Research Article

Integral-Type Operators from $F(p,q,s)$ Spaces to Zygmund-Type Spaces on the Unit Ball

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Let $H(\mathbb{B})$ denote the space of all holomorphic functions on the unit ball $\mathbb{B} \subset \mathbb{C}^n$. This paper investigates the following integral-type operator with symbol $g \in H(\mathbb{B})$, $T_g f(z) = \int_0^1 f(tz)\mathfrak{R}g(tz)dt/t$, $f \in H(\mathbb{B})$, $z \in \mathbb{B}$, where $\mathfrak{R}g(z) = \sum_{j=1}^n z_j \partial g / \partial z_j(z)$ is the radial derivative of g. We characterize the boundedness and compactness of the integral-type operators T_g from general function spaces $F(p,q,s)$ to Zygmund-type spaces $\mathcal{Z}_\mu$, where μ is normal function on $[0,1)$.

1. Introduction

Let $\mathbb{B}$ be the open unit ball of $\mathbb{C}^n$, let $\partial\mathbb{B}$ be its boundary, and let $H(\mathbb{B})$ be the family of all holomorphic functions on $\mathbb{B}$. Let $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ be points in $\mathbb{C}^n$ and $\langle z, w \rangle = z_1\overline{w}_1 + \cdots + z_n\overline{w}_n$.

Let

$$\mathfrak{R}f(z) = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z) \tag{1.1}$$

stand for the radial derivative of $f \in H(\mathbb{B})$. For $a \in \mathbb{B}$, let $g(z, a) = \log(1/|\varphi_a(z)|)$, where φ_a is the Möbius transformation of $\mathbb{B}$ satisfying $\varphi_a(0) = a$, $\varphi_a(a) = 0$, and $\varphi_a = \varphi_a^{-1}$. For $0 < p, s < \infty$, $-n - 1 < q < \infty$, we say $f \in F(p,q,s)$ provided that $f \in H(\mathbb{B})$ and

$$\|f\|_{F(p,q,s)}^p = |f(0)|^p + \sup_{a\in\mathbb{B}} \int_{\mathbb{B}} |\mathfrak{R}f(z)|^p \left(1 - |z|^2\right)^q g^s(z, a) dv(z) < \infty. \tag{1.2}$$

The space $F(p, q, s)$, introduced by Zhao in [1], is known as the general family of function spaces. For appropriate parameter values p, q, and s, $F(p, q, s)$ coincides with several classical function spaces. For instance, let $\mathbb{D}$ be the unit disk in $\mathbb{C}$, $F(p, q, s) = \mathcal{B}^{(q+2)/p}$ if $1 < s < \infty$ (see [2]), where $\mathcal{B}^{\alpha}, 0 < \alpha < \infty$, consists of those analytic functions f in $\mathbb{D}$ for which

$$\|f\|_{\mathcal{B}^{\alpha}} = \sup_{z \in \mathbb{D}} \left(1 - |z|^2\right)^{\alpha} |f'(z)| < \infty. \tag{1.3}$$

The space $F(p, p, 0)$ is the classical Bergman space $A^p = A_0^p$ (see [3]), $F(p, p - 2, 0)$ is the classical Besov space B_p, and, in particular, $F(2, 1, 0)$ is just the Hardy space H^2. The spaces $F(2, 0, s)$ are Q_s spaces, introduced by Aulaskari et al. [4, 5]. Further, $F(2, 0, 1) = BMOA$, the analytic functions of bounded mean oscillation. Note that $F(p, q, s)$ is the space of constant functions if $q + s \leq -1$. More information on the spaces $F(p, q, s)$ can be found in [6, 7].

Recall that the Bloch-type spaces (or α-Bloch space) $\mathcal{B}^{\alpha} = \mathcal{B}^{\alpha}(\mathbb{B}), \alpha > 0$, consists of all $f \in H(\mathbb{B})$ for which

$$b_{\alpha}(f) = \sup_{z \in \mathbb{B}} \left(1 - |z|^2\right)^{\alpha} |\Re f(z)| < \infty. \tag{1.4}$$

The little Bloch-type space $\mathcal{B}_0^{\alpha}(\mathbb{B}) = \mathcal{B}_0^{\alpha}$ consists of all $f \in H(\mathbb{B})$ such that

$$\lim_{|z| \to 1} \left(1 - |z|^2\right)^{\alpha} |\Re f(z)| = 0. \tag{1.5}$$

Under the norm introduced by $\|f\|_{\mathcal{B}^{\alpha}} = |f(0)| + b_{\alpha}(f)$, $\mathcal{B}^{\alpha}$ is a Banach space and $\mathcal{B}_0^{\alpha}$ is a closed subspace of $\mathcal{B}^{\alpha}$. If $\alpha = 1$, we write $\mathcal{B}$ and $\mathcal{B}_0$ for $\mathcal{B}^1$ and $\mathcal{B}_0^1$, respectively.

