(1.6)

Recently, the symmetry identities for the generalized twisted Bernoulli polynomials and the generalized twisted Bernoulli polynomials of order $k$ are studied in [4, 12].

In this paper, we study the symmetric properties of the multivariate $p$-adic invariant integral on $\mathbb{Z}_p$. From these symmetric properties, we derive the symmetry identities for the twisted generalized Euler polynomials of higher order. In [14], Kim gave the relation between the power sum polynomials and the generalized higher-order Euler polynomials. The main purpose of this paper is to give the symmetry identities for the twisted generalized Euler polynomials of higher order using the symmetric properties of the multivariate $p$-adic invariant integral on $\mathbb{Z}_p$.

## 2. Symmetry Identities for the Twisted Generalized Euler Polynomials of Higher Order

Let $\chi$ be Dirichlet's character with an odd conductor $d \in \mathbb{N}$. That is, $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$. For $\xi \in T_p$, the twisted generalized Euler polynomials attached to $\chi$, $E_{n,\chi,\xi}(x)$, are defined as

$$\int_X \chi(y)\xi^y e^{(x+y)t} d\mu(y) = \frac{2\sum_{a=0}^{d-1} (-1)^a \chi(a)\xi^a e^{at}}{\xi^d e^{dt} + 1} e^{xt} = \sum_{n=0}^{\infty} E_{n,\chi,\xi}(x)\frac{t^n}{n!} \quad \text{(see [12])}. \tag{2.1}$$

In the special case $x = 0$, $E_{n,\chi,\xi} = E_{n,\chi,\xi}(0)$ are called the $n$th twisted generalized Euler numbers attached to $\chi$.

From (2.1), we note that

$$\int_X \chi(y)\xi^y (x+y)^m d\mu(y) = E_{m,\chi,\xi}(x), \quad m \in \mathbb{N} \cup \{0\}. \tag{2.2}$$

For $n \in \mathbb{N}$ with $n \equiv 1 \pmod 2$, we have

$$\int_X \chi(x)\xi^x e^{(x+nd)t} d\mu(x) + \int_X \chi(x)\xi^x e^{xt} d\mu(x) = 2\sum_{l=0}^{nd-1} (-1)^l \chi(l)\,\xi^l e^{lt}. \tag{2.3}$$

Let $T_{k,\chi,\xi}(n) = \sum_{l=0}^{n} (-1)^l \chi(l)\xi^l l^k$. Then we see that

$$\xi^{nd} \int_X \chi(x)\xi^x e^{(x+nd)t} d\mu(x) + \int_X \chi(x)\xi^x e^{xt} d\mu(x)$$
$$= \frac{2\int_X e^{xt}\chi(x)\xi^x d\mu(x)}{\int_X e^{ndxt}\xi^{ndx} d\mu(x)} = 2\sum_{k=0}^{\infty} T_{k,\chi,\xi}(nd-1)\frac{t^k}{k!}. \tag{2.4}$$

Now we define the twisted generalized Euler polynomials $E_{n,\chi,\xi}^{(k)}(x)$ of order $k$ attached to $\chi$ as follows:

$$e^{xt}\left(\frac{2\sum_{a=0}^{d-1}(-1)^a \chi(a)\xi^a e^{at}}{\xi^d e^{dt} + 1}\right)^k = \sum_{n=0}^{\infty} E_{n,\chi,\xi}^{(k)}(x)\frac{t^n}{n!}. \tag{2.5}$$

In the special case $x = 0$, $E_{n,\chi,\xi}^{(k)} = E_{n,\chi,\xi}^{(k)}(0)$ are called the $n$th twisted generalized Euler numbers of order $k$.

Let $w_1, w_2, d \in \mathbb{N}$ with $w_1 \equiv 1, w_2 \equiv 1$, and $d \equiv 1 \pmod 2$. Then we set

$$
\begin{aligned}
J_{\chi,\xi}^{(m)}(w_1, w_2 \mid x) = {} & \left( \frac{\int_{X^m} \left( \prod_{i=1}^m \chi(x_i) \right) \xi^{(\sum_{i=1}^m x_i)w_1} e^{(\sum_{i=1}^m x_i + w_2 x)w_1 t} d\mu(x_1) \cdots d\mu(x_m)}{\int_X \xi^{dw_1 w_2 x} e^{dw_1 w_2 x t} d\mu(x)} \right) \\
& \times \left( \int_{X^m} \left( \prod_{i=1}^m \chi(x_i) \right) \xi^{(\sum_{i=1}^m x_i)w_2} e^{(\sum_{i=1}^m x_i + w_1 y)w_2 t} d\mu(x_1) \cdots d\mu(x_m) \right),
\end{aligned}
\tag{2.6}
$$

where

$$
\int_{X^m} f(x_1, \ldots, x_m) d\mu(x_1) \cdots d\mu(x_m) = \underbrace{\int_X \cdots \int_X}_{m\text{-times}} f(x_1, \ldots, x_m) d\mu(x_1) \cdots d\mu(x_m).
\tag{2.7}
$$

