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Research Article

On Hilbert-Pachpatte Multiple Integral Inequalities

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We establish some multiple integral Hilbert-Pachpatte-type inequalities. As applications, we get some inverse forms of Pachpatte's inequalities which were established in 1998.

1. Introduction

In 1934, Hilbert [1] established the following well-known integral inequality.

If $f \in L^p(0,\infty)$, $g \in L^p(0,\infty)$, $f, g \geq 0$, $p > 1$ and $1/p + 1/q = 1$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(x)}{x+y} dx\, dy \leq \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty f^p(x) dx \right)^{1/p} \left(\int_0^\infty g^q(x) dx \right)^{1/q}, \quad (1.1)$$

where $\pi / \sin(\pi/p)$ is the best value.

In recent years, considerable attention has been given to various extensions and improvements of the Hilbert inequality form different viewpoints [2–10]. In particular, Pachpatte [11] proved some inequalities similar to Hilbert's integral inequalities in 1998. In this paper, we establish some new multiple integral Hilbert-Pachpatte-type inequalities.

2. Main Results

Theorem 2.1. *Let* $h_i \geq 1$*, let* $f_i(\sigma_i) \in C^1[(x_i, 0), (0, \infty)]$*,* $i = 1, \dots, n$*, where* x_i *are positive real numbers, and define* $F_i(s_i) = \int_{s_i}^0 f_i(\sigma_i) d\sigma_i$*, for* $s_i \in (x_i, 0)$*. Then for* $1/\alpha_i + 1/\beta_i = 1$*,* $0 < \beta_i < 1$ *and* $\sum_{i=1}^n (1/\alpha_i) = 1/\alpha$*,*

$$\int_{x_1}^0 \cdots \int_{x_n}^0 \frac{\prod_{i=1}^n F_i^{h_i}(s_i)}{\left(\alpha \sum_{i=1}^n (1/\alpha_i)(-s_i)\right)^{1/\alpha}} ds_1 \cdots ds_n \geq \prod_{i=1}^n (-x_i)^{1/\alpha_i} h_i \left(\int_{x_i}^0 (s_i - x_i) \left(F_i^{h_i - 1}(s_i) f_i(s_i) \right)^{\beta_i} ds_i \right)^{1/\beta_i}. \tag{2.1}$$

Proof. From the hypotheses and in view of inverse Hölder integral inequality (see [12]), it is easy to observe that

$$\begin{aligned} \prod_{i=1}^n F_i^{h_i}(s_i) &= \prod_{i=1}^n h_i \int_{s_i}^0 F_i^{h_i-1}(\sigma_i) f_i(\sigma_i) d\sigma_i \\ &\geq \prod_{i=1}^n h_i (-s_i)^{1/\alpha_i} \left(\int_{s_i}^0 \left(F_i^{h_i-1}(\sigma_i) f_i(\sigma_i) \right)^{\beta_i} d\sigma_i \right)^{1/\beta_i}, \quad s_i \in (x_i, 0),\ i = 1, \dots, n. \end{aligned} \tag{2.2}$$

Let us note the following means inequality:

$$\prod_{i=1}^n m_i^{1/\alpha_i} \geq \left(\alpha \sum_{i=1}^n \frac{1}{\alpha_i} m_i \right)^{1/\alpha}, \quad m > 0. \tag{2.3}$$

We obtain that

$$\frac{\prod_{i=1}^n F_i^{h_i}(s_i)}{\left(\alpha \sum_{i=1}^n (1/\alpha_i)(-s_i)\right)^{1/\alpha}} \geq \prod_{i=1}^n h_i \left(\int_{s_i}^0 \left(F_i^{h_i-1}(\sigma_i) f_i(\sigma_i) \right)^{\beta_i} d\sigma_i \right)^{1/\beta_i}. \tag{2.4}$$