A positive continuous function μ on the interval $[0,1)$ is called normal if there are three constants $0 \leq \delta < 1$ and $0 < a < b$ such that

$$\frac{\mu(r)}{(1-r)^a} \text{ is decreasing on } [\delta, 1), \quad \lim_{r \to 1} \frac{\mu(r)}{(1-r)^a} = 0,$$
$$\frac{\mu(r)}{(1-r)^b} \text{ is increasing on } [\delta, 1), \quad \lim_{r \to 1} \frac{\mu(r)}{(1-r)^b} = \infty. \tag{1.6}$$

Let $\mathcal{Z} = \mathcal{Z}(\mathbb{B})$ denote the class of all $f \in H(\mathbb{B})$ such that

$$\sup_{z \in \mathbb{B}} \left(1 - |z|^2\right) \left|\Re^2 f(z)\right| < \infty. \tag{1.7}$$

Write

$$\|f\|_{\mathcal{Z}} = |f(0)| + \sup_{z \in \mathbb{B}} \left(1 - |z|^2\right) \left|\Re^2 f(z)\right|. \tag{1.8}$$

With the norm $\|\cdot\|_{\mathscr{Z}}$, $\mathscr{Z}$ is a Banach space. $\mathscr{Z}$ is called the Zygmund space (see [8]). Let $\mathscr{Z}_0$ denote the class of all $f \in H(\mathbb{B})$ such that

$$\lim_{|z|\to 1}\left(1 - |z|^2\right)\left|\mathfrak{R}^2 f(z)\right| = 0. \tag{1.9}$$

Let μ be a normal function on $[0,1)$. It is natural to extend the Zygmund space to a more general form, for an $f \in H(\mathbb{B})$, we say that f belongs to the space $\mathscr{Z}_\mu = \mathscr{Z}_\mu(\mathbb{B})$ if

$$\sup_{z\in\mathbb{B}} \mu(|z|)\left|\mathfrak{R}^2 f(z)\right| < \infty. \tag{1.10}$$

It is easy to check that $\mathscr{Z}_\mu$ becomes a Banach space under the norm

$$\|f\|_{\mathscr{Z}_\mu} = |f(0)| + \sup_{z\in\mathbb{B}} \mu(|z|)\left|\mathfrak{R}^2 f(z)\right|, \tag{1.11}$$

and $\mathscr{Z}_\mu$ will be called the Zygmund-type space.

Let $\mathscr{Z}_{\mu,0}$ denote the class of holomorphic functions $f \in \mathscr{Z}_\mu$ such that

$$\lim_{|z|\to 1} \mu(|z|)\left|\mathfrak{R}^2 f(z)\right| = 0, \tag{1.12}$$

and $\mathscr{Z}_{\mu,0}$ is called the little Zygmund-type space. When $\mu(r) = 1 - r$, from [8, page 261], we say that $f \in \mathscr{Z}_{1-r} := \mathscr{Z}$ if and only if $f \in A(\mathbb{B})$, and there exists a constant $C > 0$ such that

$$\left|f(\zeta + h) + f(\zeta - h) - 2f(\zeta)\right| < C|h|, \tag{1.13}$$

for all $\zeta \in \partial\mathbb{B}$ and $\zeta \pm h \in \partial\mathbb{B}$, where $A(\mathbb{B})$ is the ball algebra on $\mathbb{B}$.

For $g \in H(\mathbb{B})$, the following integral-type operator (so called extended Cesàro operator) is

$$T_g f(z) = \int_0^1 f(tz)\mathfrak{R}g(tz)\frac{dt}{t}, \tag{1.14}$$

where $f \in H(\mathbb{B})$ and $z \in \mathbb{B}$. Stević [9] considered the boundedness of T_g on α-Bloch spaces. Lv and Tang got the boundedness and compactness of T_g from $F(p,q,s)$ to μ-Bloch spaces for all $0 < p, s < \infty, -n - 1 < q < \infty$ (see [10]). Recently, Li and Stević discussed the boundedness of T_g from Bloch-type spaces to Zygmund-type spaces in [11]. For more information about Zygmund spaces, see [12, 13].

In this paper, we characterize the boundedness and compactness of the operator T_g from general analytic spaces $F(p,q,s)$ to Zygmund-type spaces.

In what follows, we always suppose that $0 < p, s < \infty, -n - 1 < q < \infty, q + s > -1$. Throughout this paper, constants are denoted by C; they are positive and may have different values at different places.

2. Some Auxiliary Results

In this section, we quote several auxiliary results which will be used in the proofs of our main results. The following lemma is according to Zhang [14].

Lemma 2.1. *If $f \in F(p,q,s)$, then $f \in \mathcal{B}^{(n+1+q)/p}$ and*

$$\|f\|_{\mathcal{B}^{(n+1+q)/p}} \leq \|f\|_{F(p,q,s)}. \tag{2.1}$$

Lemma 2.2 (see [9]). *For $0 < \alpha < \infty$, if $f \in B^{\alpha}$, then for any $z \in \mathbb{B}$*

$$|f(z)| \leq \begin{cases} C\|f\|_{\mathcal{B}^{\alpha}}, & 0 < \alpha < 1, \\ C\|f\|_{\mathcal{B}^{\alpha}} \log \dfrac{2}{1-|z|^2}, & \alpha = 1, \\ C\|f\|_{\mathcal{B}^{\alpha}} \left(1-|z|^2\right)^{1-\alpha}, & \alpha > 1. \end{cases} \tag{2.2}$$