From (2.6), we note that

$$
\begin{aligned}
J_{\chi,\xi}^{(m)}(w_1, w_2 \mid x) = {} & \left( \int_{X^m} \left( \prod_{i=1}^m \chi(x_i) \right) \xi^{(\sum_{i=1}^m x_i)w_1} e^{(\sum_{i=1}^m x_i)w_1 t} d\mu(x_1) \cdots d\mu(x_m) \right) e^{w_1 w_2 x t} \\
& \times \left( \frac{\int_X \chi(x_m) \xi^{w_2 x_m} e^{w_2 x_m t} d\mu(x_m)}{\int_X \xi^{dw_1 w_2 x} e^{dw_1 w_2 x t} d\mu(x)} \right) e^{w_1 w_2 y t} \\
& \times \left( \int_{X^{m-1}} \left( \prod_{i=1}^{m-1} \chi(x_i) \right) \xi^{(\sum_{i=1}^{m-1} x_i)w_2} e^{(\sum_{i=1}^{m-1} x_i)w_2 t} d\mu(x_1) \cdots d\mu(x_{m-1}) \right).
\end{aligned}
\tag{2.8}
$$

From (2.4), we can easily derive the following equation:

$$
\frac{\int_X \chi(x) \xi^x e^{xt} d\mu(x)}{\int_X \xi^{dw_1 x} e^{dw_1 x t} d\mu(x)} = \sum_{l=0}^{dw_1 - 1} (-1)^l \chi(l) \xi^l e^{lt} = \sum_{k=0}^{\infty} T_{k,\chi,\xi}(dw_1 - 1) \frac{t^k}{k!}.
\tag{2.9}
$$

It is not difficult to show that

$$
\begin{aligned}
& e^{w_1 w_2 x t} \left( \int_{X^m} \left( \prod_{i=1}^m \chi(x_i) \right) \xi^{(\sum_{i=1}^m x_i)w_1} e^{(\sum_{i=1}^m x_i)w_1 t} d\mu(x_1) \cdots d\mu(x_m) \right) \\
& = \left( \frac{2 \sum_{a=0}^{d-1} (-1)^a \chi(a) \xi^{aw_1} e^{aw_1 t}}{\xi^{dw_1} e^{dw_1 t} + 1} \right)^m e^{w_1 w_2 x t} = \sum_{k=0}^{\infty} E_{k,\chi,\xi^{w_1}}^{(m)}(w_2 x) \frac{w_1^k t^k}{k!}.
\end{aligned}
\tag{2.10}
$$

By (2.8), (2.9), and (2.10), we see that

$$
\begin{aligned}
J_{\chi,\xi}^{(m)}(w_1, w_2 \mid x) &= \left( \sum_{l=0}^{\infty} E_{l,\chi,\xi^{w_1}}^{(m)}(w_2 x) \frac{w_1^l t^l}{l!} \right) \left( \sum_{k=0}^{\infty} T_{k,\chi,\xi^{w_2}}(w_1 d - 1) \frac{w_2^k t^k}{k!} \right) \\
&\quad \times \left( \sum_{i=0}^{\infty} E_{i,\chi,\xi^{w_2}}^{(m-1)}(w_1 y) \frac{w_2^i t^i}{i!} \right) \\
&= \sum_{n=0}^{\infty} \left( \sum_{j=0}^{n} \binom{n}{j} w_2^j w_1^{n-j} E_{n-j,\chi,\xi^{w_1}}^{(m)}(w_2 x) \right. \\
&\quad \left. \times \sum_{k=0}^{j} \binom{j}{k} T_{k,\chi,\xi^{w_2}}(w_1 d - 1) E_{j-k,\chi,\xi^{w_2}}^{(m-1)}(w_1 y) \right) \frac{t^n}{n!}.
\end{aligned}
\tag{2.11}
$$

In the viewpoint of the symmetry of $J_{\chi,\xi}^{(m)}(w_1, w_2 \mid x)$ for $w_1$ and $w_2$, we have

$$
\begin{aligned}
J_{\chi,\xi}^{(m)}(w_1, w_2 \mid x) &= \sum_{n=0}^{\infty} \left( \sum_{j=0}^{n} \binom{n}{j} w_1^j w_2^{n-j} E_{n-j,\chi,\xi^{w_2}}^{(m)}(w_1 x) \right. \\
&\quad \left. \times \sum_{k=0}^{j} \binom{j}{k} T_{k,\chi,\xi^{w_1}}(w_2 d - 1) E_{j-k,\chi,\xi^{w_1}}^{(m-1)}(w_2 y) \right) \frac{t^n}{n!}.
\end{aligned}
\tag{2.12}
$$

Comparing the coefficients on both sides of (2.11) and (2.12), we obtain the following theorem.