Integrating both sides of (2.4) over s_i from x_i $(i = 1, 2, \dots, n)$ to 0 and using the special case of inverse Hölder integral inequality, we observe that

$$\begin{aligned} &\int_{x_1}^0 \cdots \int_{x_n}^0 \frac{\prod_{i=1}^n F_i^{h_i}(s_i)}{\left(\alpha \sum_{i=1}^n (1/\alpha_i)(-s_i)\right)^{1/\alpha}} ds_1 \cdots ds_n \\ &\quad \geq \prod_{i=1}^n h_i \int_{x_i}^0 \left(\int_{s_i}^0 \left(F_i^{h_i-1}(\sigma_i) f_i(\sigma_i) \right)^{\beta_i} d\sigma_i \right)^{1/\beta_i} ds_i \\ &\quad \geq \prod_{i=1}^n h_i (-x_i)^{1/\alpha_i} \left(\int_{x_i}^0 \left(\int_{s_i}^0 \left(F_i^{h_i-1}(\sigma_i) f_i(\sigma_i) \right)^{\beta_i} d\sigma_i \right) ds_i \right)^{1/\beta_i} \\ &\quad = \prod_{i=1}^n (-x_i)^{1/\alpha_i} h_i \left(\int_{x_i}^0 (s_i - x_i) \left(F_i^{h_i-1}(s_i) f_i(s_i) \right)^{\beta_i} ds_i \right)^{1/\beta_i}. \end{aligned} \tag{2.5}$$

The proof is complete. □

Remark 2.2. Taking $n = 2$, $\beta_i = 1/2$ to (2.1), (2.1) changes to

$$\int_{x_1}^{0}\int_{x_2}^{0}\frac{F_1^{h_1}(s_1)F_2^{h_2}(s_2)}{(s_1+s_2)^{-2}}ds_1\,ds_2$$
$$\geq 4h_1h_2(x_1x_2)^{-1}\left(\int_{x_1}^{0}(s_1-x_1)\left(F_1^{h_1-1}(s_1)f_1(s_1)\right)^{1/2}ds_1\right)^2 \tag{2.6}$$
$$\times\left(\int_{x_2}^{0}(s_2-x_2)\left(F_2^{h_2-1}(s_2)f_2(s_2)\right)^{1/2}ds_2\right)^2.$$

This is just an inverse inequality similar to the following inequality which was proved by Pachpatte [11]:

$$\int_0^x\int_0^y\frac{F^h(s)G^l(t)}{s+t}ds\,dt\leq\frac{1}{2}hl(xy)^{1/2}\left(\int_0^x(x-s)\left(F^{h-1}(s)f(s)\right)^2ds\right)^{1/2}$$
$$\times\left(\int_0^y(y-t)\left(G^{l-1}(t)g(t)\right)^2dt\right)^{1/2}. \tag{2.7}$$

Theorem 2.3. *Let $f_i(\sigma_i)$, $F_i(s_i)$, α_i, and β_i be as in Theorem 2.1. Let $p_i(\sigma_i)$ be n positive functions defined for $\sigma_i\in(x_i,0)$ $(i=1,2,\ldots,n)$, and define $P_i(s_i)=\int_{s_i}^0 p_i(\sigma_i)d\sigma_i$, where x_i are positive real numbers. Let ϕ_i $(i=1,2,\ldots,n)$ be n real-valued nonnegative, concave, and super-multiplicative functions defined on R_+. Then*

$$\int_{x_1}^{0}\cdots\int_{x_n}^{0}\frac{\prod_{i=1}^{n}\phi_i(F_i(s_i))}{\left(\alpha\sum_{i=1}^{n}(1/\alpha_i)(-s_i)\right)^{1/\alpha}}ds_1\cdots ds_n$$
$$\geq L(x_1,\ldots,x_n)\prod_{i=1}^{n}\left(\int_{x_i}^{0}(s_i-x_i)\left(p_i(s_i)\phi_i\left(\frac{f_i(s_i)}{p_i(s_i)}\right)\right)^{\beta_i}ds_i\right)^{1/\beta_i}, \tag{2.8}$$

where

$$L(x_1,\ldots,x_n)=\prod_{i=1}^{n}\left(\int_{x_i}^{0}\left(\frac{\phi_i(P_i(s_i))}{P_i(s_i)}\right)^{\alpha_i}ds_i\right)^{1/\alpha_i}. \tag{2.9}$$