Lemma 2.3 (see [15]). *For every $f, g \in H(\mathbb{B})$, it holds $\mathfrak{R}[T_g(f)](z) = f(z)\mathfrak{R}g(z)$.*

Lemma 2.4 (see [10]). *Let $p = n + 1 + q$. Suppose that for each $w \in \mathbb{B}$, z-variable functions g_w satisfy $|g_w(z)| \leq C/|1 - \langle z, \omega \rangle|$, then*

$$\int_{\mathbb{B}} |g_\omega(z)|^p \left(1-|z|^2\right)^q g^s(z,a)dv(z) \leq C. \tag{2.3}$$

Lemma 2.5. *Assume that $g \in H(\mathbb{B})$, $0 < p$, $s < \infty$, $-n - 1 < q < \infty$, and μ is a normal function on $[0,1)$, then $T_g : F(p,q,s) \to \mathcal{Z}_\mu(\text{or } \mathcal{Z}_{\mu,0})$ is compact if and only if $T_g : F(p,q,s) \to \mathcal{Z}_\mu(\text{or } \mathcal{Z}_{\mu,0})$ is bounded, and for any bounded sequence $\{f_k\}_{k \in \mathbb{N}}$ in $F(p,q,s)$ which converges to zero uniformly on compact subsets of $\mathbb{B}$ as $k \to \infty$, one has $\lim_{k \to \infty} \|T_g f\|_{\mathcal{Z}_\mu} = 0$.*

The proof of Lemma 2.5 follows by standard arguments (see, e.g., Lemma 3 in [16]). Hence, we omit the details.

The following lemma is similar to the proof of Lemma 1 in [17]. Hence, we omit it.

Lemma 2.6. *Let μ be a normal function. A closed set K in $\mathcal{Z}_{\mu,0}$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z| \to 1} \sup_{f \in K} \mu(|z|) \left|\mathfrak{R}^2 f(z)\right| = 0. \tag{2.4}$$

3. Main Results and Proofs

Now, we are ready to state and prove the main results in this section.

Theorem 3.1. *Let $0 < p$, $s < \infty$, $-n - 1 < q < \infty$, and let μ be normal, $g \in H(\mathbb{B})$ and $n + 1 + q \geq p$, then $T_g : F(p,q,s) \to \mathcal{Z}_\mu$ is bounded if and only if*

(i) *for $n + 1 + q > p$,*

$$M_1 = \sup_{z \in \mathbb{B}} \mu(|z|) |\Re g(z)| \left(1 - |z|^2\right)^{-(n+1+q)/p} < \infty, \tag{3.1}$$

$$M_2 = \sup_{z \in \mathbb{B}} \mu(|z|) \left|\Re^2 g(z)\right| \left(1 - |z|^2\right)^{1-(n+1+q)/p} < \infty, \tag{3.2}$$

(ii) *for $n + 1 + q = p$,*

$$M_3 = \sup_{z \in \mathbb{B}} \mu(|z|) |\Re g(z)| \left(1 - |z|^2\right)^{-1} < \infty, \tag{3.3}$$

$$M_4 = \sup_{z \in \mathbb{B}} \mu(|z|) \left|\Re^2 g(z)\right| \log \frac{2}{1 - |z|^2} < \infty. \tag{3.4}$$

Proof. (i) First, for $f, g \in H(\mathbb{B})$, suppose that $n + 1 + q > p$ and $f \in F(p, q, s)$. By Lemmas 2.1–2.3, we write $\Re^2 f = \Re(\Re f)$. We have that

$$
\begin{aligned}
\|T_g f\|_{\mathcal{Z}_\mu} &= |T_g f(0)| + \sup_{z \in \mathbb{B}} \mu(|z|) \left|\Re^2 (T_g f)(z)\right| \\
&\leq \sup_{z \in \mathbb{B}} \mu(|z|) \left(|\Re f(z)| |\Re g(z)| + |f(z)| \left|\Re^2 g(z)\right|\right) \\
&\leq \|f\|_{\mathcal{B}^{(n+1+q)/p}} \sup_{z \in \mathbb{B}} \mu(|z|) |\Re g(z)| \left(1 - |z|^2\right)^{-(n+1+q)/p} \\
&\quad + C\|f\|_{\mathcal{B}^{(n+1+q)/p}} \sup_{z \in \mathbb{B}} \mu(|z|) \left|\Re^2 g(z)\right| \left(1 - |z|^2\right)^{1-(n+1+q)/p} \\
&\leq C\|f\|_{F(p,q,s)} \sup_{z \in \mathbb{B}} \mu(|z|) |\Re g(z)| \left(1 - |z|^2\right)^{-(n+1+q)/p} \\
&\quad + C\|f\|_{F(p,q,s)} \sup_{z \in \mathbb{B}} \mu(|z|) \left|\Re^2 g(z)\right| \left(1 - |z|^2\right)^{1-(n+1+q)/p}.
\end{aligned}
\tag{3.5}
$$

Hence, (3.1) and (3.2) imply that $T_g : F(p, q, s) \to \mathcal{Z}_\mu$ is bounded.