**Theorem 2.1.** *Let $w_1, w_2, d \in \mathbb{N}$ with $w_1 \equiv 1, w_2 \equiv 1$, and $d \equiv 1$ (mod 2). For $n \in \mathbb{N} \cup \{0\}$ and $m \in \mathbb{N}$, one has*

$$
\begin{aligned}
&\sum_{j=0}^{n} \binom{n}{j} w_2^j w_1^{n-j} E_{n-j,\chi,\xi^{w_1}}^{(m)}(w_2 x) \sum_{k=0}^{j} \binom{j}{k} T_{k,\chi,\xi^{w_2}}(w_1 d - 1) E_{j-k,\chi,\xi^{w_2}}^{(m-1)}(w_1 y) \\
&= \sum_{j=0}^{n} \binom{n}{j} w_1^j w_2^{n-j} E_{n-j,\chi,\xi^{w_2}}^{(m)}(w_1 x) \sum_{k=0}^{j} \binom{j}{k} T_{k,\chi,\xi^{w_1}}(w_2 d - 1) E_{j-k,\chi,\xi^{w_1}}^{(m-1)}(w_2 y).
\end{aligned}
\tag{2.13}
$$

Let $m = 1$ and $y = 0$ in Theorem 2.1. Then we also have the following corollary.

**Corollary 2.2.** *For $w_1, w_2, d \in \mathbb{N}$ with $w_1 \equiv 1, w_2 \equiv 1$, and $d \equiv 1$ (mod 2), one has*

$$
\begin{aligned}
&\sum_{m=0}^{n} \binom{n}{m} E_{m,\chi,\xi^{w_1}}(w_2 x) w_1^m w_2^{n-m} T_{n-m,\chi,\xi^{w_2}}(w_1 d - 1) \\
&= \sum_{m=0}^{n} \binom{n}{m} E_{m,\chi,\xi^{w_2}}(w_1 x) w_1^{n-m} w_2^m T_{n-m,\chi,\xi^{w_1}}(w_2 d - 1) \quad \text{(see [2])}.
\end{aligned}
\tag{2.14}
$$

Let $\chi$ be the trivial character and $d = 1$. Then we also have the following corollary.

**Corollary 2.3.** *Let* $w_1, w_2 \in \mathbb{N}$ *with* $w_1 \equiv 1, w_2 \equiv 1 \pmod{2}$. *Then one has*

$$\sum_{j=0}^{n} \binom{n}{j} w_1^{n-j} w_2^j E_{n-j,\xi^{w_1}}(w_2 x) T_{k,\xi^{w_2}}(w_1 - 1)$$

$$= \sum_{j=0}^{n} \binom{n}{j} w_1^j w_2^{n-j} E_{n-j,\xi^{w_2}}(w_1 x) T_{k,\xi^{w_1}}(w_2 - 1), \tag{2.15}$$

*where* $E_{n,\xi}(x)$ *are the* $n$*th twisted Euler polynomials.*

If we take $w_2 = 1$ in Corollary 2.3, then we obtain the following corollary.

**Corollary 2.4** (Distribution for the twisted Euler polynomials). *For* $w_1 \in \mathbb{N}$ *with* $w_1 \equiv 1$ (mod 2), *one has*

$$E_{n,\xi}(x) = \sum_{i=0}^{n} \binom{n}{i} w_1^i E_{i,\xi^{w_1}}(x) T_{n-i,\xi}(w_1 - 1). \tag{2.16}$$