Proof. By using Jensen integral inequality (see [11]) and inverse Hölder integral inequality (see [12]) and noticing that ϕ_i ($i = 1, 2, \ldots, n$) are n real-valued super-multiplicative functions, it is easy to observe that

$$\begin{aligned}\phi_i(F_i(s_i)) &= \phi_i\left(\frac{P_i(s_i)\int_{s_i}^0 p_i(\sigma_i)(f_i(\sigma_i)/p_i(\sigma_i))d\sigma_i}{\int_{s_i}^0 p_i(\sigma_i)d\sigma_i}\right)\\ &\geq \phi_i(P_i(s_i))\phi_i\left(\frac{\int_{s_i}^0 p_i(\sigma_i)(f_i(\sigma_i)/p_i(\sigma_i))d\sigma_i}{\int_{s_i}^0 p_i(\sigma_i)d\sigma_i}\right)\\ &\geq \frac{\phi_i(P_i(s_i))}{P_i(s_i)}\int_{s_i}^0 p_i(\sigma_i)\phi_i\left(\frac{f_i(\sigma_i)}{p_i(\sigma_i)}\right)d\sigma_i\\ &\geq \left(\frac{\phi_i(P_i(s_i))}{P_i(s_i)}\right)(-s_i)^{1/\alpha_i}\left(\int_{s_i}^0\left(p_i(\sigma_i)\phi_i\left(\frac{f_i(\sigma_i)}{p_i(\sigma_i)}\right)\right)^{\beta_i}d\sigma_i\right)^{1/\beta_i}.\end{aligned} \tag{2.10}$$

In view of the means inequality and integrating two sides of (2.10) over s_i from x_i ($i = 1, 2, \ldots, n$) to 0 and noticing Hölder integral inequality, we observe that

$$\begin{aligned}&\int_{x_1}^0\cdots\int_{x_n}^0\frac{\prod_{i=1}^n\phi_i(F_i(s_i))}{\left(\alpha\sum_{i=1}^n(1/\alpha_i)(-s_i)\right)^{1/\alpha}}ds_1\cdots ds_n\\ &\geq\prod_{i=1}^n\int_{x_i}^0\left(\frac{\phi_i(P_i(s_i))}{P_i(s_i)}\right)\left(\int_{s_i}^0\left(p_i(\sigma_i)\phi_i\left(\frac{f_i(\sigma_i)}{p_i(\sigma_i)}\right)\right)^{\beta_i}d\sigma_i\right)^{1/\beta_i}ds_i\\ &\geq\prod_{i=1}^n\left(\int_{x_i}^0\left(\frac{\phi_i(P_i(s_i))}{P_i(s_i)}\right)^{\alpha_i}ds_i\right)^{1/\alpha_i}\left(\int_{x_i}^0\int_{s_i}^0\left(p_i(\sigma_i)\phi_i\left(\frac{f_i(\sigma_i)}{p_i(\sigma_i)}\right)\right)^{\beta_i}d\sigma_i ds_i\right)^{1/\beta_i}\\ &= L(x_1,\ldots,x_n)\prod_{i=1}^n\left(\int_{x_i}^0(s_i-x_i)\left(p_i(s_i)\phi_i\left(\frac{f_i(s_i)}{p_i(s_i)}\right)\right)^{\beta_i}ds_i\right)^{1/\beta_i}.\end{aligned} \tag{2.11}$$