Conversely, assume that $T_g : F(p, q, s) \to \mathcal{Z}_\mu$ is bounded. Taking the test function $f(z) \equiv 1 \in F(p, q, s)$, we see that $g \in \mathcal{Z}_\mu$, that is,

$$\sup_{z \in \mathbb{B}} \mu(|z|) \left|\Re^2 g(z)\right| < \infty. \tag{3.6}$$

For $w \in \mathbb{B}$, set

$$f_w(z) = \frac{\left(1 - |w|^2\right)^{1+(n+1+q)/p}}{(1 - \langle z, w \rangle)^{2(n+1+q)/p}} - \frac{\left(1 - |w|^2\right)}{(1 - \langle z, w \rangle)^{(n+1+q)/p}}, \quad z \in \mathbb{B}. \tag{3.7}$$

Then, $\|f_w\|_{F(p,q,s)} \leq C$ by [14] and $f_w(w) = 0$.
    Hence,

$$
\begin{aligned}
\infty > \left\| T_g f_w \right\|_{\mathcal{Z}_\mu} &\geq \sup_{z \in \mathbb{B}} \mu(|z|) \left| \mathfrak{R}^2 (T_g f_w)(z) \right| \\
&= \sup_{z \in \mathbb{B}} \mu(|z|) \left| \mathfrak{R} f_w(z) \mathfrak{R} g(z) + f_w(z) \mathfrak{R}^2 g(z) \right| \\
&\geq \mu(|w|) \left| \mathfrak{R} f_w(w) \mathfrak{R} g(w) + f_w(w) \mathfrak{R}^2 g(w) \right| \\
&= \mu(|w|) \left| \mathfrak{R} f_w(w) \right| \left| \mathfrak{R} g(w) \right| \\
&= \frac{\mu(|w|) \left| \mathfrak{R} g(w) \right| |w|^2}{\left(1 - |w|^2\right)^{(n+1+q)/p}}.
\end{aligned}
\tag{3.8}
$$

From (3.8), we have

$$\sup_{|w|>1/2} \frac{\mu(|w|) \left| \mathfrak{R} g(w) \right|}{\left(1 - |w|^2\right)^{(n+1+q)/p}} < 4 \sup_{|w|>1/2} \frac{\mu(|w|) \left| \mathfrak{R} g(w) \right| |w|^2}{\left(1 - |w|^2\right)^{(n+1+q)/p}} \leq 4 \left\| T_g f_w \right\|_{\mathcal{Z}_\mu} < \infty. \tag{3.9}$$

On the other hand, we have

$$\sup_{|w| \leq 1/2} \frac{\mu(|w|) \left| \mathfrak{R} g(w) \right|}{\left(1 - |w|^2\right)^{(n+1+q)/p}} < C \sup_{|w| \leq 1/2} \mu(|w|) \left| \mathfrak{R} g(w) \right| < \infty. \tag{3.10}$$

Combing (3.9) and (3.10), we get (3.1).
    In order to prove (3.2), let $w \in \mathbb{B}$ and set

$$h_w(z) = \frac{1 - |w|^2}{(1 - \langle z, w \rangle)^{(n+1+q)/p}}. \tag{3.11}$$

It is easy to see that $h_w(w) = 1/(1 - |w|^2)^{(n+1+q)/p-1}$, $\Re h_w(w) \approx |w|^2/(1 - |w|^2)^{(n+1+q)/p}$. We know that $h_w \in F(p, q, s)$; moreover, there is a positive constant C such that $\|h_w\|_{F(p,q,s)} \leq C$. Hence,

$$
\begin{aligned}
\infty > \|T_g h_w\|_{\mathcal{Z}_\mu} &\geq \sup_{z \in \mathbb{B}} \mu(|z|) \left| \Re^2 (T_g h_w)(z) \right| \\
&= \sup_{z \in \mathbb{B}} \mu(|z|) \left| \Re h_w(z) \Re g(z) + h_w(z) \Re^2 g(z) \right| \\
&\geq \mu(|w|) \left| \Re^2 g(w) \right| \left(1 - |w|^2 \right)^{1-(n+1+q)/p} - \frac{\mu(|w|) \left| \Re g(w) \right| |w|^2}{\left(1 - |w|^2 \right)^{(n+1+q)/p}}.
\end{aligned}
\tag{3.12}
$$

From (3.1) and (3.12), we see that (3.2) holds.

(ii) If $n+1+q = p$, then, by Lemmas 2.1 and 2.2, we have $F(p, q, s) \subseteq \mathcal{B}^1$, for $f \in F(p, q, s)$, we get

$$
\begin{aligned}
\|T_g f\|_{\mathcal{Z}_\mu} &= |T_g f(0)| + \sup_{z \in \mathbb{B}} \mu(|z|) \left| \Re^2 (T_g f)(z) \right| \\
&\leq \sup_{z \in \mathbb{B}} \mu(|z|) \left(|\Re f(z)| |\Re g(z)| + |f(z)| \left| \Re^2 g(z) \right| \right) \\
&\leq \|f\|_{\mathcal{B}^1} \sup_{z \in \mathbb{B}} \mu(|z|) |\Re g(z)| \left(1 - |z|^2 \right)^{-1} \\
&\quad + C \|f\|_{\mathcal{B}^1} \sup_{z \in \mathbb{B}} \mu(|z|) \left| \Re^2 g(z) \right| \log \frac{2}{1 - |z|^2} \\
&\leq C \|f\|_{F(p,q,s)} \sup_{z \in \mathbb{B}} \mu(|z|) |\Re g(z)| \left(1 - |z|^2 \right)^{-1} \\
&\quad + C \|f\|_{F(p,q,s)} \sup_{z \in \mathbb{B}} \mu(|z|) \left| \Re^2 g(z) \right| \log \frac{2}{1 - |z|^2}.
\end{aligned}
\tag{3.13}
$$