From (2.6), we can derive that

$$J_{\chi,\xi}^{(m)}(w_1, w_2 \mid x)$$

$$= \left( \sum_{l=0}^{w_1 d - 1} \chi(l)(-1)^l \, \xi^{w_2 l} \int_{X^m} \left( \prod_{i=1}^{m} \chi(x_i) \right) \right.$$

$$\times \xi^{(\sum_{i=1}^{m} x_i) w_1} e^{w_1 (\sum_{i=1}^{m} x_i + (w_2/w_1)l + w_2 x)t} d\mu(x_1) \cdots d\mu(x_m) \Big)$$

$$\times \left( \int_{X^{m-1}} \left( \prod_{i=1}^{m-1} \chi(x_i) \right) \xi^{(\sum_{i=1}^{m-1} x_i) w_2} e^{(\sum_{i=1}^{m-1} x_i) w_2 t} d\mu(x_1) \cdots d\mu(x_{m-1}) \right) \tag{2.17}$$

$$= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \binom{n}{k} w_1^k w_2^{n-k} E_{n-k,\chi,\xi^{w_2}}^{(m-1)}(w_1 y) \right.$$

$$\times \left. \sum_{l=0}^{w_1 d - 1} \chi(l)(-1)^l \xi^{w_2 l} E_{k,\chi,\xi^{w_1}}^{(m)} \left( w_2 x + \frac{w_2}{w_1} l \right) \right) \frac{t^n}{n!}.$$

By the symmetry property of $J_{\chi,\xi}^{(m)}(w_1, w_2 \mid x)$ in $w_1$ and $w_2$, we also see that

$$J_{\chi,\xi}^{(m)}(w_1, w_2 \mid x)$$

$$= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \binom{n}{k} w_2^k w_1^{n-k} E_{n-k,\chi,\xi^{w_1}}^{(m-1)}(w_2 y) \sum_{l=0}^{w_2 d - 1} \chi(l)(-1)^l \xi^{w_1 l} E_{k,\chi,\xi^{w_2}}^{(m)} \left( w_1 x + \frac{w_1}{w_2} l \right) \right) \frac{t^n}{n!}. \tag{2.18}$$

Comparing the coefficients on both sides of (2.17) and (2.18), we obtain the following theorem which shows the relationship between the power sums and the twisted generalized Euler polynomials of higher order.

**Theorem 2.5.** *Let $w_1, w_2, d \in \mathbb{N}$ with $w_1 \equiv 1, w_2 \equiv 1$, and $d \equiv 1 \pmod 2$. For $n \in \mathbb{N} \cup \{0\}$ and $m \in \mathbb{N}$, one has*

$$
\sum_{k=0}^{n} \binom{n}{k} w_1^k w_2^{n-k} E_{n-k,\chi,\xi^{w_2}}^{(m-1)}(w_1 y) \sum_{l=0}^{w_1 d-1} \chi(l)(-1)^l \xi^{w_2 l} E_{k,\chi,\xi^{w_1}}^{(m)}\left(w_2 x + \frac{w_2}{w_1}l\right)
$$

$$
= \sum_{k=0}^{n} \binom{n}{k} w_2^k w_1^{n-k} E_{n-k,\chi,\xi^{w_1}}^{(m-1)}(w_2 y) \sum_{l=0}^{w_2 d-1} \chi(l)(-1)^l \xi^{w_1 l} E_{k,\chi,\xi^{w_2}}^{(m)}\left(w_1 x + \frac{w_1}{w_2}l\right). \tag{2.19}
$$

If we take $x = 0$, $y = 0$, and $m = 1$ in Theorem 2.5, then we have the following identity:

$$
\sum_{k=0}^{n} \binom{n}{k} E_{k,\chi,\xi^{w_1}} w_1^k w_2^{n-k} T_{n-k,\chi,\xi^{w_2}}(w_1 d - 1)
$$

$$
= \sum_{k=0}^{n} \binom{n}{k} E_{k,\chi,\xi^{w_2}} w_2^k w_1^{n-k} T_{n-k,\chi,\xi^{w_1}}(w_2 d - 1). \tag{2.20}
$$

## Acknowledgment

## References

[1] T. Kim and Y.-H. Kim, "On the symmetric properties for the generalized twisted Bernoulli polynomials," *Journal of Inequalities and Applications*, vol. 2009, Article ID 164743, 9 pages, 2009.

[2] T. Kim, "Symmetry identities for the twisted generalized Euler polynomials," *Advanced Studies in Contemporary Mathematics*, vol. 19, no. 2, pp. 151–155, 2009.

[3] T. Kim, K. H. Park, and K.-W. Hwang, "On the identities of symmetry for the ζ-Euler polynomials of higher order," *Advances in Difference Equations*, vol. 2009, Article ID 273545, 9 pages, 2009.

[4] S.-H. Rim, Y.-H. Kim, B. Lee, and T. Kim, "Some identities of the generalized twisted Bernoulli numbers and polynomials of higher order," *Journal of Computational Analysis and Applications*, vol. 12, no. 3, pp. 695–702, 2010.