This completes the proof of Theorem 2.3. □

Remark 2.4. Taking $n = 2$, $\beta_i = 1/2$ to (2.8), (2.8) changes to

$$\begin{aligned}&\int_{x_1}^0\int_{x_2}^0\frac{\phi_1(F_1(s_1))\phi_2(F_2(s_2))}{(s_1+s_2)^{-2}}ds_1ds_2\\ &\geq L(x_1,x_2)\left(\int_{x_1}^0(s_1-x_1)\left(p_1(s_1)\phi_1\left(\frac{f_1(s_1)}{p_1(s_1)}\right)\right)^{1/2}ds_1\right)^2\\ &\quad\times\left(\int_{x_2}^0(s_2-x_2)\left(p_2(s_2)\phi_2\left(\frac{f_2(s_2)}{p_2(s_2)}\right)\right)^{1/2}ds_2\right)^2,\end{aligned} \tag{2.12}$$

where

$$L(x_1,x_2)=4\left(\int_{x_1}^{0}\left(\frac{\phi_1(P_1(s_1))}{P_1(s_1)}\right)^{-1}ds_1\right)^{-1}\left(\int_{x_2}^{0}\left(\frac{\phi_2(P_2(s_2))}{P_2(s_2)}\right)^{-1}ds_2\right)^{-1}. \tag{2.13}$$

This is just an inverse inequality similar to the following inequality which was proved by Pachpatte [11]:

$$\begin{aligned}\int_0^x\int_0^y \frac{\phi(F(s))\psi(G(t))}{s+t}ds\,dt \le L(x,y)&\left(\int_0^x(x-s)\left(p(s)\phi\left(\frac{f(s)}{p(s)}\right)\right)^2ds\right)^{1/2}\\ &\times\left(\int_0^y(y-t)\left(q(t)\psi\left(\frac{g(t)}{q(t)}\right)\right)^2dt\right)^{1/2},\end{aligned} \tag{2.14}$$

where

$$L(x,y)=\frac{1}{2}\left(\int_0^x\left(\frac{\phi(P(s))}{P(s)}\right)^2ds\right)^{1/2}\left(\int_0^y\left(\frac{\psi(Q(t))}{Q(t)}\right)^2dt\right)^{1/2}. \tag{2.15}$$

Theorem 2.5. *Let $f_i(\sigma_i)$, $p_i(\sigma_i)$, $P_i(\sigma_i)$, α_i, and β_i be as Theorem 2.3, and define $F_i(s_i)=(1/P_i(s_i))\int_{s_i}^0 p_i(\sigma_i)f_i(\sigma_i)d\sigma_i$ for $\sigma_i,s_i\in(x_i,0)$, where x_i are positive real numbers. Let ϕ_i ($i=1,2,\ldots,n$) be n real-valued, nonnegative, and concave functions on R_+. Then*

$$\begin{aligned}&\int_{x_1}^0\cdots\int_{x_n}^0\frac{\prod_{i=1}^n P_i(s_i)\phi_i(F_i(s_i))}{\left(\alpha\sum_{i=1}^n(1/\alpha_i)(-s_i)\right)^{1/\alpha}}ds_1\cdots ds_n\\ &\qquad\ge\prod_{i=1}^n x_i^{1/\alpha_i}\left(\int_{x_i}^0(s_i-x_i)\left(p_i(s_i)\phi_i(f_i(s_i))\right)^{\beta_i}ds_i\right)^{1/\beta_i}.\end{aligned} \tag{2.16}$$

Proof. From the hypotheses and by using Jensen integral inequality and the inverse Hölder integral inequality, we have

$$\begin{aligned}\phi_i(F_i(s_i))&=\phi_i\left(\frac{1}{P_i(s_i)}\int_{s_i}^0 p_i(\sigma_i)f_i(\sigma_i)d\sigma_i\right)\\ &\ge\frac{1}{P_i(s_i)}\int_{s_i}^0 p_i(\sigma_i)\phi_i\left(f_i(\sigma_i)\right)d\sigma_i\\ &\ge\frac{1}{P_i(s_i)}(-s_i)^{1/\alpha_i}\left(\int_{s_i}^0\left(p_i(\sigma_i)\phi_i\left(f_i(\sigma_i)\right)\right)^{\beta_i}d\sigma_i\right)^{1/\beta_i}.\end{aligned} \tag{2.17}$$