Applying (3.3) and (3.4) in (3.13), for the case $n + 1 + q = p$, the boundedness of the operator $T_g : F(p, q, s) \to \mathcal{Z}_\mu$ follows.

Conversely, suppose that $T_g : F(p, q, s) \to \mathcal{Z}_\mu$ is bounded. Given any $w \in \mathbb{B}$, set

$$
f_w(z) = \frac{\left(1 - |w|^2 \right)^2}{(1 - \langle z, w \rangle)^2} - \frac{\left(1 - |w|^2 \right)}{(1 - \langle z, w \rangle)}, \quad z \in \mathbb{B},
\tag{3.14}
$$

then $\|f_w\|_{F(p,q,s)} \leq C$ by [14].

By the boundedness of T_g, it is easy to see that

$$\frac{\mu(|w|)|\Re g(w)||w|^2}{\left(1-|w|^2\right)} < \infty. \tag{3.15}$$

By (3.14) and (3.15), in the same way as proving (3.1), we get that (3.3) holds.

Now, given any $w \in \mathbb{B}$, set

$$f_w(z) = \log\frac{2}{1-\langle z, w\rangle}, \quad z \in \mathbb{B}, \tag{3.16}$$

then $|\Re f_w(z)| \leq C/|1 - \langle z, w\rangle|$. Applying Lemma 2.4, we have that $\|f_w\|_{F(p,q,s)} \leq C$. Hence,

$$
\begin{aligned}
\infty > \|T_g f_w\|_{\mathscr{Z}_\mu} &\geq \sup_{z \in \mathbb{B}} \mu(|z|)\left|\Re^2(T_g f_w)(z)\right| \\
&\geq \sup_{z \in \mathbb{B}} \mu(|z|)\left|\Re f_w(z)\Re g(z) + f_w(z)\Re^2 g(z)\right| \\
&\geq \mu(|w|)\left|\Re^2 g(w)\right| \log\frac{2}{1-|w|^2} - \frac{\mu(|w|)|\Re g(w)||w|^2}{\left(1-|w|^2\right)}.
\end{aligned}
\tag{3.17}
$$

From (3.15) and (3.17), we see that (3.4) holds. The proof of this theorem is completed. $\quad\square$

Theorem 3.2. *Let* $0 < p, s < \infty$, $-n-1 < q < \infty$, *and let* μ *be normal,* $g \in H(\mathbb{B})$ *and* $n+1+q \geq p$, *then the following statements are equivalent:*

(A) $T_g : F(p, q, s) \to \mathscr{Z}_\mu$ *is compact;*

(B) $T_g : F(p, q, s) \to \mathscr{Z}_{\mu,0}$ *is compact;*

(C)   (i) *for* $n+1+q > p$,

$$\lim_{|z|\to 1} \mu(|z|)|\Re g(z)|\left(1-|z|^2\right)^{-(n+1+q)/p} = 0, \tag{3.18}$$

$$\lim_{|z|\to 1} \mu(|z|)\left|\Re^2 g(z)\right|\left(1-|z|^2\right)^{1-(n+1+q)/p} = 0, \tag{3.19}$$

(ii) *for* $n+1+q = p$,

$$\lim_{|z|\to 1} \mu(|z|)|\Re g(z)|\left(1-|z|^2\right)^{-1} = 0, \tag{3.20}$$

$$\lim_{|z|\to 1} \mu(|z|)\left|\Re^2 g(z)\right| \log\frac{2}{1-|z|^2} = 0. \tag{3.21}$$

Proof. (B) $\Rightarrow$ (A). This implication is obvious.

(A) $\Rightarrow$ (C). First, for the case $n + 1 + q > p$.

Suppose that the operator $T_g : F(p, q, s) \to \mathscr{Z}_\mu$ is compact. Let $\{z_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathbb{B}$ such that $\lim_{k\to\infty}|z_k| = 1$. Denote $f_k(z) = f_{z_k}(z), k \in \mathbb{N}$, and set

$$f_k(z) = \frac{\left(1 - |z_k|^2\right)^{1+(n+1+q)/p}}{(1 - \langle z, z_k\rangle)^{2(n+1+q)/p}} - \frac{\left(1 - |z_k|^2\right)}{(1 - \langle z, z_k\rangle)^{(n+1+q)/p}}, \quad k \in \mathbb{N}. \tag{3.22}$$