[5] T. Kim, "Symmetry of power sum polynomials and multivariate fermionic *p*-adic invariant integral on $\mathbb{Z}_p$," *Russian Journal of Mathematical Physics*, vol. 16, no. 1, pp. 93–96, 2009.

[6] M. Cenkci, Y. Simsek, and V. Kurt, "Multiple two-variable *p*-adic *q*-L-function and its behavior at *s* = 0," *Russian Journal of Mathematical Physics*, vol. 15, no. 4, pp. 447–459, 2008.

[7] Y. Simsek, "On *p*-adic twisted -L-functions related to generalized twisted Bernoulli numbers," *Russian Journal of Mathematical Physics*, vol. 13, no. 3, pp. 340–348, 2006.

[8] T. Kim and C. Adiga, "Sums of products of generalized Bernoulli numbers," *International Mathematical Journal*, vol. 5, no. 1, pp. 1–7, 2004.

[9] I. N. Cangul, V. Kurt, H. Ozden, and Y. Simsek, "On the higher-order *w*-*q*-Genocchi numbers," *Advanced Studies in Contemporary Mathematics*, vol. 19, no. 1, pp. 39–57, 2009.

[10] H. Ozden, I. N. Cangul, and Y. Simsek, "Remarks on *q*-Bernoulli numbers associated with Daehee numbers," *Advanced Studies in Contemporary Mathematics*, vol. 18, no. 1, pp. 41–48, 2009.

[11] M. Cenkci, "The *p*-adic generalized twisted (*h*, *q*)-*ζ*-*l*-function and its applicaitons," *Advanced Studies in Contemporary Mathematics*, vol. 15, no. 1, pp. 37–47, 2007.

[12] K. H. Park, "On interpolation functions of the generalized twisted (*h*, *q*)-Euler polynomials," *Journal of Inequalities and Applications*, vol. 2009, Article ID 946569, 17 pages, 2009.

[13] S.-H. Rim, K. H. Park, and E. J. Moon, "On Genocchi numbers and polynomials," *Abstract and Applied Analysis*, vol. 2008, Article ID 898471, 7 pages, 2008.

[14] T. Kim, "Symmetry properties of the generalized higher-order Euler polynomials," *Advanced Studies in Contemporary Mathematics*, vol. 20, no. 2, pp. 1–5, 2010.

[15] T. Kim, "Note on the Euler *q*-zeta functions," *Journal of Number Theory*, vol. 129, no. 7, pp. 1798–1804, 2009.

[16] T. Kim, "On a *q*-analogue of the *p*-adic log gamma functions and related integrals," *Journal of Number Theory*, vol. 76, no. 2, pp. 320–329, 1999.

[17] T. Kim, "*q*-generalized Euler numbers and polynomials," *Russian Journal of Mathematical Physics*, vol. 13, no. 3, pp. 293–298, 2006.

[18] T. Kim, "Multiple *p*-adic *L*-function," *Russian Journal of Mathematical Physics*, vol. 13, no. 2, pp. 151–157, 2006.

[19] H. M. Srivastava, T. Kim, and Y. Simsek, "*q*-Bernoulli numbers and polynomials associated with multiple *q*-zeta functions and basic *L*-series," *Russian Journal of Mathematical Physics*, vol. 12, no. 2, pp. 241–268, 2005.

[20] T. Kim, "Power series and asymptotic series associated with the *q*-analog of the two-variable *p*-adic *L*-function," *Russian Journal of Mathematical Physics*, vol. 12, no. 2, pp. 186–196, 2005.

[21] T. Kim, "Analytic continuation of multiple *q*-zeta functions and their values at negative integers," *Russian Journal of Mathematical Physics*, vol. 11, no. 1, pp. 71–76, 2004.

[22] T. Kim, "Non-Archimedean *q*-integrals associated with multiple Changhee *q*-Bernoulli polynomials," *Russian Journal of Mathematical Physics*, vol. 10, no. 1, pp. 91–98, 2003.

[23] T. Kim, "On Euler-Barnes multiple zeta functions," *Russian Journal of Mathematical Physics*, vol. 10, no. 3, pp. 261–267, 2003.

[24] T. Kim, "*q*-Volkenborn integration," *Russian Journal of Mathematical Physics*, vol. 9, no. 3, pp. 288–299, 2002.

[25] T. Kim, "Note on Dedekind type DC sums," *Advanced Studies in Contemporary Mathematics*, vol. 18, no. 2, pp. 249–260, 2009.