Hence

$$\begin{aligned}
&\int_{x_1}^{0}\cdots\int_{x_n}^{0}\frac{\prod_{i=1}^{n}P_i(s_i)\phi_i(F_i(s_i))}{\left(\alpha\sum_{i=1}^{n}(1/\alpha_i)(-s_i)\right)^{1/\alpha}}ds_1\cdots ds_n\\
&\quad\ge\prod_{i=1}^{n}\int_{x_i}^{0}\left(\int_{s_i}^{0}\left(p_i(\sigma_i)\phi_i(f_i(\sigma_i))\right)^{\beta_i}d\sigma_i\right)^{1/\beta_i}ds_i\\
&\quad\ge\prod_{i=1}^{n}x_i^{1/\alpha_i}\left(\int_{x_i}^{0}\int_{s_i}^{0}\left(p_i(\sigma_i)\phi_i(f_i(\sigma_i))\right)^{\beta_i}d\sigma_i\,ds_i\right)^{1/\beta_i}\\
&\quad=\prod_{i=1}^{n}(-x_i)^{1/\alpha_i}\left(\int_{x_i}^{0}(s_i-x_i)\left(p_i(s_i)\phi_i(f_i(s_i))\right)^{\beta_i}ds_i\right)^{1/\beta_i}.
\end{aligned}\tag{2.18}$$

□

Remark 2.6. Taking $n=2$, $\beta_i=1/2$ to (2.16), (2.16) changes to

$$\begin{aligned}
&\int_{x_1}^{0}\int_{x_2}^{0}\frac{P_1(s_1)P_2(s_2)\phi_1(F_1(s_1))\phi_2(F_2(s_2))}{(s_1+s_2)^{-2}}ds_1ds_2\\
&\quad\ge 4(x_1x_2)^{-1}\left(\int_{x_1}^{0}(s_1-x_1)\left(p_1(s_1)\phi_1(f_1(s_1))\right)^{1/2}ds_1\right)^{2}\\
&\qquad\times\left(\int_{x_2}^{0}(s_2-x_2)\left(p_2(s_2)\phi_2(f_2(s_2))\right)^{1/2}ds_2\right)^{2}.
\end{aligned}\tag{2.19}$$

This is just an inverse inequality similar to the following inequality which was proved by Pachpatte [11]:

$$\begin{aligned}
&\int_{0}^{x}\int_{0}^{y}\frac{P(s)Q(t)\phi(F(s))\psi(G(t))}{s+t}ds\,dt\\
&\quad\le\frac{1}{2}(xy)^{1/2}\left(\int_{0}^{x}(x-s)\left(p(s)\phi(f(s))\right)^{2}ds\right)^{1/2}\left(\int_{0}^{y}(y-t)\left(q(t)\psi(g(t))\right)^{2}dt\right)^{1/2}.
\end{aligned}\tag{2.20}$$

Remark 2.7. In (2.20), if $p_1(s_1)=p_2(s_2)=1$, then $P_1(s_1)=s_1$, $P_2(s_2)=s_2$. Therefore (2.20) changes to

$$\begin{aligned}
&\int_{x_1}^{0}\int_{x_2}^{0}\frac{\phi_1(F_1(s_1))\phi_2(F_2(s_2))}{(s_1+s_2)^{-2}}ds_1ds_2\\
&\quad\ge 4(x_1x_2)^{-1}\left(\int_{x_1}^{0}(s_1-x_1)\left(\phi_1(f_1(s_1))\right)^{1/2}ds_1\right)^{2}\left(\int_{x_2}^{0}(s_2-x_2)\left(\phi_2(f_2(s_2))\right)^{1/2}ds_2\right)^{2}.
\end{aligned}\tag{2.21}$$

This is just an inverse inequality similar to the following Inequality which was proved by Pachpatte [11]:

$$\int_0^x \int_0^y \frac{\phi(F(s))\psi(G(t))}{(st)^{-1}(s+t)} ds\, dt \leq \frac{1}{2}(xy)^{1/2}\left(\int_0^x (x-s)\left(\phi\left(f(s)\right)\right)^2 ds\right)^{1/2}\left(\int_0^y (y-t)\left(\psi\left(g(t)\right)\right)^2 dt\right)^{1/2}. \tag{2.22}$$

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