It is easy to see that $f_k \in F(p, q, s)$ for $k \in \mathbb{N}$ and $f_k \to 0$ uniformly on compact subsets of $\mathbb{B}$ as $k \to \infty$. By Lemma 2.5, it follows that

$$\lim_{k\to\infty} \left\|T_g f_k\right\|_{\mathscr{Z}_\mu} = 0. \tag{3.23}$$

By Lemma 2.3, we have

$$\begin{aligned}
\left\|T_g f_k\right\|_{\mathscr{Z}_\mu} &\geq \sup_{z\in\mathbb{B}} \mu(|z|)\left|\mathfrak{R}^2\left(T_g f_k\right)(z)\right| \\
&= \sup_{z\in\mathbb{B}} \mu(|z|)\left|\mathfrak{R}f_k(z)\mathfrak{R}g(z) + f_k(z)\mathfrak{R}^2 g(z)\right| \\
&\geq \mu(|z_k|)\left|\mathfrak{R}f_k(z_k)\mathfrak{R}g(z_k)\right| + f_k(z_k)\mathfrak{R}^2 g(z_k)\bigg| \\
&= \mu(|z_k|)\left|\mathfrak{R}f_k(z_k)\right|\left|\mathfrak{R}g(z_k)\right| \\
&= \frac{\mu(|z_k|)\mathfrak{R}g(z_k)|z_k|^2}{\left(1 - |z_k|^2\right)^{(n+1+q)/p}}.
\end{aligned} \tag{3.24}$$

From (3.23) and (3.24), we obtain

$$\lim_{k\to\infty} \frac{\mu(|z_k|)\mathfrak{R}g(z_k)}{\left(1 - |z_k|^2\right)^{(n+1+q)/p}} = \lim_{k\to\infty} \frac{\mu(|z_k|)\mathfrak{R}g(z_k)|z_k|^2}{\left(1 - |z_k|^2\right)^{(n+1+q)/p}} = 0, \tag{3.25}$$

which means that (3.18) holds.

Similarly, we take the test function

$$f_k(z) = \frac{\left(1 - |z_k|^2\right)^2}{(1 - \langle z, z_k\rangle)^2} - \frac{\left(1 - |z_k|^2\right)}{(1 - \langle z, z_k\rangle)}, \quad k \in \mathbb{N}. \tag{3.26}$$

Then, $f_k \in F(p, q, s)$ for $k \in \mathbb{N}$ and $f_k \to 0$ uniformly on compact subsets of $\mathbb{B}$ as $k \to \infty$. We obtain that (3.20) holds for the case $n + 1 + q = p$.

For proving (3.19), we set

$$h_k(z) = \frac{1 - |z_k|^2}{(1 - \langle z, z_k \rangle)^{(n+1+q)/p}}, \quad z \in \mathbb{B}, \tag{3.27}$$

then $\|h_k\|_{F(p,q,s)} \leq C$, and $\{h_k\}_{k \in \mathbb{N}}$ converges to 0 uniformly on any compact subsets of $\mathbb{B}$ as $k \to \infty$. By Lemma 2.5, it yields

$$\lim_{k \to \infty} \|T_g h_k\|_{\mathscr{Z}_\mu} = 0. \tag{3.28}$$

Further, we have

$$
\begin{aligned}
\|T_g h_k\|_{\mathscr{Z}_\mu} &\geq \sup_{z \in \mathbb{B}} \mu(|z|) \left| \mathfrak{R}^2 (T_g h_k)(z) \right| \\
&= \sup_{z \in \mathbb{B}} \mu(|z|) \left| \mathfrak{R} h_k(z) \mathfrak{R} g(z) + h_k(z) \mathfrak{R}^2 g(z) \right| \\
&\geq \mu(|z_k|) \left| \mathfrak{R} h_k(z_k) \mathfrak{R} g(z_k) + h_k(z_k) \mathfrak{R}^2 g(z_k) \right| \\
&\geq \mu(|z_k|) \left| \mathfrak{R}^2 g(z_k) \right| \left(1 - |z_k|^2\right)^{1-(n+1+q)/p} - \frac{\mu(|z_k|) \left| \mathfrak{R} g(z_k) \right| |z_k|^2}{\left(1 - |z_k|^2\right)^{(n+1+q)/p}}.
\end{aligned}
\tag{3.29}
$$

From (3.25), (3.28), and (3.29), it follows that

$$\lim_{k \to \infty} \mu(|z_k|) \left| \mathfrak{R}^2 g(z_k) \right| \left(1 - |z_k|^2\right)^{1-(n+1+q)/p} = 0, \tag{3.30}$$

which implies that (3.19) holds.

(ii) Second, for the case $n + 1 + q = p$, take the test function

$$f_k(z) = \frac{(\log(2/(1 - \langle z, z_k \rangle)))^2}{\log\left(2/\left(1 - |z_k|^2\right)\right)}, \quad z \in \mathbb{B}. \tag{3.31}$$

Then, $\|f_k\|_{F(p,q,s)} \leq C$ by Lemma 2.4 and $f_k \to 0$ uniformly on any compact subset of $\mathbb{B}$. By Lemma 2.5 and condition (A), we have

$$\|T_g f_k\|_{\mathscr{Z}_\mu} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{3.32}$$

Hence, we have that

$$
\begin{aligned}
\left\| T_g f_k \right\|_{\mathcal{Z}_\mu} &\geq \sup_{z \in \mathbb{B}} \mu(|z|) \left| \mathfrak{R}^2 (T_g f_k)(z) \right| \\
&= \sup_{z \in \mathbb{B}} \mu(|z|) \left| \mathfrak{R} f_k(z) \mathfrak{R} g(z) + f_k(z) \mathfrak{R}^2 g(z) \right| \\
&\geq \mu(|z_k|) \left| \mathfrak{R}^2 g(z_k) \right| \log \frac{2}{1 - |z_k|^2} - 2 \frac{\mu(|z_k|) |\mathfrak{R} g(z_k)| |z_k|^2}{\left(1 - |z_k|^2 \right)}.
\end{aligned}
\tag{3.33}
$$

From (3.20), (3.32), and (3.33), it follows that

$$
\lim_{k \to \infty} \mu(|z_k|) \left| \mathfrak{R}^2 g(z_k) \right| \log \frac{2}{1 - |z_k|^2} = 0,
\tag{3.34}
$$

which implies that (3.21) holds.

   (C) $\Rightarrow$ (B). Suppose that (3.18) and (3.19) hold for $f \in F(p,q,s)$. By Lemmas 2.1 and 2.2, we have that

$$
\begin{aligned}
\mu(|z|) \left| \mathfrak{R}^2 (T_g f)(z) \right| &= \mu(|z|) \left| \mathfrak{R} f(z) \mathfrak{R} g(z) + f(z) \mathfrak{R}^2 g(z) \right| \\
&\leq C \|f\|_{F(p,q,s)} \mu(|z|) |\mathfrak{R} g(z)| \left(1 - |z|^2 \right)^{-(n+1+q)/p} \\
&\quad + C \|f\|_{F(p,q,s)} \mu(|z|) \left| \mathfrak{R}^2 g(z) \right| \left(1 - |z|^2 \right)^{1-(n+1+q)/p}.
\end{aligned}
\tag{3.35}
$$

Note that (3.18) and (3.19) imply that

$$
\lim_{|z| \to 1} \mu(|z|) \left| \mathfrak{R}^2 g(z) \right| = 0.
\tag{3.36}
$$

Further, they also imply that (3.1) and (3.2) hold. From this and Theorem 3.1, it follows that set $T_g(\{ f : \|f\|_{F(p,q,s)} \leq 1 \})$ is bounded. Using these facts, (3.18), and (3.19), we have

$$
\lim_{|z| \to 1} \sup_{\|f\|_{F(p,q,s)} \leq 1} \mu(|z|) \left| \mathfrak{R}^2 (T_g f)(z) \right| = 0.
\tag{3.37}
$$

Similarly, we obtain that (3.37) holds for the case $n + 1 + q = p$ by (3.20) and (3.21). Exploiting Lemma 2.6, the compactness of the operator $T_g : F(p,q,s) \to \mathcal{Z}_{\mu,0}$ follows. The proof of this theorem is completed. $\qquad\square$

   Finally, we consider the case $n + 1 + q < p$.

Theorem 3.3. *Let $0 < p, s < \infty$, $-n - 1 < q < \infty$, and let μ be normal, $g \in H(\mathbb{B})$, $n + 1 + q < p$, then the following statements are equivalent:*

(A) $T_g : F(p, q, s) \to \mathscr{Z}_\mu$ *is bounded;*

(B) $g \in \mathscr{Z}_\mu$ *and*

$$\sup_{z \in \mathbb{B}} \mu(|z|) |\Re g(z)| \left(1 - |z|^2\right)^{-(n+1+q)/p} < \infty. \tag{3.38}$$

The proof of Theorem 3.3 follows by the proof of Theorem 3.1. So, we omit the details here.

Theorem 3.4. *Let $0 < p, s < \infty$, $-n - 1 < q < \infty$, and let μ be normal, $g \in H(\mathbb{B})$ and $n + 1 + q < p$, then the following statements are equivalent:*

(A) $T_g : F(p, q, s) \to \mathscr{Z}_\mu$ *is compact;*

(B) $g \in \mathscr{Z}_\mu$ *and*

$$\lim_{|z| \to 1} \mu(|z|) |\Re g(z)| \left(1 - |z|^2\right)^{-(n+1+q)/p} = 0. \tag{3.39}$$

Proof. (A) $\Rightarrow$ (B). We assume that $T_g : F(p, q, s) \to \mathscr{Z}_\mu$ is compact. For $f \equiv 1$, we obtain that $g \in \mathscr{Z}_\mu$. Exploiting the test function in (3.22), similarly to the proof of Theorem 3.2, we obtain that (3.39) holds. As a consequence, it follows that

$$\lim_{|z| \to 1} \mu(|z|) |\Re g(z)| = 0. \tag{3.40}$$

(B) $\Rightarrow$ (A). Assume that $\{f_k\}_{k \in \mathbb{N}}$ is a sequence in $F(p, q, s)$ such that $\sup_{k \in \mathbb{N}} \|f_k\|_{F(p,q,s)} \leq L < \infty$, and $f_k \to 0$ uniformly on compact of $\mathbb{B}$ as $k \to \infty$. By Lemma 2.1 and [18, Lemma 4.2],

$$\lim_{k \to \infty} \sup_{z \in \mathbb{B}} |f_k(z)| = 0. \tag{3.41}$$

From (3.39), we have that for every $\varepsilon > 0$, there is a $\delta \in (0, 1)$, such that, for every $\delta < |z| < 1$,

$$\frac{\mu(|z|) |\Re g(z)|}{\left(1 - |z|^2\right)^{(n+1+q)/p}} < \varepsilon, \tag{3.42}$$

and from (3.39) that

$$G_\mu = \sup_{z \in \mathbb{B}} \mu(|z|) |\Re g(z)| < \infty. \tag{3.43}$$

Hence,

$$
\begin{aligned}
\mu(|z|)\left|\mathfrak{R}^2(T_g f_k)(z)\right| &= \mu(|z|)\left|\mathfrak{R} f_k(z)\mathfrak{R}g(z) + f_k(z)\mathfrak{R}^2 g(z)\right| \\
&\le \sup_{|z|\le\delta}\mu(|z|)\left|\mathfrak{R} f_k(z)\right|\left|\mathfrak{R}g(z)\right| + \sup_{\delta<|z|<1}\mu(|z|)\left|\mathfrak{R} f_k(z)\right|\left|\mathfrak{R}g(z)\right| \\
&\quad + \|g\|_{\mathcal{Z}_\mu}\sup_{z\in\mathbb{B}}\left|f_k(z)\right| \\
&\le G_\mu\sup_{|z|\le\delta}\left|\mathfrak{R} f_k(z)\right| + \|f_k\|_{\mathcal{B}^{(n+1+q)/p}}\sup_{\delta<|z|<1}\frac{\mu(|z|)\left|\mathfrak{R}g(z)\right|}{\left(1-|z|^2\right)^{(n+1+q)/p}} \\
&\quad + \|g\|_{\mathcal{Z}_\mu}\sup_{z\in\mathbb{B}}\left|f_k(z)\right| \\
&\le G_\mu\sup_{|z|\le\delta}\left|\mathfrak{R} f_k(z)\right| + \varepsilon L + \|g\|_{\mathcal{Z}_\mu}\sup_{z\in\mathbb{B}}\left|f_k(z)\right|.
\end{aligned}
\tag{3.44}
$$

Since $f_k \to 0$ on compact subsets of $\mathbb{B}$ by the Cauchy estimate, it follows that $\mathfrak{R} f_k \to 0$ on compact subsets of $\mathbb{B}$, in particular on $|z| \le \delta$. Taking in (3.44), the supremum over $z \in \mathbb{B}$, letting $k \to \infty$, using the above-mentioned facts, $T_g f_k(0) = 0$, and since ε is an arbitrary positive number, we obtain

$$
\lim_{k\to\infty}\|T_g f_k\|_{\mathcal{Z}_\mu} = 0.
\tag{3.45}
$$

Hence, by Lemma 2.5, the compactness of the operator $T_g : F(p,q,s) \to \mathcal{Z}_\mu$ follows. The proof of this theorem is completed. □

Theorem 3.5. *Let $0 < p, s < \infty$, $-n - 1 < q < \infty$, and let μ be normal, $g \in H(\mathbb{B})$ and $n + 1 + q < p$, then the following statements are equivalent:*

(A) $T_g : F(p,q,s) \to \mathcal{Z}_{\mu,0}$ *is compact;*

(B) $g \in \mathcal{Z}_{\mu,0}$ *and*

$$
\lim_{|z|\to1}\mu(|z|)\left|\mathfrak{R}g(z)\right|\left(1-|z|^2\right)^{-(n+1+q)/p} = 0.
\tag{3.46}
$$

Proof. (A) $\Rightarrow$ (B). For $f \equiv 1$, we obtain that $g \in \mathcal{Z}_{\mu,0}$. In the same way as in Theorem 3.4, we get that (3.46) holds.

(B) $\Rightarrow$ (A). By Lemmas 2.1 and 2.2, we have that

$$
\begin{aligned}
\mu(|z|)\left|\mathfrak{R}^2(T_g f)(z)\right| &= \mu(|z|)\left|\mathfrak{R} f(z)\mathfrak{R}g(z) + f(z)\mathfrak{R}^2 g(z)\right| \\
&\le C\|f\|_{F(p,q,s)}\mu(|z|)\left|\mathfrak{R}g(z)\right|\left(1-|z|^2\right)^{-(n+1+q)/p} \\
&\quad + C\|f\|_{F(p,q,s)}\mu(|z|)\left|\mathfrak{R}^2 g(z)\right|.
\end{aligned}
\tag{3.47}
$$

This along with Theorem 3.2 implies that $T_g(\{f : \|f\|_{F(p,q,s)} \leq 1\})$ is bounded. Taking the supremum over the unit ball in $F(p,q,s)$, letting $|z| \to 1$ in (3.46), using the condition (B), and finally by applying Lemma 2.6, we get the compactness of the operator $T_g : F(p,q,s) \to \mathcal{Z}_{\mu,0}$. This completes the proof of the theorem. $\qquad\square$

Acknowledgments

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